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Existence and multiplicity of positive solutions for a coupled system of perturbed nonlinear fractional differential equations

Kazem Ghanbari*
Department of Mathematics, Sahand University of Technology, Tabriz, Iran.
E-mail: kghanbari@sut.ac.ir

Yousef Gholami
Department of Mathematics, Sahand University of Technology, Tabriz, Iran.
E-mail: y_gholami@sut.ac.ir

Abstract In this paper, we consider a coupled system of nonlinear fractional differential equations (FDEs), such that both equations have a particular perturbed terms. Using *Leray-Schauder* fixed point theorem, we investigate the existence and multiplicity of positive solutions for this system.

Keywords. fractional derivative, fixed point theorem, positive solution.

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1. Introduction

During 1695 to 20th century, three hundred years after foundation of fractional calculus nobody expected that this wonderful branch of calculus introduces applicable theory in almost all fields such as basic sciences, engineering, social sciences, medicine, economics, dynamical processes and so on (see more details in monographes [6, 9, 10]). Every interested researcher can find a large number of attractive subjects in various fields of fractional calculus and related applications such as solvability, existence and multiplicity of positive solutions for given boundary value problems of fractional differential equations. For more details see [2, 3, 4, 5, 8, 12, 13, 14, 15, 16] and references therein.

Coupled systems of differential equations arise naturally in extensive volume of scientific problems such as dynamical systems, social researches and biological relation between different kinds of animals such as prey and predator problem (see [1, 11, 18] and references cited therein).

To the best of our knowledge, there are a few papers concerning existence and multiplicity of positive solutions for coupled systems of nonlinear fractional differential equations with negative perturbed term. X. Zhang et. al.

*Corresponding Author.

[17] considered the existence and multiplicity of positive solutions for perturbed boundary value problem of the following form

$$\begin{cases} -D_{0+}^{\alpha}u(t) = p(t)f(t,u(t)) - q(t), & 0<t<1,\ \alpha\in(2,3) \\ u(0)=u^{'}(0)=u(1)=0, \end{cases}$$

where D_{0+}^{α} is $Riemann-Liouville$ fractional derivative of order α.

In this paper we consider a coupled system of perturbed fractional differential equation of the form

$$\begin{cases} {}^{c}D_{0+}^{\alpha}u(t) = a(t)f(v(t),u(t)) - p_1(t), & t\in(0,1) \\ {}^{c}D_{0+}^{\beta}v(t) = b(t)g(u(t),v(t)) - p_2(t), & \alpha,\beta\in(2,3) \end{cases} \tag{1.1}$$

with boundary conditions

$$\begin{cases} u(0)=0=v(0) \\ u^{'}(0)+u^{''}(0)=0=v^{'}(0)+v^{''}(0) \\ u^{'}(1)+u^{''}(1)=0=v^{'}(1)+v^{''}(1) \end{cases} \tag{1.2}$$

where, ${}^{c}D_{0+}^{\alpha}$ is $Caputo$ fractional derivative of order α and p_1,p_2 are perturbed terms.

We assume the following conditions hold throughout this paper:

(H_1) $a,b:[0,1]\to(0,\infty)$ and $\int_0^1 a(s)ds$, $\int_0^1 b(s)ds<\infty$.
(H_2) $f,g\in C\left([0,+\infty)\times[0,+\infty),(0,+\infty)\right)$. Particularly $f(0,0)$ and $g(0,0)$ do not vanish identically for $t\in(0,1)$.
(H_3) $p_i:[0,1]\to[0,+\infty)$ and $p_i\in L^1(0,1)$.

2. Preliminary materials

In this section, we first state some fundamental definitions and lemmas from fractional calculus and then, considering fixed point theory, we represent the $Leray-Schauder$ fixed point index theorem and some related lemmas.

Definition 2.1. [6] The $Riemann-Liouville$ fractional integral of order $\alpha>0$ for given integrable function $u:(0,1)\to\mathbb{R}^+$ is defined by

$$I_{0+}^{\alpha}u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}u(s)ds. \tag{2.1}$$

Definition 2.2. [10] The $Caputo$ fractional derivative of order $\alpha>0$ for given function $u\in C^n((0,1),\mathbb{R}^+)(n\in\mathbb{Z}^+\cup 0)$ is defined as follow

$$\begin{cases} {}^{c}D_{0+}^{\alpha}u(t) = \int_0^t \frac{(t-s)^{n-\alpha-1}}{\Gamma(n-\alpha)}u^{(n)}(s)ds, & n=[\alpha]+1(n\notin\mathbb{Z}^+\cup 0) \\ u^{(n)}(t), & n=\alpha(n\in\mathbb{Z}^+\cup 0). \end{cases} \tag{2.2}$$

Lemma 2.3. [6] *Assume* $u \in C^n[0,1]$, $D^{\alpha}_{0+} \in C(0,1) \cap L(0.1)$ *for* $\alpha > 0$. *Then*

$$I^{\alpha}_{0+}\,{}^{c}D^{\alpha}_{0+}u(t) = u(t) + c_1 + c_2 t + .. + c_n t^{n-1}.$$

Moreover, fractional differential equation

$${}^{c}D^{\alpha}_{0+}u(t) = 0$$

has the unique solution

$$u(t) = c_1 + c_2 t + .. + c_n t^{n-1}$$

, *where* $n = [\alpha] + 1$ *and for every* $i = 1, 2, .., n;\ c_i \in \mathbb{R}$.

Lemma 2.4. *Let* $h \in C(0,1)$ *such that* $0 < \int_0^1 h(s)ds < +\infty$. *Then the boundary value problem*

$$\begin{cases} {}^{c}D^{\alpha}_{0+}u(t) = h(t),\ \ t \in (0,1), \alpha \in (2,3) \\ \\ u(0) = 0, \\ u^{'}(0) + u^{''}(0) = 0, \\ u^{'}(1) + u^{''}(1) = 0, \end{cases} \tag{2.3}$$

has the unique solution

$$u(t) = \int_0^1 G(t,s)h(s)ds, \tag{2.4}$$

where

$$G(t,s) = \frac{1}{2\Gamma(\alpha)} \begin{cases} (2t - t^2)\left[(\alpha-1)(1-s)^{\alpha-2} + (\alpha-1)(\alpha-2)(1-s)^{\alpha-3}\right] \\ +(t-s)^{\alpha-1},\ \ 0 < s \le t < 1 \\ \\ (2t - t^2)\left[(\alpha-1)(1-s)^{\alpha-2} + (\alpha-1)(\alpha-2)(1-s)^{\alpha-3}\right], \\ 0 < t \le s < 1. \end{cases} \tag{2.5}$$

The function $G(t,s)$ *is called the* $Green's\ function$ *of boundary value problem* (2.3).

Proof. Using Lemma 2.3, we can reduce the equation

$${}^{c}D^{\alpha}_{0+}u(t) = h(t),\ \ t \in (0,1), \alpha \in (2,3)$$

to the following integral equation

$$u(t) = -c_0 - c_1 t - c_2 t^2 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds. \tag{2.6}$$

Now considering the first boundary condition $u(0) = 0$, we find that $c_0 = 0$. Using second boundary condition $u^{'}(0)+u^{''}(0) = 0$, we deduce that $c_2 = -\frac{1}{2}c_1$. Finally imposing the last boundary condition $u^{'}(1) + u^{''}(1) = 0$, we find

$$c_0 = 0, \quad c_2 = -\frac{1}{2}c_1 = \int_0^1 \frac{(1-s)^{\alpha-2} + (\alpha-2)(1-s)^{\alpha-3}}{2\Gamma(\alpha)} h(s)ds.$$

Substituting the constant coefficients $c_0,\ c_1,\ c_2$ in (2.6), we find

$$\begin{aligned}
u(t) &= \int_0^1 \frac{(2t-t^2)\left[(\alpha-1)(1-s)^{\alpha-2} + (\alpha-1)(\alpha-2)(1-s)^{\alpha-3}\right]}{2\Gamma(\alpha)} h(s)ds \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{2\Gamma(\alpha)} h(s)ds \\
&= \int_0^t \frac{(2t-t^2)\left[(\alpha-1)(1-s)^{\alpha-2} + (\alpha-1)(\alpha-2)(1-s)^{\alpha-3}\right]}{2\Gamma(\alpha)} h(s)ds \\
&\quad + \int_0^t \frac{(t-s)^{\alpha-1}}{2\Gamma(\alpha)} h(s)ds \\
&\quad + \int_t^1 \frac{(2t-t^2)\left[(\alpha-1)(1-s)^{\alpha-2} + (\alpha-1)(\alpha-2)(1-s)^{\alpha-3}\right]}{2\Gamma(\alpha)} h(s)ds \\
&= \int_0^1 G(t,s)h(s)ds.
\end{aligned}$$

Construction procedure of coefficients $c_0,\ c_1,\ c_2$, shows that the boundary value problem (2.3) has the unique solution (2.4). This completes the proof. □

Lemma 2.5. *The $Green's\ function$ (2.5) has the following properties:*

- (P_1) *for every* $t, s \in (0,1)$*, we have* $G(t,s) > 0$*,* $G(t,s) \in C\left((0,1)\times(0,1)\right)$*.*
- (P_2) *for all* $t, s \in (0,1)$*,* $\frac{\partial G(t,s)}{\partial t} \geq 0$*.*

Proof. the proof is immediate by the construction of $Green's\ function$ in (2.5). □

Lemma 2.6. *There exist a positive function* $\gamma(s) \in C(0,1)$*, such that:*

$$\min_{t\in[p,q]} G(t,s) \geq \gamma(s) \max_{t\in[0,1]} G(t,s),\ \ p, q \in [\frac{1}{4}, \frac{3}{4}], p < q.$$

Proof. By definition of the $Green's\ function$ $G(t,s)$ in (2.5), we assume:

$$\begin{aligned}
g_1(t,s) &= \frac{(2t-t^2)\left[(\alpha-1)(1-s)^{\alpha-2} + (\alpha-1)(\alpha-2)(1-s)^{\alpha-3}\right]}{2\Gamma(\alpha)} \\
&\quad + \frac{(t-s)^{\alpha-1}}{2\Gamma(\alpha)},\ \ s \leq t
\end{aligned}$$

and

$$g_2(t,s) = \frac{(2t-t^2)\left[(\alpha-1)(1-s)^{\alpha-2} + (\alpha-1)(\alpha-2)(1-s)^{\alpha-3}\right]}{2\Gamma(\alpha)},\ t \le s.$$

Hence monotonic property of $G(t,s)$ in second part of Lemma 2.5 insures that, both of functions $g_1(t,s)$ and $g_2(t,s)$ are increasing with respect to first variable t. So we have:

$$\begin{aligned}\min_{t\in[p,q]} g_1(t,s) &\ge \frac{(2p-p^2)\left[(\alpha-1)(1-s)^{\alpha-2} + (\alpha-1)(\alpha-2)(1-s)^{\alpha-3}\right]}{2\gamma(\alpha)} \\ &\quad + \frac{(p-s)^{\alpha-1}}{2\Gamma(\alpha)}, \\ \max_{t\in[0,1]} g_1(t,s) &\le \frac{(\alpha-1)(1-s)^{\alpha-2} + (\alpha-1)(\alpha-2)(1-s)^{\alpha-3} + (1-s)^{\alpha-1}}{2\Gamma(\alpha)}.\end{aligned}$$

Also

$$\begin{aligned}\min_{t\in[p,q]} g_2(t,s) &\ge \frac{(2p-p^2)\left[(\alpha-1)(1-s)^{\alpha-2} + (\alpha-1)(\alpha-2)(1-s)^{\alpha-3}\right]}{2\Gamma(\alpha)}, \\ \max_{t\in[0,1]} g_2(t,s) &\le \frac{(\alpha-1)(1-s)^{\alpha-2} + (\alpha-1)(\alpha-2)(1-s)^{\alpha-3}}{2\Gamma(\alpha)}.\end{aligned}$$

Setting

$$\gamma(s) = \frac{m(s)}{M(s)},$$

where

$$\begin{aligned} m(s) &= \min_{t\in[p,q]} \{g_1(t,s), g_2(t,s)\} = g_2(p,s), \\ M(s) &= \max_{t\in[0,1]} \{g_1(t,s), g_2(t,s)\} = g_1(1,s),\end{aligned}$$

we conclude that for $s \in (0,1)$

$$\gamma(s) = \frac{(2p-p^2)[(\alpha-1)(1-s)^{\alpha-2} + (\alpha-1)(\alpha-2)(1-s)^{\alpha-3}]}{(\alpha-1)(1-s)^{\alpha-2} + (\alpha-1)(\alpha-2)(1-s)^{\alpha-3} + (1-s)^{\alpha-1}}. \tag{2.7}$$

Clearly we can observe that $\gamma(s)$ is a positive and continuous function in $(0,1)$, which completes the proof. □

Remark 2.7. From definition of $\gamma(s)$ in (2.7), simple calculation shows that $\gamma(s) > 7/32$.

Remark 2.8. Consider the following boundary value problem:

$$\begin{cases} {}^{c}D_{0^+}^{\alpha} u(t) = 2p(t) \\ \\ u(0) = 0, \\ u^{'}(0) + u^{''}(0) = 0, \\ u^{'}(1) + u^{''}(1) = 0. \end{cases} \tag{2.8}$$

From Lemma 2.4, we know that boundary value problem (2.8) has the unique solution

$$w(t) = 2\int_0^1 G(t,s)p(s)ds.$$

Now considering the boundary value problem

$$\begin{cases} {}^{c}D_{0^+}^{\alpha} u(t) = a(t)f(v(t),(u-w)(t)) + p(t) \\ \\ u(0) = 0, \\ u^{'}(0) + u^{''}(0) = 0, \\ u^{'}(1) + u^{''}(1) = 0, \end{cases} \tag{2.9}$$

by (H_2), it is clear that for every $t \in (0,1)$, $u(t) \geq w(t)$. So if $z(t)$ is a positive solution of BVP (2.9), then by linear property of fractional differential operators, $(z-w)(t)$ is the positive solution of the following BVP

$$\begin{cases} {}^{c}D_{0^+}^{\alpha} u(t) = a(t)f(v(t),u(t)) - p(t) \\ \\ u(0) = 0, \\ u^{'}(0) + u^{''}(0) = 0, \\ u^{'}(1) + u^{''}(1) = 0. \end{cases} \tag{2.10}$$

Thus, in order to investigate positive solutions of the BVP (2.10), it is enough to find the positive solutions of the BVP (2.9).

In this part, first we introduce the following *Banach* space. Let $E = C[0,1]$ with the max-norm

$$||u|| = \max_{t\in[0,1]} |u(t)|.$$

Now assume that special *Banach* space of this investigation and its norm, defined as follow:

$$B = E \times E, \qquad ||(u,v)|| = ||u|| + ||v||, \tag{2.11}$$

also the partial order of *Banach* space B, is given by

$$(u_1,u_2) \leq (v_1,v_2)$$

when

$$u_1(t) \leq v_1(t)\ ,\ u_2(t) \leq v_2(t)$$

for $(u_1, u_2), (v_1, v_2) \in B,\ t \in (0, 1)$.
Let

$$T_1 : B \to E, \quad T_1(u, v) = \int_0^1 G(t, s)[a(s)f(v(s), (u - w)(s)) + p_1(s)]ds,$$
$$T_2 : B \to E, \quad T_2(u, v) = \int_0^1 G(t, s)[b(s)g(u(s), (v - w)(s)) + p_2(s)]ds,$$

be two Hammerstein integral operators and define

$$T : B \to B, \qquad T(u, v) = (T_1(u, v), T_2(u, v)). \tag{2.12}$$

Finally we define cone $K \subset B$ as follows

$$K = \left\{ (u, v) \in B \;\middle|\; (u(t), v(t)) \geq 0, t \in (0, 1), \begin{matrix} \min\limits_{t\in[p,q]} u(t) \geq \frac{7}{32}||u|| \\ \min\limits_{t\in[p,q]} v(t) \geq \frac{7}{32}||v|| \end{matrix} \right\}. \tag{2.13}$$

Lemma 2.9. *Assume that the conditions $(H_1) - (H_3)$ hold. Then $T(K) \subset K$.*

Proof. From definition of Hammerstein integral operators T_1, T_2, also by definition of cone K in (2.13), obviously we can see that for every $(u, v) \in K$:

$$T_1(u, v), T_2(u, v) \geq 0. \tag{2.14}$$

Using (2.12) it is clear that $T(u, v) \geq 0$. Also for all $(u, v) \in K$, we have:

$$\begin{aligned} \min_{t\in[p,q]} T_1(u, v) &= \min_{t\in[p,q]} \int_0^1 G(t, s) \left[a(s)f(v(s), (u - w)(s)) + p_1(s)\right] ds \\ &\geq \int_0^1 \min_{t\in[p,q]} G(t, s) \left[a(s)f(v(s), (u - w)(s) + p_1(s)\right] ds \\ &\geq \frac{7}{32} \int_0^1 \max_{t\in[0,1]} G(t, s) \left[a(s)f(v(s), (u - w)(s) + p_1(s)\right] ds \\ &\geq \frac{7}{32} \max_{t\in[0,1]} \int_0^1 G(t, s) \left[a(s)f(v(s), (u - w)(s) + p_1(s)\right] ds \\ &= \frac{7}{32}||T_1(u, v)||. \end{aligned} \tag{2.15}$$

Similarly, we can show that for every $(u, v) \in K$

$$\min_{t\in[p,q]} T_2(u, v) \geq \frac{7}{32}||T_2(u, v)||. \tag{2.16}$$

It follows from (2.14)-(2.16) that if $(u, v) \in K$, then $T(u, v) \in K$. This completes the proof. ☐

Lemma 2.10. *Suppose that conditions $(H_1) - (H_3)$ hold. Then the integral operator $T : K \to K$, is completely continuous.*

Proof. First of all, by Lemma 2.9 we conclude that, operator $T : K \to K$ is well defined. Now in the following three steps, we show that $T : K \to K$ is completely continuous:

(S_1) Uniformly boundedness of T.
(S_2) Continuity of T.
(S_3) Equicontinuity of T.

(PS_1) Let $\Omega \subset K$ is bounded. Thus there exist a positive constant M such that for all $(u, v) \in \Omega$, we have $||(u, v)|| \leq M$. Equivalently we have $||u||, ||v|| \leq M$. Assume

$$L_1 = \max_{\substack{t\in[0,1]\\ u,v\in[0,M]}} (a(t)f(v(t), (u - w)(t)) + p_1(t)) + 1,$$

so for $(u, v) \in \Omega$, we have

$$\begin{aligned} |T_1(u, v)| &= \int_0^1 G(t, s) \left[a(s)f(s, v(s), (u - w)(s)) + p_1(s)\right] ds \\ &\leq L_1 \int_0^1 G(t, s)ds < +\infty. \end{aligned} \tag{2.17}$$

Hence T_1 is uniformly bounded on Ω.

(PS_2) Consider the sequence $\{(u_n, v_n)\} \subset \Omega$ where

$$\lim_{n\to\infty} (u_n, v_n) = (u, v) \in \Omega. \tag{2.18}$$

Since $\int_0^1 \left[a(s)f(v(s), (u - w)(s)) + p_1(s)\right] ds < \infty$, by *Lebesgue Dominated Convergence* theorem, we deduce that when $n \to \infty$

$$\int_0^1 \left[a(s)f(v_n(s), (u_n - w)(s)) + p_1(s)\right] ds$$
$$- \int_0^1 \left[a(s)f(v(s), (u - w)(s)) + p_1(s)\right] ds \to 0.$$

Hence

$$||T_1(u_n, v_n) - T_1(u, v)|| \to 0, \quad (n \to \infty). \tag{2.19}$$

Thus T is continuous on Ω.

(PS_3) For $(u, v) \in \Omega$ and for all $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$ and from (PS_1), we have:

$$
\begin{aligned}
&||T_1(u(t_2), v(t_2)) - T_1(u(t_1), v(t_1))|| \\
&\leq \int_0^1 [G(t_2, s) - G(t_1, s)] \left[a(s)f(v(s), (u - w)(s)) + p_1(s)\right] ds \\
&\leq L_1 \int_0^1 [G(t_2, s) - G(t_1, s)] \, ds \\
&+ \leq L_1 \int_0^{t_1} [G(t_2, s) - G(t_1, s)] \, ds + L_1 \int_{t_1}^{t_2} [G(t_2, s) - G(t_1, s)] \, ds \\
&\quad + L_1 \int_{t_2}^1 [G(t_2, s) - G(t_1, s)] \, ds \\
&= L_1 \int_0^{t_1} \frac{[(t_2 - t_1)(2 - (t_2 + t_1))] \left[(\alpha - 1)(1 - s)^{\alpha-2} + (\alpha - 1)(\alpha - 2)(1 - s)^{\alpha-3}\right]}{2\Gamma\alpha} ds \\
&\quad + L_1 \int_0^{t_1} \frac{\left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}\right]}{2\Gamma(\alpha)} ds \\
&\quad + L_1 \int_{t_1}^{t_2} \frac{[(t_2 - t_1)(2 - (t_2 + t_1))] \left[(\alpha - 1)(1 - s)^{\alpha-2} + (\alpha - 1)(\alpha - 2)(1 - s)^{\alpha-3}\right]}{2\Gamma\alpha} ds \\
&\quad + L_1 \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{2\Gamma(\alpha)} ds \\
&\quad + L_1 \int_{t_2}^1 \frac{[(t_2 - t_1)(2 - (t_2 + t_1))] \left[(\alpha - 1)(1 - s)^{\alpha-2} + (\alpha - 1)(\alpha - 2)(1 - s)^{\alpha-3}\right]}{2\Gamma(\alpha)} ds.
\end{aligned}
$$

Thus, if uniformly $t_1 \to t_2$, then

$$||T_1(u(t_2), v(t_2)) - T_1(u(t_1), v(t_1))|| \longrightarrow 0. \tag{2.20}$$

Hence Hammerstein integral operator T_1 is equicontinuous on Ω.

$(PS_1) - (PS_3)$ and *Arzela-Ascoli* theorem, show that integral operator $T_1 : K \to K$ is completely continuous. Similarly we can prove that, integral operator $T_2 : K \to K$ is also completely continuous, which implies that operator $T : K \to K$ is completely continuous. The proof is complete. □

Theorem 2.11 (Leray-Schauder Fixed Point Index). [7] *Assume that K is a cone in* Banach *space X. Let D be an open bounded subset of X with $D_K = D \cap K \neq \varnothing$ and $\overline{D_K} \neq K$. Suppose that $T : \overline{D_K} \to K$ is a compact map such that for all $x \in \partial D_K, x \neq Tx$. Then the following results hold:*

(i) If $||Tx|| \leq ||x||$ for $x \in \partial D_K$, then $i_K(T, D_K) = 1$.
(ii) If there exist $e \in K \backslash \{0\}$ such that $x \neq Tx + \lambda e$ for all $x \in \partial D_K$ and all $\lambda > 0$, then $i_K(T, D_K) = 0$.

(*iii*) *Let* D_0 *be open in* X *such that* $\overline{D}_0 \subset D_K$. *If* $i_K(T, D_K) = 1$ *and* $i_K(T, D_{0K}) = 0$, *then* T *has a fixed point in* $D_K \backslash \overline{D_{0K}}$. *The same result holds if* $i_K(T, D_K) = 0$ *and* $i_K(T, D_{0K}) = 1$.

The sets $K_{\mu,\nu}, \Omega_{\mu,\nu}$, $(\mu, \nu > 0)$ are defined as follows:

$$K_{\mu,\nu} = \{(u,v) \in K : \ ||(u,v)|| < \mu + \nu\}$$

$$\begin{aligned}\Omega_{\mu,\nu} &= \{(u,v) \in K : \ l(u,v) < \frac{7}{32}(\mu + \nu)\} \\ &= \{(u,v) : \ \frac{7}{32}||(u,v)|| \le l(u,v) < \frac{7}{32}(\mu + \nu)\},\end{aligned}$$

where $l : K \to [0, +\infty)$ with $l(u,v) = \min\{(u(t) + v(t)) : t \in [p,q]\}$.

Lemma 2.12. [7] *The set* $\Omega_{\mu,\nu}$ *defined above has the following properties;*

(*a*) $\Omega_{\mu,\nu}$ *is open with respect to* K.
(*b*) $K_{\frac{7}{32}(\mu,\nu)} \subset \Omega_{\mu,\nu} \subset K_{\mu,\nu}$.
(*c*) $(u,v) \in \partial\Omega_{\mu,\nu}$ *if and only if* $l(u,v) = \frac{7}{32}(\mu+\nu)$.
(*d*) *If* $(u,v) \in \partial\Omega_{\mu,\nu}$, *then* $\frac{7}{32}(\mu,\nu) \le (u,v) \le (\mu,\nu)$ *for* $t \in [p,q]$.

Remark 2.13. Let

$$\begin{aligned}
f^{\mu,\nu}_{\frac{7}{32}(\mu,\nu)} &= \min\left\{ \frac{f(v(t),u(t))}{\mu+\nu} \ \middle| \ t \in [p,q], u \in [\frac{7}{32}\mu, \mu], v \in [0,+\infty) \right\}, \\
f^{\mu,\nu}_0 &= \max\left\{ \frac{f(v(t),u(t))}{\mu+\nu} \ \middle| \ t \in [0,1], u \in [0,\mu], v \in [0,+\infty) \right\}, \\
g^{\mu,\nu}_{\frac{7}{32}(\mu,\nu)} &= \min\left\{ \frac{g(u(t),v(t))}{\mu+\nu} \ \middle| \ t \in [p,q], u \in [0,+\infty), v \in [\frac{7}{32}\nu, \nu] \right\}, \\
g^{\mu,\nu}_0 &= \max\left\{ \frac{g(u(t),v(t))}{\mu+\nu} \ \middle| \ t \in [0,1], u \in [0,+\infty), v \in [0,\nu] \right\}.
\end{aligned}$$

Remark 2.14. According to Lemma 2.6, assume that

$$\begin{aligned}
m_a &= \left(4\int_0^1 M(s)a(s)ds\right)^{-1}, \quad M_a = \left(\frac{7}{32}\int_p^q M(s)a(s)ds\right)^{-1}, \\
m_b &= \left(4\int_0^1 M(s)b(s)ds\right)^{-1}, \quad M_b = \left(\frac{7}{32}\int_p^q M(s)b(s)ds\right)^{-1}.
\end{aligned} \tag{2.21}$$

3. Main Results

Lemma 3.1. *Assume that conditions* $(H_1) - (H_3)$ *and the conditions*

$$\begin{aligned}
f^{\mu,\nu}_{\frac{7}{32}(\mu,\nu)} &\ge \frac{7}{32}M_a \ , \ u \neq T_1(u,v), \qquad (u,v) \in \partial\Omega_{\mu,\nu}, \\
g^{\mu,\nu}_{\frac{7}{32}(\mu,\nu)} &\ge \frac{7}{32}M_b \ , \ v \neq T_2(u,v), \qquad (u,v) \in \partial\Omega_{\mu,\nu}
\end{aligned} \tag{3.1}$$

hold. Then $i_K(T, \Omega_{\mu,\nu}) = 0$.

Proof. Suppose that $e(t) = (1,1)$ for $t \in [0,1]$. Thus $e \in K$. We claim that

$$(u,v) \neq T(u,v) + \theta e, \qquad (u,v) \in \partial\Omega_{\mu,\nu},\ \theta > 0.$$

Otherwise, there exist a $(u_0, v_0) \in \partial\Omega_{\mu,\nu}$ and $\theta > 0$ such that $u_0 = T_1(u_0,v_0)+\theta$ and $v_0 = T_2(u_0,v_0)+\theta$. Using condition (3.1) and Lemma 2.12(c), we conclude that for $t \in [p,q]$,

$$\begin{aligned} u_0(t) &= \int_0^1 G(t,s)\left[a(s)f(v_0(s),(u_0-w)(s)) + p_1(s)\right]ds + \theta \\ &\geq \int_p^q G(t,s)a(s)f(v_0(s),(u_0-w)(s))ds + \theta \\ &\geq \frac{7}{32}(\mu+\nu)M_a\left(\frac{7}{32}\int_p^q M(s)a(s)ds\right) + \theta > \frac{7}{32}(\mu+\nu) + \theta. \end{aligned}$$

Similarly we can prove that $v_0(t) > \frac{7}{32}(\mu+\nu)+\theta$. This implies that $l(u_0,v_0) > \frac{7}{32}(\mu+\nu)$, which is contradiction with Lemma 2.12(c). Hence from Theorem 2.11(ii), we deduce that $i_K(T,\Omega_{\mu,\nu}) = 0$. □

Assume that the following condition holds:

(H_4)

$$\int_0^1 M(s)p_1(s)ds \leq \frac{\mu+\nu}{4}\ ,\ \int_0^1 M(s)p_2(s)ds \leq \frac{\mu+\nu}{4}.$$

Lemma 3.2. *Let conditions* $(H_1)-(H_4)$ *and the following conditions hold.*

$$\begin{aligned} &f_0^{\mu,\nu} \leq m_a,\ u \neq T_1(u,v), \qquad (u,v) \in \partial K_{\mu,\nu}, \\ &g_0^{\mu,\nu} \leq m_b,\ v \neq T_2(u,v), \qquad (u,v) \in \partial K_{\mu,\nu}. \end{aligned} \tag{3.2}$$

Then $i_K(T, K_{\mu,\nu}) = 1$.

Proof. Considering notation 2.13 and conditions (3.2), for $(u,v) \in \partial K_{\mu,\nu}$ we have

$$\begin{aligned} T_1(u,v) &= \int_0^1 G(t,s)[a(s)f(v(s),(u-w)(s)) + p_1(s)]ds \\ &\leq \int_0^1 M(s)a(s)f(v(s),(u-w)(s))ds + \int_0^1 M(s)p_1(s)ds \\ &\leq \frac{\mu+\nu}{4}m_a\left(4\int_0^1 M(s)a(s)ds\right) + \frac{\mu+\nu}{4} \\ &= \frac{\mu+\nu}{2} = \frac{||(u,v)||}{2}. \end{aligned}$$

This implies that $||T_1(u,v)|| \leq \frac{||(u,v)||}{2}$. Similarly we can show that $||T_2(u,v)|| \leq \frac{||(u,v)||}{2}$. Thus $||T(u,v)|| \leq ||(u,v)||$. Applying Theorem 2.11($i$), we conclude that $i_K(T, K_{\mu,\nu}) = 1$. □

Theorem 3.3. *Assume that conditions $(H_1)-(H_4)$ are satisfied also one of the following conditions hold :*

(C_1) *There exist $\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3 \in (0,+\infty)$ with $(\mu_1,\nu_1) < \frac{7}{32}(\mu_2,\nu_2)$ and $(\mu_2,\nu_2) < (\mu_3,\nu_3)$ such that*

$$f_0^{\mu_1,\nu_1} \leq m_a,\ f^{\mu_2,\nu_2}_{\frac{7}{32}(\mu_2,\nu_2)} \geq \frac{7}{32}M_a,\ u \neq T_1(u,v), \quad (u,v) \in \partial\Omega_{\mu_2,\nu_2},$$
$$f_0^{\mu_3,\nu_3} \leq m_a.$$

Also

$$g_0^{\mu_1,\nu_1} \leq m_b,\ g^{\mu_2,\nu_2}_{\frac{7}{32}(\mu_2,\nu_2)} \geq \frac{7}{32}M_b,\ v \neq T_2(u,v), \quad (u,v) \in \partial\Omega_{\mu_2,\nu_2},$$
$$g_0^{\mu_3,\nu_3} \leq m_b.$$

(C_2) *There exist $\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3 \in (0,+\infty)$ with $(\mu_1,\nu_1) < (\mu_2,\nu_2) < \frac{7}{32}(\mu_3,\nu_3)$ such that*

$$f^{\mu_1,\nu_1}_{\frac{7}{32}(\mu_1,\nu_1)} \geq \frac{7}{32}M_a,\ f_0^{\mu_2,\nu_2} \leq m_a,\ u \neq T_1(u,v), \quad (u,v) \in \partial K_{\mu_2,\nu_2},$$
$$f^{\mu_3,\nu_3}_{\frac{7}{32}(\mu_3,\nu_3)} \geq \frac{7}{32}M_a.$$

Also

$$g^{\mu_1,\nu_1}_{\frac{7}{32}(\mu_1,\nu_1)} \geq \frac{7}{32}M_b,\ g_0^{\mu_2,\nu_2} \leq m_b,\ v \neq T_2(u,v), \quad (u,v) \in \partial K_{\mu_2,\nu_2},$$
$$g^{\mu_3,\nu_3}_{\frac{7}{32}(\mu_3,\nu_3)} \geq \frac{7}{32}M_b.$$

Then the coupled system (1.1),(1.2) has two positive solutions in K.

Proof. Assume that condition (C_2) holds. We prove that operator T has two fixed points $(u_*,v_*), (u^*,v^*)$ in $K_{\mu_2,\nu_2}\backslash\overline{\Omega_{\mu_1,\nu_1}}$ and $\Omega_{\mu_3,\nu_3}\backslash K_{\mu_2,\nu_2}$. According to Lemma 3.1 and Lemma 3.2, we have

$$i_K(T, K_{\mu_2,\nu_2}) = 1\ ,\ i_K(T,\Omega_{\mu_1,\nu_1}) = i_K(T,\Omega_{\mu_3,\nu_3}) = 0.$$

Using Lemma 2.12(b), we conclude that $K_{\mu_1,\nu_1} \subset K_{\mu_2,\nu_2} \subset \Omega_{\mu_3,\nu_3}$.

$$i_K(T, K_{\mu_2,\nu_2}\backslash\overline{\Omega_{\mu_1,\nu_1}}) = i_K(T, K_{\mu_2,\nu_2}) - i_K(T,\Omega_{\mu_1,\nu_1}) = 1$$
$$i_K(T, \Omega_{\mu_3,\nu_3}\backslash\overline{K_{\mu_2,\nu_2}}) = i_K(T,\Omega_{\mu_3,\nu_3}) - i_K(T, K_{\mu_2,\nu_2}) = -1.$$

Thus considering Theorem 2.11(iii), operator T has two fixed points (u_*,v_*) and (u^*,v^*) in $K_{\mu_2,\nu_2}\backslash\overline{\Omega_{\mu_1,\nu_1}}$ and $\Omega_{\mu_3,\nu_3}\backslash\overline{K_{\mu_2,\nu_2}}$. We can prove the same result, when condition (C_1) occurs. Using Remark 2.8, we deduce that coupled system (1.1),(1.2) has two positive solutions in K. This completes the proof. □

Remark 3.4. If in (C_2), $f^{\mu_1,\nu_1}_{\frac{7}{32}(\mu_1,\nu_1)} \geq \frac{7}{32}M_a$ is replaced by $f^{\mu_1,\nu_1}_{\frac{7}{32}(\mu_1,\nu_1)} > \frac{7}{32}M_a$ and $g^{\mu_1,\nu_1}_{\frac{7}{32}(\mu_1,\nu_1)} \geq \frac{7}{32}M_b$ is replaced by $g^{\mu_1,\nu_1}_{\frac{7}{32}(\mu_1,\nu_1)} > \frac{7}{32}M_b$, then the coupled system (1.1),(1.2) has a third solution $(\widehat{u},\widehat{v}) \in K$. See more details in Theorem 2.11 [7].

Considering Theorem 3.3, we have the following lemma.

Lemma 3.5. *Let the conditions $(H_1)-(H_4)$ and one of the following conditions hold :*

(D_1) *There exist $\mu_1,\mu_2,\nu_1,\nu_2 \in (0,+\infty)$ with $(\mu_1,\nu_1) < \frac{7}{32}(\mu_2,\nu_2)$ such that*

$$f^{\mu_1,\nu_1}_0 \leq m_a, \quad f^{\mu_2,\nu_2}_{\frac{7}{32}(\mu_2,\nu_2)} \geq \frac{7}{32}M_a,$$
$$g^{\mu_1,\nu_1}_0 \leq m_b, \quad g^{\mu_2,\nu_2}_{\frac{7}{32}(\mu_2,\nu_2)} \geq \frac{7}{32}M_b.$$

(D_2) *There exist $\mu_1,\mu_2,\nu_1,\nu_2 \in (0,+\infty)$ with $(\mu_1,\nu_1) < (\mu_2,\nu_2)$ such that*

$$f^{\mu_1,\nu_1}_{\frac{7}{32}(\mu_1,\nu_1)} \geq \frac{7}{32}M_a, \quad f^{\mu_2,\nu_2}_0 \leq m_a,$$
$$g^{\mu_1,\nu_1}_{\frac{7}{32}(\mu_1,\nu_1)} \geq \frac{7}{32}M_b, \quad g^{\mu_2,\nu_2}_0 \leq m_b.$$

Then the coupled system (1.1),(1.2) has one positive solution in K.

Remark 3.6. We can prove that if $f^{\mu,\nu}_{\frac{7}{32}(\mu,\nu)} > \frac{7}{32}M_a$, then $u \neq T_1(u,v)$, $(u,v) \in \partial\Omega_{\mu,\nu}$. Also if $f^{\mu,\nu}_0 < m_a$, then $u \neq T_1(u,v)$, $(u,v) \in \partial K_{\mu,\nu}$. Similarly we can derive this result for operator T_2.(For more details see [7] pp. 694-695.)

4. Example

Consider the following fractional coupled system

$$\begin{cases} {}^cD^{\frac{11}{4}}_{0^+}u(t) = a(t)f(v(t),u(t)) - p_1(t), \\ {}^cD^{\frac{9}{4}}_{0^+}v(t) = b(t)g(u(t),v(t)) - p_2(t), \quad t \in (0,1) \end{cases} \tag{4.1}$$

with boundary conditions

$$\begin{cases} u(0) = 0 = v(0) \\ u'(0) + u''(0) = 0 = v'(0) + v''(0) \\ u'(1) + u''(1) = 0 = v'(1) + v''(1) \end{cases} \tag{4.2}$$

where

$$f(v,u)=\begin{cases}10^{-6}\sin^4(v)+u^{\frac{1}{3}}, & (u,v)\in[0,10^3]\times[0,+\infty)\\ 10^{-6}\sin^4(v)+u^{\frac{1}{3}}+10^5(u-10^3), & (u,v)\in[10^3,10^5]\times[0,+\infty)\\ 10^{-6}\sin^4(v)+u^{\frac{1}{3}}+9.9\times10^9, & (u,v)\in[10^5,+\infty)\times[0,+\infty)\end{cases}$$

and

$$g(u,v)=\begin{cases}10^{-8}\cos^2(u)+v^{\frac{1}{2}}, & (u,v)\in[0,+\infty)\times[0,10^7]\\ 10^{-8}\cos^2(u)+v^{\frac{1}{2}}+10^7(v-10^7), & (u,v)\in[0,+\infty)\times[10^7,10^{\frac{15}{2}}]\\ 10^{-8}\cos^2(u)+v^{\frac{1}{2}}+(\sqrt{10}-1)\times10^{14}, & (u,v)\in[0,+\infty)\times[10^{\frac{15}{2}},+\infty)\end{cases}$$

also

$$a(t)=2\Gamma(\frac{11}{4})(1-t)^{\frac{1}{4}},\quad b(t)=2\Gamma(\frac{9}{4})(1-t)^{\frac{1}{2}}$$
$$p_1(t)=(1-t)^{-\frac{1}{4}},\quad p_2(t)=(1-t)^{-\frac{1}{8}}.$$

Choosing

$$\mu_1=10^{-10},\quad \mu_2=10^2,\quad \mu_3=10^7,$$
$$\nu_1=10^{-15},\quad \nu_2=10,\quad \nu_3=10^8,$$

simple calculation shows that condition (H_4) holds as follow

$$\int_0^1 M(s)p_1(s)ds<\frac{4}{\Gamma(\frac{11}{4})}<\frac{\mu_2+\nu_2}{4},\int_0^1 M(s)p_2(s)ds<\frac{4}{\Gamma(\frac{9}{4})}<\frac{\mu_2+\nu_2}{4},$$
$$m_a=\frac{12}{121},\quad M_a=\frac{373248}{66591},\quad m_b=\frac{231}{1381},\quad M_b=\frac{32\times10^6}{3311119}.$$

Now, with direct computation we have

$f(v,u) \geq \sqrt[3]{\frac{7}{32}} \times 10^{-\frac{10}{3}} > \frac{7}{32}(\mu_1+\nu_1)M_a, u \in [\frac{7}{32}\mu_1, \mu_1], v \in [0,\nu_1].$

$So\ f^{\mu_1,\nu_1}_{\frac{7}{32}(\mu_1,\nu_1)} > \frac{7}{32}M_a.$

$f(v,u) \leq 10^{-6} + 10^{\frac{2}{3}} \leq (\mu_2+\nu_2)m_a,\ u \in [0,\mu_2], v \in [0,\nu_2].$

$So\ f^{\mu_2,\nu_2}_0 \leq m_a.$

$f(v.u) \geq \sqrt[3]{\frac{7}{32}} \times 10^{\frac{7}{3}} + 9.99 \times 10^9 > \frac{7}{32}(\mu_3+\nu_3)M_a,\ u \in [\frac{7}{32}\mu_3, \mu_3],$

$v \in [0,\nu_3].\ So\ f^{\mu_3,\nu_3}_{\frac{7}{32}(\mu_3,\nu_3)} > \frac{7}{32}M_a,$

$g(u,v) \geq \sqrt{\frac{7}{32}} \times 10^{-\frac{15}{2}} > \frac{7}{32}(\mu_1+\nu_1)M_b, \quad u \in [0,\mu_1], v \in [\frac{7}{32}\nu_1, \nu_1].$

$Hence\ g^{\mu_1,\nu_1}_{\frac{7}{32}(\mu_1,\nu_1)} > \frac{7}{32}M_b.$

$g(u,v) \leq 10^{-8} + \sqrt{10} \leq (\mu_2+\nu_2)m_b, \quad u \in [0,\mu_2], v \in [0,\nu_2].$

$Hence\ g^{\mu_2,\nu_2}_0 \leq m_b.$

$g(u,v) \geq \sqrt{\frac{7}{32}} \times 10^4 + (\sqrt{10}-1) \times 10^{14} > \frac{7}{32}(\mu_3+\nu_3)M_b, u \in [0,\mu_3], v \in [\frac{7}{32}\nu_3, \nu_3].$

$Hence\ g^{\mu_3,\nu_3}_{\frac{7}{32}(\mu_3,\nu_3)} > \frac{7}{32}M_b.$

Using Theorem 3.3 and Remark 3.4 and Remark 3.6, we conclude that operator T has three fixed points in K. Finally using Remark 2.8 we conclude that coupled system (4.1),(4.2) has three positive solutions in K.

5. Conclusion

In this paper, the existence and multiplicity of positive solutions for coupled system of nonlinear fractional BVPs with negatively perturbed terms, have been studied. Employing fixed point technique, existence and multiplicity results of positive solutions for BVP (2.9), implies the same results for BVP (2.10).

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