Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 1, No. 1, 2013, pp. 55-70

Parameter determination in a parabolic inverse problem in general dimensions

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Abstract It is well known that the parabolic partial differential equations in two or more space dimensions with overspecified boundary data, feature in the mathematical modeling of many phenomena. In this article, an inverse problem of determining an unknown time-dependent source term of a parabolic equation in general dimensions is considered. Employing some transformations, we change the inverse problem to a Volterra integral equation of convolution-type. By using an explicit procedure based on Sinc function properties, the resulting integral equation is replaced by a system of linear algebraic equations. The convergence analysis is included, and it is shown that the error in the approximate solution is bounded in the infinity norm by the condition number and the norm of the inverse of the coefficient matrix multiplied by a factor that decays exponentially with the size of the system. Some numerical examples are given to demonstrate the computational efficiency of the method.

Keywords. Parabolic equation, Inverse problem, Sinc function.

2010 Mathematics Subject Classification. 35K10, 45D05, 35R30.

1. Introduction

Over the last few years, it has become increasingly apparent that many physical phenomena can be described in terms of parabolic partial differential equations with source control parameters. This type of equations arise, for example, in the study of heat conduction processes, thermoelasticity, chemical diffusion and control theory [2-17]. Growing attention is being paid to the development, analysis and implementation of accurate methods for the numerical solution of parabolic inverse problems, i.e. for the determination of some unknown function $p(t)$ in the parabolic partial differential equations.

Suppose that Ω is a bounded simply connected domain in $\mathbb{R}^m$ with boundary $\partial\Omega \in C^2$. We consider the initial boundary value problem for a parabolic equation in the cylinder $Q = \Omega \times (0, T)$, where $T > 0$, with lateral surface

$S = \partial\Omega \times (0, T)$. In this problem, not only the temperature but also the component of the source dependent of time is assumed unknown, while additional information , or overdetermination, is given. The main contribution of this article is to employ the Sinc-collocation method to approximate the solution of the following inverse problem, i.e. to find a pair of functions $\{u, p\}$ in the following parabolic equation

$$\frac{\partial u(\mathbf{x},t)}{\partial t} = \Delta u(\mathbf{x},t) + p(t)u(\mathbf{x},t) + q(\mathbf{x},t), \qquad (\mathbf{x},t) \in Q, \tag{1.1}$$

with the initial-boundary conditions

$$u(\mathbf{x},0) = f(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{1.2}$$

$$u(\mathbf{x},t) = 0, \qquad (\mathbf{x},t) \in S, \tag{1.3}$$

subject to the overspecification

$$u(\mathbf{x}^*,t) = E(t), \qquad (\mathbf{x}^*,t) \in Q, \tag{1.4}$$

or

$$\int_\Omega u(\mathbf{x},t)d^m\mathbf{x} = E(t), \qquad t \in (0,T), \tag{1.5}$$

where Δ is Laplace operator and f, q and E are given functions.

The inverse problem above and other similar problems have been studied by many authors [2-5], [21-23]. Under some suitable assumptions on the data, it was shown in [3, 18] that a global solution holds for the one-dimensional case. For problem (1.1)-(1.3) and (1.5), a classical global solution has been obtained in [22], where the potential theoretic representation of the solution was employed. In [4], the authors considered the inverse problem in more general form

$$\frac{\partial u(\mathbf{x},t)}{\partial t} = \Delta u(\mathbf{x},t) + p(t)u(\mathbf{x},t) + q(\mathbf{x},t,u,u_x,p(t)), \qquad (\mathbf{x},t) \in Q,$$
$$u(\mathbf{x},0) = f(\mathbf{x}), \qquad \mathbf{x} \in \Omega,$$
$$u(\mathbf{x},t) = g(x,t), \qquad (\mathbf{x},t) \in S,$$

subject to the overspecification

$$u(\mathbf{x}^*,t) = E(t), \qquad (\mathbf{x}^*,t) \in Q,$$

or

$$\int_{G(t)} \Phi(x,t)u(\mathbf{x},t)d^m\mathbf{x} = E(t), \qquad t \in (0,T),$$

where F, f, g, G and E are known functions. For each of the two problems stated above, they demonstrated the existence, uniqueness and continuous dependence upon the data under some reasonable assumptions on the data. Also, various numerical methods are developed to find the numerical affects of this problem in one-dimensional case [7-11, 25]. The articles [12-17] have investigated the two or three-dimensional versions.

Sinc methods have increasingly been recognized as powerful tools for attacking problems in applied mathematics [1, 19, 20, 24, 26, 27]. By comparison to the finite difference, finite element and boundary element methods, the Sinc collocation approach has been shown to be more suitable in handling singularities, boundary layers and semi-infinite domains. Furthermore, the residual error entailed in the Sinc collocation method is known to exhibit an exponential convergence rate.

The remainder of the present paper is divided into four sections. In section 2, we describe the collocation procedure by means of Sinc method for approximating convolution integrals. Section 3 contains the transformation of the inverse problem into a Volterra integral equation and the construction of the new Sinc-collocation method to replace the integral equation by an explicit system of linear algebraic equations. In section 4, convergence and error estimates are proved. Finally, some numerical results are presented in section 5.

2. Collocating convolutions

The Sinc function is defined on the whole real line by

$$sinc(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

The translated Sinc functions with evenly spaced nodes are given by

$$S(k,h)(x) = sinc(\frac{x-kh}{h}),$$

and are called the kth Sinc functions where k is an integer and h is a step size appropriately chosen depending on a given positive integer N. For purposes of Sinc approximation, consider first the case of a finite interval (a,b).Define ϕ by $w = \phi = \log[(z-a)/(b-z)]$; this function ϕ provides a conformal transformation of the "eye-shaped" region $\mathcal{D} = \{z \in \mathbb{C} : |arg[(z-a)/(b-z)]| < d\}$, onto the strip D_d defined by $\mathcal{D}_d = \{z \in \mathbb{C} : \ |Im(z)| < d\}$. The same function ϕ also provided a one-to-one transformation of (a,b) onto the real line $\mathbb{R}$. The Sinc points are defined for $h > 0$ and $k = 0, \pm1, \pm2, ...$, by $z_k = \phi^{-1}(kh) = (a + be^{kh})/(1 + e^{kh})$.

There are three important spaces of functions, $\mathcal{H}^1(\mathcal{D})$, $\mathcal{L}_\alpha(\mathcal{D})$ and $\mathcal{M}_\alpha(\mathcal{D})$ associated with Sinc approximation on the interval (a,b).
Let $\mathcal{H}^1(\mathcal{D})$ denote the family of all functions f that are analytic in $\mathcal{D}$, such

that

$$\int_{\partial\mathcal{D}} |f(z)||dz| < \infty.$$

Corresponding to number α, let $\mathcal{L}_\alpha(\mathcal{D})$ be the set of all analytic functions f, for which there exists a constant c_1, such that

$$|f(z)| \leq c_1 \frac{|\rho(z)|^\alpha}{(1+|\rho(z)|)^{2\alpha}}, \qquad z \in \mathcal{D}.$$

The family $\mathcal{M}_\alpha(\mathcal{D})$ consists of all functions f that are analytic in $\mathcal{D}$ and continuous in $\overline{\mathcal{D}}$ such that $g \in \mathcal{L}_\alpha(\mathcal{D})$, where

$$g(z) = f(z) - \frac{f(a) + \rho(z) f(b)}{1+\rho(z)}.$$

Now, we describe the Sinc-collocation procedure for approximating convolution μ of functions f and g, defined by the integral

$$\mu(x) = \int_a^x f(x-t)g(t)dt, \qquad x \in (a,b). \tag{2.1}$$

The method of the present section provides an explicit procedure for accurate approximation of μ when either of f or g has singularities at one of both endpoints of its interval of definition, or in the case that μ has singularities at one or both of the endpoints of (a,b) [27].

We assume that $g \in \mathcal{H}^1(\mathcal{D})$, and that f is analytic in a domain $\mathcal{D}_f$, with ϕ_f denoting a conformal mapping of $\mathcal{D}_f$ onto $\mathcal{D}_d$, and $\phi_f : (0,c) \to \mathbb{R}$, c being an arbitrary number on the interval $[2(b-a), \infty]$. Corresponding to a positive integer N we set $m = 2N+1$, and we determine h via the formula $h = (\frac{\pi d}{\alpha N})^{\frac{1}{2}}$. Let

$$\delta_{jk}^{(-1)} = \frac{1}{2} + \int_0^{j-k} \frac{\sin(\pi x)}{\pi x} dx,$$

then we define a matrix whose (j,k)th entry is given by $\delta_{jk}^{(-1)}$ as $I^{(-1)} = [\delta_{jk}^{(-1)}]$, and the square matrix A_m is obtained by

$$A_m = hI^{(-1)} diag[\frac{1}{\phi'(z_{-N})}, ..., \frac{1}{\phi'(z_N)}].$$

Throughout this paper, the Laplace transformation means the function F defined by

$$F(s) = \int_0^c f(t) e^{-\frac{t}{s}} dt,$$

where c defined as above, and we shall assume that the Laplace transformation exists for some $c \in [2(b-a), \infty]$, for all s on the right half of the complex plane i.e., $\Omega^+ = \{z \in \mathbb{C} : Re(z) > 0\}$.

Now, by above assumptions, we describe the approximation procedure for μ in (2.1). If the nonsingular matrix X_m and complex numbers s_j are determined such that

$$A_m = X_m diag[s_{-N}, ..., s_N]X_m^{-1},$$

then, square matrix $F(A_m)$ may be defined via the equation

$$F(A_m) = X_m diag[F(s_{-N}), ..., F(s_N)]X_m^{-1}.$$

Now, define column vectors G_m and P_m by

$$G_m = [g(z_{-N}), ..., g(z_N)]^T,$$

$$P_m = [\mu_{-N}, ..., \mu_N]^T = F(A_m)G_m,$$

then, the component μ_j of vector P_m approximates the value $\mu(x)$ at the Sinc point $x = z_j$. Thus, the approximation of μ on (a, b) takes the form

$$\mu(x) \approx \sum_{j=-N}^{N} \mu_j\omega_j(x) = \{F(A_m)G_m\}^T W(x), \qquad x \in (a, b), \tag{2.2}$$

where $W(x) = [\omega_{-N}(x), ..., \omega_N(x)]^T$, and $\{\omega_j\}$ is a Sinc basis as follows

$$\lambda_j(x) = S(j, h)o\phi(x), \qquad j = -N, ..., N,$$

$$\omega_j(x) = \lambda_j(x), \qquad j = -N + 1, ..., N - 1,$$

$$\omega_{-N}(x) = \{1 + e^{-Nh}\}\{\frac{1}{1 + \rho(x)} - \sum_{j=-N+1}^{N} \frac{\lambda_j(x)}{1 + e^{jh}}\},$$

$$\omega_N(x) = \{1 + e^{-Nh}\}\{\frac{\rho(x)}{1 + \rho(x)} - \sum_{j=-N}^{N-1} \frac{e^{jh}\lambda_j(x)}{1 + e^{jh}}\}.$$

Note that the functions ω_j defined above satisfy the relation $\omega_j(z_k) = \delta_{jk}$.

Theorem 2.1. [27] *Let μ be defined as (2.1) where the Laplace transformation of f with $c \geq 2(b - a)$ exists for all s in Ω^+, and let $F(s) = \mathcal{O}(s)$ as $s \to \infty$ in Ω^+. Let $g \in \mathcal{H}^1(\mathcal{D})$, and let α and α_f, be positive constants such that $0 < \alpha \leq 1$. Set*

$$P(r, \tau) = \int_a^\tau f(r + \tau - \eta)g(\eta)d\eta,$$

and assume that $P(r, .) \in \mathcal{M}_\alpha(\mathcal{D}')$, uniformly, for $r \in [0, b - a]$, and also that

$$|P_r(r, \tau)| \leq c_2 \frac{[\rho_f(r)]^{\alpha_f}\phi_f'(r)}{[1 + \rho_f(r)]^{2\alpha_f}},$$

for all $r \in [0, b-a]$, *and for all* $\tau \in \mathcal{D}$, *with* c_2 *a constant independent of* r *and* τ. *Let* $h = (\frac{\pi d}{\alpha N})^{\frac{1}{2}}$, *then there exists a constant* c_3 *which is independent of* N *such that*

$$\|\mu - \{F(A_m)G_m\}^T W\|_\infty \leq c_3 N^{\frac{1}{2}} e^{-(\pi d \alpha N)^{\frac{1}{2}}}.$$

3. Method discussion

In this section the Sinc-collocation method is implemented for solving the parabolic inverse problem. For this end, first, we change the inverse problem to a Volterra integral equation of convolution type, then by using the collocation procedure for approximating convolution integrals described in previous section, the numerical solution for resulting integral equation will be given.

Employing a pair of transformations

$$r(t) = \exp\{-\int_0^t p(s)ds\},$$
$$v(\mathbf{x}, t) = r(t)u(\mathbf{x}, t),$$

the problem (1.1)-(1.5) will become

$$\frac{\partial v(\mathbf{x}, t)}{\partial t} = \Delta v(\mathbf{x}, t) + r(t)q(\mathbf{x}, t), \qquad (\mathbf{x}, t) \in Q, \tag{3.1}$$

$$v(\mathbf{x}, 0) = f(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{3.2}$$

$$v(\mathbf{x}, t) = 0, \qquad (\mathbf{x}, t) \in S, \tag{3.3}$$

$$v(\mathbf{x}^*, t) = r(t)E(t), \qquad (\mathbf{x}^*, t) \in Q, \tag{3.4}$$

or

$$\int_\Omega v(\mathbf{x}, t)d^m\mathbf{x} = r(t)E(t), \qquad t \in (0, T). \tag{3.5}$$

Obviously, if we obtain $\{v, r\}$ from (3.1)-(3.3) and (3.4) or (3.5) then $\{u, p\}$ can be found as

$$u(\mathbf{x}, t) = \frac{v(\mathbf{x}, t)}{r(t)},$$
$$p(t) = -\frac{r'(t)}{r(t)}.$$

If we assume that the function $r(t)$ is known, then, the direct problem (3.1)-(3.3) has the following solution [6]

$$v(\mathbf{x}, t) = v_0(\mathbf{x}, t) + \int_0^t \int_\Omega G(\mathbf{x}, \mathbf{y}, t-\tau)q(\mathbf{y}, \tau)r(\tau)d^m\mathbf{y}d\tau, \tag{3.6}$$

where

$$v_0(\mathbf{x},t) = \int_\Omega G(\mathbf{x},\mathbf{y},t) f(\mathbf{y}) d^m\mathbf{y},$$

$$G(\mathbf{x},\mathbf{y},t) = \theta(\mathbf{x}-\mathbf{y},t) - \theta(\mathbf{x}+\mathbf{y},t),$$

and

$$\theta(\mathbf{x},t) = \sum_{\mathbf{n}\in\mathbb{Z}^m} K(\mathbf{x}+2\mathbf{n},t),$$

with

$$K(\mathbf{x},t) = \frac{1}{(4\pi t)^{m/2}} \exp\{-\frac{\|\mathbf{x}\|^2}{4t}\}.$$

Replacing $\mathbf{x} = \mathbf{x}^*$ in (3.6) and using additional condition (3.4), or integrating (3.6) on Ω and using additional condition (3.5) yields the following integral equation

$$r(t)E(t) = g(t) + \int_0^t \Psi(t-\tau,\tau) r(\tau) d\tau, \tag{3.7}$$

where for problem (3.1)-(3.4), $g(t) = v_0(\mathbf{x}^*,t)$, and the function Ψ can be found by

$$\int_\Omega G(\mathbf{x}^*,\mathbf{y},t-\tau) q(\mathbf{y},\tau) d^m\mathbf{y} = \Psi(t-\tau,\tau),$$

or for problem (3.1)-(3.3) and (3.5), $g(t) = \int_\Omega v_0(\mathbf{x},t) d^m\mathbf{x}$, and the function Ψ can be found by

$$\int_\Omega \int_\Omega G(\mathbf{x},\mathbf{y},t-\tau) q(\mathbf{y},\tau) d^m\mathbf{x} d^m\mathbf{y} = \Psi(t-\tau,\tau).$$

Taking them into account, we can obtain

$$\Psi(t-\tau,\tau) = \sum_{n=1}^{\infty} h_n(t-\tau)\varphi_n(\tau),$$

where $h_n(t) = \exp\{-n^2\pi^2 t\}$, and there exists a constant K such that

$$|\varphi_n(\tau)| \le K \sup_{(y,\tau)\in Q} |q(y,\tau)|, \qquad \tau \in (0,T), \quad n \in \mathbb{N}.$$

For a positive integer M denote

$$\Psi_M(t-\tau,\tau) = \sum_{n=1}^{M} h_n(t-\tau)\varphi_n(\tau),$$

we may write

$$\int_0^t |\Psi(t-\tau,\tau) - \Psi_M(t-\tau,\tau)||r(\tau)| d\tau \le \frac{K}{\pi^2}\psi'(M+1) \sup_{\tau\in(0,T)} |r(\tau)| \sup_{(y,\tau)\in Q} |q(y,\tau)|,$$

where ψ is the digamma function. Thus, by solving the following approximating integral equation

$$r(t)E(t) = g(t) + \int_0^t \Psi_M(t-\tau,\tau)r(\tau)d\tau, \tag{3.8}$$

we obtain an approximation for $r(t)$, then, if we replace it in (3.6), we will have the approximate solution for $v(\mathbf{x},t)$. Therefore, in the remaining part of this paper we try to solve the integral equation (3.8), by using the Sinc-collocation method.

Let $\Gamma = (0,T)$, By exploiting the above definitions, we take

$$\mathcal{D} = \{z \in \mathbb{C} : |arg(\frac{z}{T-z})| < d \leq \frac{\pi}{2}\},$$
$$\phi(x) = \ln(\frac{x}{T-x}),$$

and the Sinc grid points are

$$x_k = \frac{Te^{kh}}{1+e^{kh}}, \quad k = -N, ..., N.$$

Consider the integral equation (3.8), the "Laplace transformation" for h_n, with $c = \infty$, can be determined as

$$H_n(s) = \int_0^\infty h_n(t)e^{-\frac{t}{s}}dt = \frac{s}{1+n^2\pi^2 s}, \qquad s \in \Omega^+, \ \ n \in \mathbb{N}.$$

By using the numerical procedure in the previous section for convolution integrals we may write

$$\int_0^t h_n(t-\tau)\varphi_n(\tau)r(\tau)d\tau \approx \{H_n(A_m)\bar{\varphi_n}R\}^T W(t), \tag{3.9}$$

where $\bar{\varphi}$ is the diagonal matrix defined by $\bar{\varphi_n} = diag[\varphi_n(x_{-N}), ..., \varphi_n(x_N)]$, and

$$R = [r(x_{-N}), ..., r(x_N)]^T. \tag{3.10}$$

Substituting (3.9) in equation (3.8), we obtain

$$E(t)r(t) - \sum_{n=1}^{M} \{H_n(A_m)\bar{\varphi_n}R\}^T W(t) = g(t). \tag{3.11}$$

Equation (3.11) is collocated at $2N+1$ points. For suitable collocation points, we use the Sinc grid points x_k, $k = -N, ..., N$. Thus we have $2N+1$ linear algebraic equations which can be solved for the unknown coefficients $r(x_j)$, $j = -N, ..., N$. This system in the matrix form is given by

$$BR = Y, \tag{3.12}$$

where

$$B = \bar{E} - \sum_{n=1}^{M} H_n(A_m)\bar{\varphi_n},$$

and

$$Y = [g(x_{-N}), ..., g(x_N)]^T.$$

By solving the linear system (3.12), we obtain approximate solutions r_j, $j = -N, ..., N$. Then we employ a method similar to Nyström' s idea and write

$$\widetilde{r}_m(t) = \frac{1}{E(t)}\{g(t) + \sum_{n=1}^{M}\{H_n(A_m)\bar{\varphi_n}\widetilde{R}\}^T W(t)\}, \tag{3.13}$$

where $m = 2N + 1$ and

$$\widetilde{R} = [r_{-N}, ..., r_N]^T. \tag{3.14}$$

4. Convergence analysis

In this section, we discuss the convergence of the Sinc-collocation method described in the previous section. We start with the following definition.

Definition 4.1. Let $\mathcal{G}_\alpha(\mathcal{D})$, with $0 < \alpha \leq 1$, denote the family of all functions $g \in \mathcal{H}^1(\mathcal{D})$, such that

$$|\Theta_r(r, \tau, n)| \leq C_1^{(n)} \frac{r^{\xi_n - 1}}{(1+r)^{2\xi_n}}, \qquad r \in (0, T), \quad \tau \in \mathcal{D}, \quad n \in \mathbb{N},$$

with $C_1^{(n)}$ and ξ_n constants independent of r and τ, where

$$\Theta(r, \tau, n) = \int_0^\tau h_n(r + \tau - \eta) g(\eta) d\eta,$$

and $\Theta(r, ., n) \in \mathcal{M}_\alpha(\mathcal{D}')$, uniformly, for $r \in (0, T)$.

The following auxiliary lemma will be needed in the subsequent analysis.

Lemma 4.2. *Suppose that R and $\widetilde{R}$ are defined by (3.10) and (3.14), respectively. Then for $r\varphi_n \in \mathcal{G}_\alpha(\mathcal{D})$,*

$$\|R - \widetilde{R}\| \leq C_2 \|B^{-1}\| N^{\frac{1}{2}} e^{-(\pi d \alpha N)^{\frac{1}{2}}},$$

where C_2 is a positive constant which is independent of N.

Proof. First, we have

$$\|R - \widetilde{R}\| = \|R - B^{-1}Y\| \leq \|B^{-1}\| \|BR - Y\|.$$

Let s_k be the kth component of the vector $S = BR - Y$, that is,

$$s_k = \{BR\}_k - g(x_k),$$

where $\{BR\}_k$ is the kth component of the vector BR. From (3.8), we have

$$g(x_k) = r(x_k)E(x_k) - \sum_{n=1}^{M} \int_0^{x_k} h_n(t-\tau)\varphi_n(\tau)r(\tau)d\tau,$$

and from (3.12)

$$\{BR\}_k = E(x_k)r(x_k) - \sum_{n=1}^{M} \{H_n(A_m)\bar{\varphi_n}R\}^T W(x_k).$$

Since

$$\begin{aligned}\|BR - Y\| &= \max_{-N\leq k\leq N} |s_k| \\ &\leq \sum_{n=1}^{M} \max_{-N\leq k\leq N} |\int_0^{x_k} h_n(t-\tau)\varphi_n(\tau)r(\tau)d\tau - \{H_n(A_m)\bar{\varphi_n}R\}^T W(x_k)|,\end{aligned}$$

using theorem 1, there exists a constant C_2 independent of N such that

$$\|BR - Y\| \leq C_2 N^{\frac{1}{2}} e^{-(\pi d\alpha N)^{\frac{1}{2}}}.$$

Therefore we have

$$\|R - \widetilde{R}\| \leq C_2 \|B^{-1}\| N^{\frac{1}{2}} e^{-(\pi d\alpha N)^{\frac{1}{2}}}.$$

Theorem 4.3. *Suppose that $r(t)$ is the exact solution of integral equation (3.8), and let $\widetilde{r}_m(t)$ be the approximate solution of equation (3.8) given by (3.13). Then for $r\varphi_n \in \mathcal{G}_\alpha(\mathcal{D})$, there exists a positive constant C independent of N such that*

$$\sup_{t\in(0,T)} |r(t) - \widetilde{r}_m(t)| \leq C\lambda_1\{Cond(B) + \lambda_2\|B^{-1}\| + 1\} N^{\frac{1}{2}} e^{-(\pi d\alpha N)^{\frac{1}{2}}},$$

where

$$\lambda_1 = \sup_{t\in(0,T)} |\frac{1}{E(t)}|, \qquad \lambda_2 = \sup_{t\in(0,T)} |E(t)|.$$

Proof. Define

$$r_m(t) = \frac{1}{E(t)}\{g(t) + \sum_{n=1}^{M}\{H_n(A_m)\bar{\varphi_n}R\}^T W(t)\}, \tag{4.1}$$

then

$$|r(t) - \widetilde{r}_m(t)| \leq |r(t) - r_m(t)| + |r_m(t) - \widetilde{r}_m(t)|. \tag{4.2}$$

By theorem 1, there exists a constant C_3 such that

$$\sup_{t\in(0,T)} |r(t)-r_m(t)| \le \lambda_1 \sum_{n=1}^{M} \sup_{t\in(0,T)} |\int_0^t h_n(t-\tau)\varphi_n(\tau)r(\tau)d\tau - \{H_n(A_m)\bar{\varphi_n}R\}^T W(t)|$$

$$\le \lambda_1 C_3 N^{\frac{1}{2}} e^{-(\pi d\alpha N)^{\frac{1}{2}}}, \qquad (4.3)$$

where

$$\lambda_1 = \sup_{t\in(0,T)} |\frac{1}{E(t)}|.$$

Also we have

$$\sup_{t\in(0,T)} |r_m(t)-\widetilde{r}_m(t)| \le \lambda_1 \sum_{n=1}^{M} \sup_{t\in(0,T)} |\{H_n(A_m)\bar{\varphi_n}R\}^T W(t) - \{H_n(A_m)\bar{\varphi_n}\widetilde{R}\}^T W(t)|$$

$$= \lambda_1 \sup_{t\in(0,T)} |\{R-\widetilde{R}\}^T \sum_{n=1}^{M} \{H_n(A_m)\bar{\varphi_n}\}^T W(t)|$$

$$\le \lambda_1 \|R-\widetilde{R}\| \| \sum_{n=1}^{M} H_n(A_m)\bar{\varphi_n}\|$$

$$= \lambda_1 \|R-\widetilde{R}\| \|B-\bar{E}\| \le \lambda_1 \|R-\widetilde{R}\| \{\|B\| + \lambda_2\},$$

where

$$\lambda_2 = \sup_{t\in(0,T)} |E(t)|.$$

Using Lemma 1, we have

$$\sup_{t\in(0,T)} |r_m(t)-\widetilde{r}_m(t)| \le C_4 \lambda_1 \{\|B\| + \lambda_2\} \|B^{-1}\| N^{\frac{1}{2}} e^{-(\pi d\alpha N)^{\frac{1}{2}}}. \qquad (4.4)$$

Finally, substituting (4.3) and (4.4) in (4.4), the proof is completed.

5. Numerical results

In this section, we illustrate the use of our algorithm by displaying the results obtained from its application to some test problems. In these examples we take $\alpha = 1$ and $d = \frac{\pi}{2}$, and therefore $h = \frac{\pi}{\sqrt{2N}}$. In practical application, data contain random noise. We will illustrate the effect of the solution in virtue of the noisy data

$$E_\delta(t) = E(t)(1 + \delta \sin 50t),$$

$$f_\delta(x) = f(x)(1 + \delta \sin 50x),$$

where δ is the noise parameter.

Example 1. Consider the following inverse problem

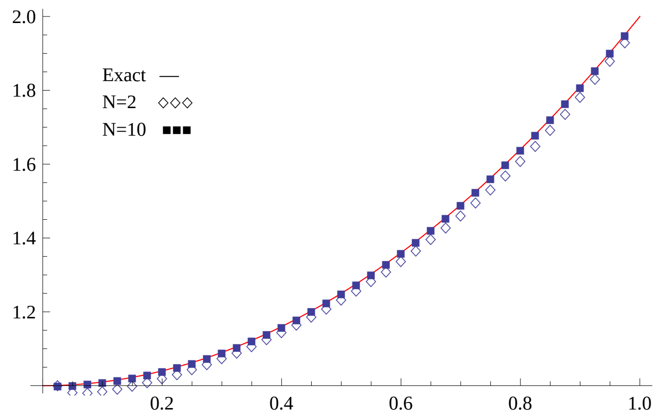

FIGURE 1. Plot of exact and approximate solutions for p(t) in Example 1.

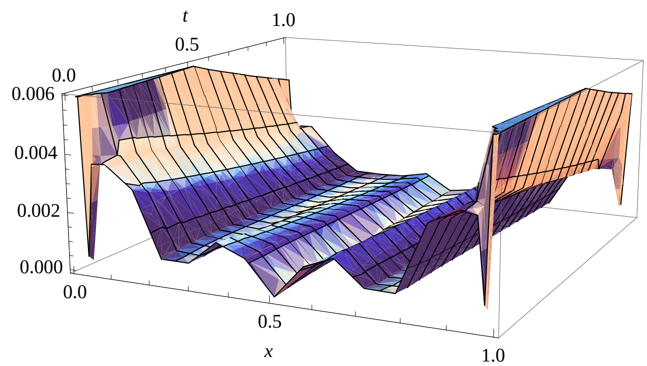

FIGURE 2. Error of approximation of u(x,t) with $N = 15$ in Example 1.

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + p(t)u(x,t) + e^{-t}(-2\cos x - (1+t^2)(x-1)\sin x), && 0 < x, t < 1,\\
u(x,0) &= (x-1)\sin x, && 0 < x < 1,\\
u(0,t) &= u(1,t) = 0, && 0 < t \leq 1,\\
\int_0^1 u(x,t)dx &= e^{-t}(\sin 1 - 1), && 0 \leq t \leq 1,
\end{aligned}$$

for which the exact solution is $u(x,t) = e^{-t}(x-1)\sin x$, and $p(t) = 1+t^2$. The example has been solved by taking different values of N. The approximation of $p(t)$ for $N = 2$ and $N = 10$, is shown in Figure 1. In Figure 2, the error of approximation of $u(x,t)$ for $N = 15$ is plotted. Also, the exact solution $p(t)$ together with the numerical one for various values of the noise parameter $\delta = 1\%, 2\%$ and 5% are shown in Figure 3.

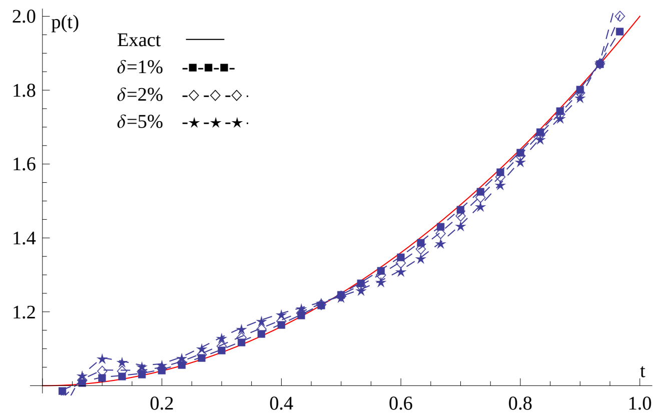

FIGURE 3. Plot of exact and approximate solution for p(t) with noisy data in Example 1 when N=10.

TABLE 1. Results for $u(x,t)$ when t=0.5 in Example 2.

(x,y)	$u(x,y,0.5)$ exact	$N=5$ error	$N=10$ error	$N=15$ error	$N=20$ error	$N=25$ error
(0.1,0.1)	-0.124643	7.1×10^{-3}	4.2×10^{-3}	4.8×10^{-5}	7.6×10^{-6}	2.0×10^{-6}
(0.2,0.2)	-0.421483	6.2×10^{-4}	2.3×10^{-4}	3.8×10^{-6}	7.0×10^{-6}	1.4×10^{-6}
(0.3,0.3)	-0.76141	5.4×10^{-4}	1.3×10^{-4}	8.2×10^{-6}	3.5×10^{-6}	7.2×10^{-7}
(0.4,0.4)	-1.02296	3.7×10^{-4}	1.6×10^{-4}	1.2×10^{-5}	5.9×10^{-6}	7.9×10^{-7}
(0.5,0.5)	-1.12042	5.5×10^{-4}	3.3×10^{-4}	4.8×10^{-5}	8.3×10^{-6}	2.2×10^{-6}
(0.6,0.6)	-1.02296	2.1×10^{-3}	7.9×10^{-4}	5.6×10^{-5}	4.5×10^{-6}	1.4×10^{-6}
(0.7,07)	-0.76141	8.0×10^{-3}	4.0×10^{-4}	4.2×10^{-5}	7.1×10^{-6}	2.5×10^{-6}
(0.8,0.8)	-0.421483	3.6×10^{-3}	1.9×10^{-3}	1.7×10^{-4}	5.7×10^{-5}	7.4×10^{-6}
(0.9,0.9)	-0.124643	8.7×10^{-3}	5.2×10^{-3}	7.1×10^{-4}	8.2×10^{-5}	9.6×10^{-6}

Example 2. Consider the following inverse problem

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+p(t)u(x,y,t)+2e^{t+1}\sin\pi y(-1+t(x-1)x), \qquad 0<x,y,t<1,$$

$$u(x,y,0)=e(x-1)x\sin\pi y, \qquad 0<x,y<1,$$

$$u(0,y,t)=u(1,y,t)=u(x,0,t)=u(x,1,t)=0 \qquad 0<t\le 1,$$

$$u(0.2,0.5,t)=-\frac{4}{25}e^{t+1}, \qquad 0\le t\le 1,$$

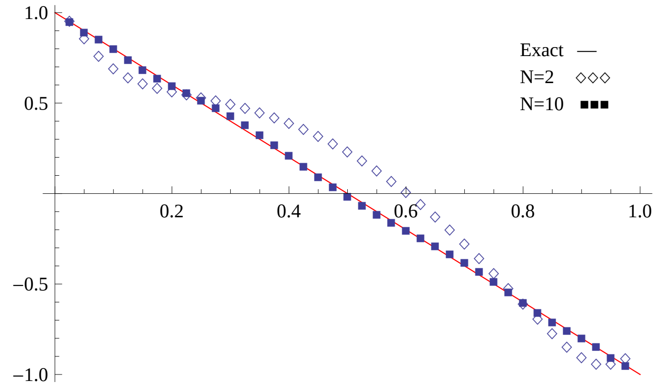

FIGURE 4. Plot of exact and approximate solutions for p(t) in Example 2.

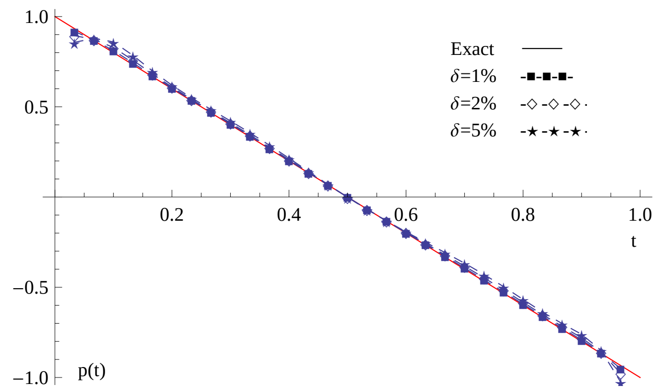

FIGURE 5. Plot of exact and approximate solution for p(t) with noisy data in Example 2 when N=10.

for which the exact solution is $u(x,t) = e^{t+1}x(x-1)\sin \pi y$, and $p(t) = 1 - 2t$. The example has been solved by taking different values of N. We report the absolute value of the errors of $u(x,y,t)$ for $N = 5, 10, 15, 20$ and 25 in Table 1. The approximation of $p(t)$ for $N = 2$ and $N = 10$, is shown in Figure 4. Also, the exact solution $p(t)$ together with the numerical one for various values of the noise parameter $\delta = 1\%, 2\%$ and 5% are shown in Figure 5.

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