Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 3, No. 1, 2015, pp. 51-58

A new family of four-step fifteenth-order root-finding methods with high efficiency index

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Abstract In this paper a new family of fifteenth-order methods with high efficiency index is presented. This family include four evaluations of the function and one evaluation of its first derivative per iteration. Therefore, this family of methods has the efficiency index which equals 1.71877. In order to show the applicability and validity of the class, some numerical examples are discussed.

Keywords. Nonlinear equations, Ostrowski's method, Order of convergence, Efficiency index.

2010 Mathematics Subject Classification. 65H05.

1. Introduction

Consider a nonlinear equation

$$f(x) = 0, \tag{1.1}$$

where $f : D \subset \mathbb{R} \to \mathbb{R}$ is a scalar function on an open interval D. In order to approximate the simple root α of (1.1), it is suitable to use iteration methods. The first famous iterative method was attributed by Newton and it is defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{1.2}$$

this is an important and basic method (see [8,12]), which converges quadratically. In recent years many iterative methods have been proposed [1-11,13,14]. These methods

Received: 1 February 2015 ; Accepted: 7 October 2015.

are mostly based on the Newton's method. The famous Ostrowski's method [8] is an example of fourth order multipoint methods, which is given as

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)} \frac{f(y_n)}{f'(x_n)}. \end{cases}$$

Recently, based on Ostrowski's method, some higher-order multipoint methods have been proposed for solving nonlinear equations. Soleymani and Sharifi in [9] suggested fifteenth-order methods given by

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ w_n = z_n - \frac{f(z_n)f[x_n,y_n]}{f[x_n,z_n]f[y_n,z_n]} \left(1 + \frac{f(z_n)}{f(x_n)}\right), \\ x_{n+1} = w_n - \frac{f(w_n)}{f[x_n,w_n]+(f[y_n,x_n,z_n]-f[y_n,x_n,w_n]-f[z_n,x_n,w_n])(x_n-w_n)}. \end{cases} \tag{1.3}$$

Zafar and Bibi in [14] presented new iterative methods of convergence order fourteen to approximate the simple roots of non-linear equations:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \frac{(x_n-y_n)f(y_n)}{f(x_n)-2f(y_n)}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)} - \frac{f(w_n)}{f'(w_n)}, \end{cases} \tag{1.4}$$

where

$$w_n = z_n - \frac{f(z_n)}{f'(z_n)},$$
$$f'(z_n) = f[z_n, y_n] + f[z_n, x_n, x_n](z_n - y_n),$$
$$f'(w_n) = f[x_n, w_n] + (f[y_n, x_n, z_n] - f[y_n, x_n, w_n] - f[z_n, x_n, w_n])(x_n - w_n).$$

In this paper, we present a new family with high order of convergence and high efficiency index. The convergence analysis is provided to establish their fifteenth-order of convergence. In terms of computational cost, they require the evaluations of only four functions and one first-order derivative per iteration. This gives 1.71877 as efficiency index of the presented methods. The new methods are comparable with Soleymani and Sharifi's method, Zafar and Bibi's method and Newton's method. The efficacy of the methods is tested on a number of numerical examples.

2. New class of iteration methods

In this section, we construct a new class of four-step methods with fifteenth-order of convergence. Due to this, we consider

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ s_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)}\frac{f(y_n)}{f'(x_n)}, \\ z_n = s_n - \{K(t_1) \times L(t_2) \times P(t_3)\}\frac{f(s_n)f[x_n,y_n]}{f[x_n,s_n]f[y_n,s_n]}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \end{cases}$$

where

$$t_1 = \frac{f(s_n)}{f(x_n)}, \quad t_2 = \frac{f(y_n)}{f(x_n)}, \quad t_3 = \frac{f(s_n)}{f(y_n)},$$

and $K(t_1), L(t_2), P(t_3)$ are three real-valued weight functions. The first three steps are the eighth-order convergent iterative method of [7]. This scheme includes four evaluations of the function and two evaluations of its first derivative. $f'(z_n)$ should be annihilated as the order of convergence remains at the highest level by the smallest use of number of evaluations per iteration. Hence, to improve the efficiency index, we approximate $f'(z_n)$ by

$$f'(z_n) \approx f[z_n,s_n] + f[z_n,s_n,y_n](z_n - s_n) + f[z_n,s_n,y_n,x_n](z_n - s_n)(z_n - y_n).$$

Now, we present a new family of fifteenth-order Ostrowski-type iterative methods as follows:

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \\ s_n = y_n - \frac{f(x_n)}{f(x_n)-2f(y_n)} \frac{f(y_n)}{f'(x_n)}, \\ z_n = s_n - \{K(t_1) \times L(t_2) \times P(t_3)\} \frac{f(s_n)f[x_n,y_n]}{f[x_n,s_n]f[y_n,s_n]}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[z_n,s_n]+f[z_n,s_n,y_n](z_n-s_n)+f[z_n,s_n,y_n,x_n](z_n-s_n)(z_n-y_n)}, \end{cases} \tag{2.1}$$

where

$$t_1 = \frac{f(s)}{f(x)}, \quad t_2 = \frac{f(y)}{f(x)}, \quad t_3 = \frac{f(s)}{f(y)},$$

without the index n, should be chosen such that the order of convergence arrives fifteen. The existence of the weight functions in scheme (2.1) is clear, for example

$$\begin{aligned} &K_1(t) = \sin(t) + \cos(t), \qquad L_1(t) = t^4 e^t + 1, \qquad P_1(t) = e^{t^2}, \\ &K_2(t) = e^t - 1 + \cos(t), \qquad L_2(t) = e^{t^4}, \qquad P_2(t) = 1 - t + \sin(t), \\ &K_3(t) = 1 + \sin(t), \quad L_3(t) = 1 + t^4\cos(t), \quad P_3(t) = \cos(t). \end{aligned}$$

Convergence analysis on scheme (2.1) will be best characterized in Theorem 2.1:

Theorem 2.1. *Suppose $\alpha \in D$ be a simple zero of a sufficiently differentiable function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and $f'(\alpha) \neq 0$. If the initial point x_0 is sufficiently close to α, then the sequence x_n generated by any method of the family (2.1) has fifteen-order of convergence to α if K, L and P are any functions with*

$$\begin{aligned} &K(0) = K'(0) = 1, \\ &L(0) = 1, \quad L'(0) = L''(0) = L'''(0) = 0, \quad \left|L^{(4)}(0)\right| < \infty, \\ &P(0) = 1, \quad P'(0) = 0, \quad |\mathrm{P}''(0)| < \infty, \end{aligned} \tag{2.2}$$

and the error equation is given by:

$$\begin{aligned} e_{n+1} = x_{n+1} - \alpha = &-\tfrac{1}{24}c_2^3(c_2^2 - c_3)^2 c_4 \left(\left(L^{(4)}(0) + 12(-6 + \mathrm{P}''(0))\right) c_2^4 \right. \\ &\left. -24(-4 + \mathrm{P}''(0))c_2^2 c_3 + 12\mathrm{P}''(0)c_3^2 - 24c_2c_4\right) e_n^{15} + O(e_n^{16}). \end{aligned}$$

(2.3)

Proof. Let α be the simple root of $f(x)$, i.e., $f(\alpha) = 0, f'(\alpha) \neq 0$, and the error equation is $e_n = x_n - \alpha$.

By Taylor's expansion of $f(x_n)$ about $x = \alpha$ and putting $f(\alpha) = 0$, we have

$$f(x_n) = f'(\alpha)\left(e_n + c_2 e_n^2 + c_3 e_n^3 + \cdots + c_{15} e_n^{15}\right) + O(e_n^{16}),$$

where

$$c_k = \frac{f^{(k)}(\alpha)}{k!\, f'(\alpha)}, \qquad k = 2, 3, \cdots$$

Accordingly, we attain

$$y_n - \alpha = c_2 e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + \cdots + O(e_n^{16}).$$

Similarly

$$s_n - \alpha = (c_2^3 - c_2 c_3)e_n^4 - 2(2c_2^4 - 4c_2^2 c_3 + c_3^2 + c_2 c_4)e_n^5 + \cdots + O(e_n^{16}).$$

We also have

$$f(s_n) = f'(\alpha)\left((c_2^3 - c_2 c_3)e_n^4 - 2(2c_2^4 - 4c_2^2 c_3 + c_3^2 + c_2 c_4)e_n^5\right) + \cdots + O(e_n^{16})$$

Now using symbolic computation in the third step of (2.1) and the real-valued weight functions as in (2.2), we attain

$$\begin{aligned} z_n - \alpha = -\tfrac{1}{24}\big(&c_2(c_2^2 - c_3)((L^{(4)}(0) + 12(-6 + P''(0)))c_2^4 \\ &-24(-4 + P''(0))c_2^2 c_3 + 12P''(0)c_3^2 - 24c_2 c_4)\big)\, e_n^8 + \cdots + O(e_n^{16}). \end{aligned} \tag{2.4}$$

By considering (2.4), we will obtain the error equation (2.3). This shows that the order of convergence of the new family of Ostrowski-type iterative methods (2.1) is fifteen. This completes the proof. $\square$

3. Numerical results

In this section, the new methods (2.1) are employed to solve some nonlinear equations and compared with Soleymani and Sharifi's method (SSM), Zafar and Bibi's method (ZBM), and Newton's method (NM). The test functions of $f(x)$ are listed as follows: Numerical computations reported here have been carried out in a

Table 1. Test functions and their roots

Functions	Roots
$f_1(x) = x^2 - e^x - 3x + 2$	$\alpha \approx 0.25753028543986079$
$f_2(x) = x^5 + x - 10000$	$\alpha \approx 6.3087771299726893$
$f_3(x) = x^5 + x^4 + 4x^2 - 15$	$\alpha \approx 1.3474280989683053$
$f_4(x) = 10xe^{-x^2} - 1$	$\alpha \approx 1.6796306104284499$
$f_5(x) = \cos(x) - x$	$\alpha \approx 0.73908513321516064$
$f_6(x) = e^{-x^2+x+2} - 1$	$\alpha \approx -1$
$f_7(x) = \log(x^2 + x + 2) - x + 1$	$\alpha \approx 4.1525907367571583$
$f_8(x) = e^x + x - 20$	$\alpha \approx 2.8424389537844472$
$f_9(x) = \arcsin(x^2 - 1) - \frac{1}{2}x + 1$	$\alpha \approx 0.59481096839836918$
$f_{10}(x) = 2x\cos(x) + x - 3$	$\alpha \approx -3.0346643069740450$
$f_{11}(x) = (x - 1)^6 - 1$	$\alpha \approx 2$
$f_{12}(x) = \cos^2(x) - \frac{x}{5}$	$\alpha \approx 2.3202042744957261$

$Mathematica$ 8.0 environment. Table 2 shows the difference of the root α and the approximation x_n to α, where α is the exact root computed with 800 significant digits ($Digits := 800$) and x_n is calculated by using the same Total Number of Function Evaluations (TNFE) for all methods. $|x_n - \alpha|$ and the absolute values of the function ($|f(x_n)|$) are also shown in Table 2.

4. Conclusions

In this work, we have suggested new higher-order iterative methods. The convergence order of these methods is fifteen, and consist of four evaluations of the function and one evaluation of its first derivative per iteration, so they have an efficiency index equal to $15^{\frac{1}{5}} = 1.71877$. Computational results and comparison with the existing well known methods confirm robust and efficient of our methods.

TABLE 2. Comparison of various iterative methods under the same total number of function evaluations (TNFE = 10).

	(2.1), K_1, L_1, P_1	(2.1), K_2, L_2, P_2	(2.1), K_3, L_3, P_3	(SSM)	(ZBM)	(NM)
$f_1, x_0 = 0$						
$\|x_n - \alpha\|$	0.52e − 290	0.18e − 292	0.42e − 294	0.88e − 293	0.24e − 254	0.41e − 49
$\|f(x_n)\|$	0.20e − 289	0.67e − 292	0.16e − 293	0.33e − 292	0.91e − 254	0.16e − 48
$f_2, x_0 = -1$						
$\|x_n - \alpha\|$	0.11e − 48	0.65e − 47	0.63e − 46	0.12e − 45	0.28e − 36	0.28e − 5
$\|f(x_n)\|$	0.90e − 45	0.51e − 43	0.50e − 42	0.94e − 42	0.23e − 32	0.23e − 1
$f_3, x_0 = 2$						
$\|x_n - \alpha\|$	0.14e − 239	0.17e − 239	0.47e − 241	0.21e − 248	0.43e − 200	0.11e − 30
$\|f(x_n)\|$	0.51e − 238	0.63e − 238	0.17e − 239	0.79e − 247	0.16e − 198	0.43e − 29
$f_4, x_0 = 1.6$						
$\|x_n - \alpha\|$	0.84e − 211	0.44e − 211	0.55e − 214	0.22e − 201	0.34e − 184	0.21e − 28
$\|f(x_n)\|$	0.23e − 210	0.12e − 210	0.15e − 213	0.60e − 201	0.95e − 184	0.594e − 28
$f_5, x_0 = 1.8$						
$\|x_n - \alpha\|$	0.56e − 272	0.40e − 261	0.74e − 259	0.98e − 261	0.23e − 229	0.90e − 41
$\|f(x_n)\|$	0.94e − 272	0.67e − 261	0.12e − 258	0.16e − 260	0.39e − 229	0.15e − 40
$f_6, x_0 = 1$						
$\|x_n - \alpha\|$	0.57e − 112	0.16e − 99	0.70e − 95	0.44e − 98	0.77e − 98	0.54e − 13
$\|f(x_n)\|$	0.17e − 111	0.49e − 99	0.21e − 94	0.132e − 97	0.23e − 97	0.16e − 12
$f_7, x_0 = -0.5$						
$\|x_n - \alpha\|$	0.24e − 230	0.12e − 224	0.150e − 223	0.81e − 224	0.17e − 193	0.88e − 36
$\|f(x_n)\|$	0.14e − 230	0.72e − 225	0.90e − 224	0.49e − 224	0.10e − 193	0.53e − 36
$f_8, x_0 = 3.2$						
$\|x_n - \alpha\|$	0.63e − 151	0.12e − 146	0.15e − 145	0.15e − 142	0.25e − 115	0.45e − 17
$\|f(x_n)\|$	0.12e − 149	0.21e − 145	0.28e − 144	0.27e − 141	0.46e − 114	0.82e − 16
$f_9, x_0 = 1.8$						
$\|x_n - \alpha\|$	0.21e − 149	0.68e − 156	0.41e − 169	0.15e − 160	0.82e − 138	0.25e − 26
$\|f(x_n)\|$	0.22e − 149	0.72e − 156	0.43e − 169	0.16e − 160	0.86e − 138	0.27e − 26
$f_{10}, x_0 = 1$						
$\|x_n - \alpha\|$	0.15e − 296	0.49e − 286	0.71e − 283	0.23e − 283	0.93e − 252	0.35e − 40
$\|f(x_n)\|$	0.25e − 296	0.81e − 286	0.12e − 282	0.37e − 283	0.15e − 251	0.57e − 40
$f_{11}, x_0 = -0.5$						
$\|x_n - \alpha\|$	0.20e − 31	0.14e − 30	0.76e − 30	0.22e − 29	0.21e − 12	0.60e − 4
$\|f(x_n)\|$	0.12e − 30	0.86e − 30	0.46e − 29	0.13e − 28	0.13e − 11	0.36e − 3
$f_{12}, x_0 = 3.2$						
$\|x_n - \alpha\|$	0.44e − 190	0.24e − 189	0.76e − 182	0.11e − 187	0.23e − 161	0.42e − 43
$\|f(x_n)\|$	0.35e − 190	0.19e − 189	0.61e − 182	0.84e − 188	0.19e − 161	0.34e − 43

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