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Analytical solutions for the fractional Klein-Gordon equation

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Abstract In this paper, we solve a inhomogeneous fractional Klein-Gordon equation by the method of separating variables. We apply the method for three boundary conditions, contain Dirichlet, Neumann, and Robin boundary conditions, and solve some examples to illustrate the effectiveness of the method.

Keywords. Fractional Klein-Gordon equation, Mittag-Leffler, Method of separating variables, Caputo derivative.

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1. Introduction

Partial fractional differential equation frequently appear in variety of applications [9, 10], so lots of scientists have been focused on solving them, but unfortunately most of these equations do not have exact solutions, so for solving them numerical and approximative methods must be used. For instance variational method [11], supper symmetric method [15, 3, 2], finite difference method (FDM), and functional variable method [7], are some of these methods. In [14], a novel predictor-corrector method, called Jacobian-predictor-corrector approach, for the numerical solutions of fractional ordinary differential equations, which are based on the polynomial interpolation, was presented. The authors of [13], proposed the Cantor-type cylindrical-coordinate method in order to investigate a family of local fractional differential operators on Cantor sets. In [1], the authors studied an initial value problem for a fractional differential equation using the Riemann-Liouville fractional derivative. The Klein-Gordon equation has important applications in mathematical physics such as solid-state physics, nonlinear optics, and quantum mechanics [4, 5, 11, 12].

In this paper we consider the fractional Klein-Gordon equation as follows

$$D_t^{\alpha} u(x,t) - u_{xx}(x,t) + u(x,t) = f(x,t). \tag{1.1}$$

In (1.1) by $\alpha = 2$ we can get the PDE Klein-Gordon equation. We try to solve (1.1) by the method of separating of variables. For this purpose in section 2 we give some preliminaries, section 3 is contained the exact solution of inhomogeneous fractional Klein-Gordon equation with three boundary conditions, Drichlet, Neumann, and Robin boundary condition, then in section 4 we propose some examples related to section 3.

2. Preliminaries

In this section, we introduce some necessary definitions from fractional calculus. You can find some extra details to fractional derivatives [6, 9, 10].

Definition 1.([6]) A function $f : \mathbb{R} \to \mathbb{R}^+$ is said to be in the space C_ν, with $\nu \in \mathbb{R}$, if it can be written as $f(x) = x^p f_1(x)$ with $p > \nu$, $f_1(x) \in C[0, \infty)$ and it is said to be in the space C_ν^m if $f^{(m)} \in C_\nu$ for $m \in \mathbb{N} \bigcup \{0\}$.

Definition 2.([8]) The Riemann-Liouville fractional integral of $f \in C_\nu$ with order $\alpha > 0$ and $\nu \geq -1$ is defined as:

$$\begin{gathered} J^\alpha f(t) = \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha > 0, \quad t > 0, \\ J^0 f(t) = f(t). \end{gathered} \tag{2.1}$$

Definition 3.([6]) The Riemann-Liouville fractional derivative of $f \in C_{-1}^m$ with order $\alpha > 0$ and $m \in \mathbb{N} \bigcup \{0\}$, is defined as:

$$D_t^\alpha f(t) = \tfrac{d^m}{dt^m} J^{m-\alpha} f(t), \quad m-1 < \alpha \leq m, \quad m \in \mathbb{N}. \tag{2.2}$$

Definition 4.([6]) The Caputo fractional derivative of $f \in C_{-1}^m$ with order $\alpha > 0$ and $m \in \mathbb{N} \bigcup \{0\}$, is defined as:

$${}^C D_t^\alpha f(t) = \left\{ \begin{array}{ll} J^{m-\alpha} f^{(m)}(t), & m-1 < \alpha \leq m, \quad m \in \mathbb{N} \\ \frac{d^m f(t)}{dt^m}, & \alpha = m. \end{array} \right. \tag{2.3}$$

Definition 5.([6]) A two-parameter Mittag-Leffler function is defined by the following series

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}. \tag{2.4}$$

Definition 6.([8]) A multivariate Mittag-Leffler function is defined as

$$\begin{aligned} & E_{(a_1, a_2, \cdots, a_n), b}(z_1, z_2, \cdots, z_n) \\ & = \sum_{k=0}^{\infty} \sum_{l_1 + l_2 + \cdots + l_n = k} \frac{k!}{l_1! \times l_2! \times \cdots \times l_n!} \frac{\prod_{i=1}^{n} z_i^{l_i}}{\Gamma(b + \sum_{i=1}^{n} a_i l_i)}, \end{aligned} \tag{2.5}$$

where $b > 0$, $l_1, l_2, \cdots, \ l_n \geq 0$, $|z_i| < \infty, i = 1, 2, \cdots, n$.

Definition 7. Let us define the Laplace-transform (LT) operator φ on a function $u(x,t)$, $(t \geq 0)$ by

$$\varphi\{u(x,t); t \mapsto s\} = \int_0^\infty e^{-st} u(x,t) dt, \tag{2.6}$$

and denote it by $\varphi\{u(x,t);t\mapsto s\} = L(u(x,t))$, where s is the LT parameter. For our purpose here, we shall take s to be real and positive.
Consequently, the LT of Mittag-Leffler function has the following form

$$L(E_{\alpha,\beta}(t)) = \int_0^\infty e^{-st}E_{\alpha,\beta}(t)dt = \sum_{k=0}^{\infty}\frac{1}{s^{k+1}\Gamma(\alpha k+\beta)}. \tag{2.7}$$

Lemma 2.1([8]). Let $\mu > \mu_1 > \mu_2 > \ldots > \mu_n \geq 0$, $m_i - 1 < \mu_i \leq m_i$, $m_i \in \mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$, $d_i \in \mathbb{R}, i = 1,2,\ldots,n$. Consider the initial value problem

$$\begin{cases} (D^\mu y)(x) - \displaystyle\sum_{i=1}^{n}\lambda_i(D^{\mu_i}y)(x) = g(x), \\ y^{(k)}(0) = c_k \in \mathbb{R},\ k = 0,1,\ldots,m-1,\ m-1<\mu\leq m, \end{cases}$$

where the function g(x) is assumed to lie in C_{-1}, if $\mu \in \mathbb{N}$, in C^1_{-1}, if $\mu \notin \mathbb{N}$ and the unknown function $y(x)$ is to be determined in the space C^m_{-1}. This has solution

$$y(x) = y_g(x) + \sum_{k=0}^{m-1}c_k u_k(x),\ \ x\geq 0,$$

where

$$y_g(x) = \int_0^x t^{\mu-1}E_{(.),\mu}(t)g(x-t)dt$$

and

$$u_k(x) = \frac{x^k}{k!} + \sum_{i=l_k+1}^{n} d_i x^{k+\mu-\mu_i}E_{(.),k+1+\mu-\mu_i}(x),\quad k = 0,1,\ldots,m-1,$$

fulfills the initial conditions $u_k^{(l)}(0) = \delta_{kl},\ k,l = 0,1,\ldots,m-1$. The function

$$E_{(.),\sigma}(x) = E_{(\mu-\mu_1,\ldots,\mu-\mu_n),\sigma}(d_1x^{\mu-\mu_1},\ldots,d_nx^{\mu-\mu_n}),$$

is a particular case of the multivariate Mittag-Leffler function (see [8]) and the natural numbers $l_k, k = 0,1,\ldots,m-1$, are determined from the condition

$$\begin{cases} m_{l_k} \geq k+1, \\ m_{l_k+1} \leq k. \end{cases}$$

In the case $m_i \leq k, i = 1,2,\ldots,n$, we set $l_k := 0$, and if $m_i \geq k+1, i = 1,2,\ldots,n$, then $l_k := n$.

3. Nonhomogeneous fractional Klein-Gordon equation

3.1. **Dirichlet boundary condition.** In this section, we determine the solution of the fractional Klein-Gordon equation (1.1), with the initial condition

$$u(x,0) = \phi(x),\quad u_t(x,0) = \psi(x),\quad 0\leq x\leq L, \tag{3.1}$$

and the inhomogeneous boundary conditions

$$u(0,t) = \mu_1(t),\quad u(L,t) = \mu_2(t),\quad t\geq 0,$$

where $\mu_1(t)$ and $\mu_2(t)$ are nonzero smooth functions with order-one continuous derivative and $\phi(x)$ and $\psi(x)$ are a continuous functions satisfying $\phi(0) = \mu_1(0)$ and $\phi(L) = \mu_2(0)$. Now, for applying the method of separating variables with nonhomogeneous boundary conditions, we first transform it into homogeneous ones by taking

$$u(x,t) = W_1(x,t) + V_1(x,t),$$

where $W_1(x,t)$ is an unknown function and

$$V_1(x,t) = \frac{\mu_2(t) - \mu_1(t)}{L}x + \mu_1(t), \tag{3.2}$$

which satisfies the following boundary conditions

$$V_1(0,t) = \mu_1(t), \quad V_1(L,t) = \mu_2(t). \tag{3.3}$$

Furthermore $W_1(x,t)$ satisfies the problem with homogeneous boundary conditions as follows:

$$\begin{cases} D_t^\alpha W_1(x,t) - \dfrac{\partial^2 W_1(x,t)}{\partial x^2} + W_1(x,t) = \tilde{f}(x,t), \\ W_1(x,0) = g_1(x), \quad (W_1)_t(x,0) = g_2(x), \quad 0 \le x \le L, \\ W_1(0,t) = 0, \quad W_1(L,t) = 0, \quad t \ge 0, \end{cases} \tag{3.4}$$

where

$$\tilde{f}(x,t) = f(x,t) - D_t^\alpha V_1(x,t) + \frac{\partial^2 V_1(x,t)}{\partial x^2} - V_1(x,t), \tag{3.5}$$
$$g_1(x) = \phi(x) - \frac{\mu_2(0) - \mu_1(0)}{L} - \mu_1(0),$$
$$g_2(x) = \psi(x) - \frac{\mu_2'(0) - \mu_1'(0)}{L} - \mu_1'(0).$$

For solving the corresponding homogeneous equation in (1.1), with proposed method we set $W_1(x,t) = X(x)T(t)$ in (3.4), we get the following linear ODE for $X(x)$:

$$X''(x) - \lambda X(x) = 0, \quad X(0) = 0, \quad X(L) = 0 \tag{3.6}$$

and a fractional linear ode with the Caputo derivative for $T(t)$

$$D_t^\alpha T(t) + (1-\lambda)T(t) = 0, \tag{3.7}$$

in which λ is a negative constant.

Hence the Sturm-Liouville problem, which is given by (3.6), has eigenvalues λ_n and corresponding eigenfunctions $X_n(t)$ as follows:

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(t) = \sin\frac{n\pi}{L}x, \quad n = 1, 2, \cdots. \tag{3.8}$$

Now, we want to find a solution of the inhomogeneous problem in (3.4) of the form

$$W_1(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin(\frac{n\pi}{L}x). \tag{3.9}$$

We suppose that the series can be differentiated term by term. For determining $B_n(t)$, we expand $\tilde{f}(x,t)$ as a Fourier series by the eigenfunctions $\{\sin(\frac{n\pi}{L}x)\}$ as follows

$$\tilde{f}(x,t) = \sum_{n=1}^{\infty} \tilde{f}_n(t)\sin(\frac{n\pi}{L}x), \tag{3.10}$$

in which

$$\tilde{f}_n(t) = \frac{2}{L}\int_0^L \tilde{f}(x,t)\sin(\frac{n\pi}{L}x)dx. \tag{3.11}$$

Now, if we substitute (3.9) and (3.10) into (3.4), we have

$$\begin{aligned}\sum_{n=1}^{\infty} D_t^{\alpha} B_n(t)\sin(\frac{n\pi}{L}x) + \left(\frac{n^2\pi^2}{L^2}+1\right)\sum_{n=1}^{\infty} B_n(t)\sin(\frac{n\pi}{L}x)\\ = \sum_{n=1}^{\infty}\tilde{f}_n(t)\sin(\frac{n\pi}{L}x).\end{aligned} \tag{3.12}$$

By orthogonality properties of $\sin(\frac{n\pi}{L}x)$, we obtain

$$D_t^{\alpha} B_n(t) + \left(1+\frac{n^2\pi^2}{L^2}\right)B_n(t) = \tilde{f}_n(t). \tag{3.13}$$

Since $W_1(x,t)$ satisfies the initial condition in (3.4), we must have

$$\sum_{n=1}^{\infty} B_n(0)\sin(\frac{n\pi}{L}x) = g_1(x), \tag{3.14}$$

$$\sum_{n=1}^{\infty} \frac{\partial B_n}{\partial t}(0)\sin(\frac{n\pi x}{L}) = g_2(x), \tag{3.15}$$

which gives

$$\begin{aligned} B_n(0) &= \frac{2}{L}\int_0^L g_1(x)\sin(\frac{n\pi}{L}x)dx, \\ \frac{\partial B_n}{\partial t}(0) &= \frac{2}{L}\int_0^L g_2(x)\sin(\frac{n\pi x}{L})dx, \quad n=1,2,\cdots \end{aligned} \tag{3.16}$$

For each value of n, (3.14) and (3.16) give a fractional initial value problem.

Lemma 2.1 implies that the fractional initial value problem has the following solution

$$\begin{aligned} B_n(t) = & \int_0^t \tau^{\alpha-1}E_{(\alpha,\alpha)}(-(\frac{n\pi}{L})^2-1)\tau^{\alpha}\tilde{f}_n(t-\tau)d\tau \\ & +B_n(0)u_0(t)+B_n'(0)u_1(0). \end{aligned}$$

Therefore we obtain the solution of the (3.4) with the form

$$\begin{aligned} W_1(x,t) &= \sum_{n=1}^{\infty} B_n(t)\sin(\frac{n\pi x}{L}) \\ &= \sum_{n=1}^{\infty}\sin(\frac{n\pi x}{L})\left[\int_0^t \tau^{\alpha-1}E_{(\alpha,\alpha)}(-((\frac{n\pi}{L})^2-1)\tau^{\alpha}\tilde{f}_n(t-\tau)d\tau\right]. \end{aligned} \tag{3.17}$$

Hence the solution of (1.1) is the form as

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} B_n(t)\sin(\frac{n\pi x}{L}) + \mu_1(t) + \frac{\mu_2(t)-\mu_1(t)}{L}x \\ &= \sum_{n=1}^{\infty}\sin(\frac{n\pi x}{L})\left[\int_0^t \tau^{\alpha-1}E_{(\alpha,\alpha)}(-((\frac{n\pi}{L})^2-1)\tau^{\alpha}\tilde{f}_n(t-\tau)d\tau\right] \\ &\quad +\mu_1(t) + \tfrac{\mu_2(t)-\mu_1(t)}{L}x. \end{aligned} \tag{3.18}$$

3.2. **Neumann boundary condition.** In this subsection, we obtain the solution of the fractional Klein-Gordon equation (1.1) with the initial and Neumann boundary conditions

$$\begin{aligned} &u(x,0)=\phi(x), \quad u_t(x,0)=\psi(x), \quad 0\le x\le L, \\ &u_x(0,t)=\mu_1(t), \quad u_x(L,t)=\mu_2(t), \quad t\ge 0, \end{aligned} \tag{3.19}$$

in which $\phi(x),\ \psi(x),\ \mu_1(t),\ \mu_2(t)$ are as defined in subsection 3.1.

For solving the problem with inhomogeneous boundary conditions, in a similar way, transform it into a homogeneous boundary condition. Thus we suppose that

$$u(x,t) = W_2(x,t) + V_2(x,t),$$

where $\tilde{W}_2(x,t)$ is an unknown function and

$$V_2(x,t) = \frac{\mu_2(t)-\mu_1(t)}{2L}x^2 + \mu_1(t)x, \tag{3.20}$$

which satisfies the following boundary conditions:

$$(V_2)_x(0,t) = \mu_1(t), \quad (V_2)_x(L,t) = \mu_2(t) \tag{3.21}$$

and the function $W_2(x,t)$ satisfies in problem with homogeneous boundary conditions as follows:

$$\begin{cases} D_t^{\alpha}W_2(x,t) - \dfrac{\partial^2 W_2(x,t)}{\partial x^2} + W_2(x,t) = \tilde{f}(x,t), \\ W_2(x,0) = g_1(x), \quad \frac{\partial W_2(x,0)}{\partial t} = g_2(x), \quad 0\le x\le L, \\ (W_2)_x(0,t)=0, \quad (W_2)_x(L,t)=0, \quad t\ge 0, \end{cases} \tag{3.22}$$

in which

$$\begin{aligned} \tilde{f}(x,t) &= f(x,t) - D_t^{\alpha}V_2(x,t) + \frac{\mu_2(t)-\mu_1(t)}{L} - \frac{\mu_2(t)-\mu_1(t)}{2L}x^2 - \mu_1(t)x, \\ g_1(x) &= \phi(x) - \frac{x^2}{2L}[\mu_2(0)-\mu_1(0)] - \mu_1(0)x, \\ g_2(x) &= \psi(x) - \frac{x^2}{2L}[\mu_2'(0)-\mu_1'(0)] - \mu_1'(0)x. \end{aligned} \tag{3.23}$$

Now, for solving the corresponding homogeneous equation in (3.22) by the method of separating variables (assuming that $\tilde{f}(x,t) = 0$), we suppose that $W_2(x,t) = X(x)T(t)$ and substituting it in (3.22), we obtain a linear ode for $X(x)$ and a linear FDE for $T(t)$ as

$$X''(x) + \lambda X(x) = 0, \quad X(0) = X(L) = 0,$$
$$D_t^{\alpha} T(t) + (1-\lambda)T(t) = 0. \tag{3.24}$$

The Sturm-Liouville problem, which is given by (3.24), has eigenvalues and corresponding eigenfunctions as follows:

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad X_n(x) = \cos(\frac{n\pi x}{L}), \quad n = 1, 2, \dots,$$
$$. \tag{3.25}$$

Now we are going to find a solution of the inhomogeneous problem in (3.22) which takes the form

$$W_2(x,t) = \sum_{n=1}^{\infty} B_n(t) \cos(\frac{n\pi}{L}x). \tag{3.26}$$

For determining $B_n(t)$, we expand $\tilde{f}(x,t)$, as a Fourier series by the eigenfunctions $\{\cos(\frac{n\pi}{L}x)\}$, as follows:

$$\tilde{f}(x,t) = \sum_{n=1}^{\infty} \tilde{f}_n(t) \cos(\frac{n\pi}{L}x), \tag{3.27}$$

in which the Fourier coefficients are

$$\tilde{f}_n(t) = \frac{2}{L}\int_0^L \tilde{f}(x,t)\cos(\frac{n\pi}{L}x)dx. \tag{3.28}$$

Then substituting (3.26), (3.27) into (3.22) implies

$$D_t^{\alpha} B_n(t) + \left(1 + \frac{n^2\pi^2}{L^2}\right) B_n(t) = \tilde{f}_n(t), \tag{3.29}$$

Since $W_2(x,t)$ fulfills the initial conditions in (3.22), we have

$$\sum_{n=1}^{\infty} B_n(0)\cos(\frac{n\pi}{L}x) = g_1(x), \tag{3.30}$$
$$\sum_{n=1}^{\infty} \frac{\partial B_n}{\partial t}(0)\cos(\frac{n\pi x}{L}) = g_2(x), \tag{3.31}$$

which yields

$$B_n(0) = \frac{2}{L}\int_0^L g_1(x)\cos(\frac{n\pi}{L}x)dx, \tag{3.32}$$
$$\frac{\partial B_n}{\partial t}(0) = \frac{2}{L}\int_0^L g_2(x)\cos(\frac{n\pi x}{L})dx.$$

Hence lemma 2.1 implies that the fractional initial value problem has the solution as

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} B_n(t)\cos(\frac{n\pi x}{L}) = \mu_1(t)x + \frac{\mu_2(t)-\mu_1(t)}{2L}x^2 \\ &+ \sum_{n=1}^{\infty} \cos(\frac{n\pi}{L}x)\left[\int_0^t \tau^{\alpha-1}E_{(\alpha,\alpha)}(-(n\pi)^2-1)\tau^{\alpha}\tilde{f}_n(t-\tau)d\tau\right]. \end{aligned} \tag{3.33}$$

3.3. **Robin boundary condition.** In this subsection, we want to find the solution of the (1.1) with the Robin boundary conditions

$$\begin{aligned} &u(x,0)=\phi(x), \quad u_t(x,0)=\psi(x), \quad 0\le x\le L, \\ &u(0,t)+\alpha_1 u_x(0,t)=\mu_1(t), \quad u(L,t)+\beta_1 u_x(L,t)=\mu_2(t), \quad t\ge 0. \end{aligned} \tag{3.34}$$

where α_1 and β_1 are nonzero constants. for solving this equation, we should translate the nonhomogeneous boundary conditions to the homogenous ones. So

$$u(x,t) = W_3(x,t) + V_3(x,t),$$

where $W_3(x,t)$ is a new unknown function and

$$V_3(x,t) = \frac{\mu_1(t)-\mu_2(t)}{\alpha_1-\beta_1-L}x - \frac{(L+\beta_1)\mu_1(t)-\alpha_1\mu_2(t)}{\alpha_1-\beta_1-L}, \tag{3.35}$$

so we will have

$$\begin{cases} V_3(0,t)+\alpha_1(V_3)_x(0,t)=\mu_1(t), \\ V_3(L,t)+\beta_1(V_3)_x(L,t)=\mu_2(t). \end{cases}$$

The function $W_3(x,t)$ is the solution of problem with homogeneous boundary conditions:

$$\begin{cases} D_t^{\alpha}W_3(x,t)+(W_3)_{xx}(x,t)+W_3(x,t)=\tilde{f}(x,t), \\ W_3(x,0)=g_1(x), \quad (W_3)_t(x,0)=g_2(x), \quad 0\le x\le L, \\ W_3(0,t)+\alpha_1(W_3)_x(0,t)=0, \\ W_3(L,t)+\beta_1(W_3)_x(L,t)=0. \end{cases} \tag{3.36}$$

where

$$\tilde{f}(x,t) = f(x,t) - D_t^{\alpha}V_3(x,t) + (V_3)_{xx}(x,t) - V_3(x,t), \tag{3.37}$$

and

$$\begin{aligned} &g_1(x)=\phi(x)-V_3(x,t), \\ &g_2(x)=\psi(x)-\frac{\mu_1^{'}(0)-\mu_2^{'}(0)}{\alpha_1-\beta_1-L}x+\frac{(L+\beta_1)\mu_1^{'}(0)-\alpha_1\mu_2^{'}(0)}{\alpha_1-\beta_1-L}. \end{aligned} \tag{3.38}$$

Like two previous subsections first we solve the homogeneous equation ($\tilde{f}(x,t)=0$). By assuming $W_3(x,t)=X(x)T(t)$, and substituting it in (3.36), an ordinary linear differential equations obtained for $X(x)$ as follows:

$$\begin{cases} X''(x)+\lambda^2X(x)=0, \\ X(0)+\alpha_1X^{'}(0)=0, \quad X(L)+\beta_1X^{'}(L)=0. \end{cases}$$

Note that the eigenvalues are countable and can be listed in a sequence $\lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots$, with

$$\lim_{n\to\infty} \lambda_n = \infty, \tag{3.39}$$

and corresponding eigenfunctions are

$$X_n(x) = -\alpha_1\lambda_n \cos(\lambda_n x) + \sin(\lambda_n x), \qquad n = 1, 2, \ldots. \tag{3.40}$$

we can show that

$$\begin{aligned}\int_0^L X_n^2(x)dx &= \int_0^L -\alpha_1\lambda_n \cos(\lambda_n x) + \sin(\lambda_n x)^2 dx\\ &= \frac{\alpha_1^2\lambda_n - 1}{2}.\frac{\tan\lambda_n L}{1+\tan^2\lambda_n L} - \frac{\alpha_1\tan^2\lambda_n L}{1+\tan^2\lambda_n L} + \frac{(\alpha_1^2\lambda_1^2+1)L}{2},\end{aligned} \tag{3.41}$$

where

$$\tan\lambda_n L = \frac{(\alpha_1-\beta_1)\lambda_n}{1+\alpha_1\beta_1\lambda_n^2}.$$

The soloution for (3.36) is of the form

$$W_3(x,t) = \sum_{n=1}^{\infty} B_n(t)X_n(t), \tag{3.42}$$

with following initial conditions:

$$\begin{cases} W_3(x,0) = \sum_{n=1}^{\infty} B_n(0)(-\alpha_1\lambda_n\cos\lambda_n x + \sin\lambda_n x) = g_1(x),\\ (W_3)_t(x,0) = \sum_{n=1}^{\infty} B_n^{'}(0)(-\alpha_1\lambda_n\cos\lambda_n x + \sin\lambda_n x) = g_2(x).\end{cases} \tag{3.43}$$

We assume that the series can be differentiated term by term. In order to determine $B_n(t)$, we expand $\tilde{f}(x,t)$ as follows

$$\tilde{f}(x,t) = \sum_{n=1}^{\infty} \tilde{f}_n(t)(-\alpha_1\mu_n\cos\mu_n x + \sin\mu_n x), \tag{3.44}$$

where

$$\tilde{f}_n(t) = b_n^{-1}\int_0^L \tilde{f}(\xi,t)(\alpha_1\mu_n\cos\mu_n\xi + \sin\mu_n\xi)d\xi, \tag{3.45}$$

then

$$\begin{aligned}&\sum_{n=1}^{\infty} D^\alpha B_n(t)X_n + \lambda_n^2\sum_{n=1}^{\infty} B_n(t)X_n + \sum_{n=1}^{\infty} B_n(t)X_n\\ &= \sum_{n=1}^{\infty} \tilde{f}_n(t)X_n.\end{aligned} \tag{3.46}$$

By orthogonality properties of $X_n(x)$, we get

$$D_t^\alpha B_n(t) + (1+\lambda_n^2)B_n(t) = \tilde{f}_n(t). \tag{3.47}$$

According to lemma 2.1

$$B_n(t) = \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(-(\lambda_n^2+1)\tau^\alpha)\tilde{f}_n(t-\tau)d\tau$$
$$+u_0(t)B_n(0) + u_1(t)B_n^{'}(0), \tag{3.48}$$

where

$$u_0(t) = 1 - (1+\lambda_n^2)E_{\alpha,\alpha+1}(-(\lambda_n^2+1)t^\alpha),$$
$$u_1(t) = t - (1+\lambda_n^2)E_{\alpha,\alpha+2}(-(\lambda_n^2+1)t^\alpha),$$

so

$$u(x,t) = W_3(x,t) + V_3(x,t)$$
$$= \sum_{n=1}^{\infty} B_n(t)X_n + \frac{\mu_1(t)-\mu_2(t)}{\alpha_1-\beta_1-L}x - \frac{(L+\beta_1)\mu_1(t)-\alpha_1\mu_2(t)}{\alpha_1-\beta_1-L}. \tag{3.49}$$

4. Examples

In this section, we consider three examples with different initial and boundary conditions and source terms. We show that the solution obtained above agree with those established in these examples.

Example 1.

Consider the following inhomogeneous fractional Klein-Gordon equation

$$D_t^\alpha u(x,t) - \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) = f(x,t), \tag{4.1}$$

with the initial condition and Dirichlet boundary conditions as follows:

$$\begin{aligned} &u(x,0) = 0, \quad u_t(x,0) = 0, \quad 0 \le x \le 1, \\ &u(0,t) = t^3, \quad u(1,t) = t^3 + t^2, \quad t \ge 0, \end{aligned} \tag{4.2}$$

in which

$$f(x,t) = \sin(3\pi x)\frac{\Gamma(3)}{\Gamma(3-\alpha)}t^{2-\alpha} + x\frac{\Gamma(3)}{\Gamma(3-\alpha)}t^{2-\alpha} +$$
$$+\frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha} + (9\pi^2+1)t^2\sin(3\pi x) + t^2x + t^3.$$

For solving the problem, we first transform it into a homogeneous boundary condition. Therefore suppose that

$$u(x,t) = W_1(x,t) + V_1(x,t) = W_1(x,t) + t^2x + t^3,$$

where $W_1(x,t)$ is an unknown function and satisfies in problem with homogeneous boundary conditions as:

$$\begin{cases} D_t^\alpha W_1(x,t) - \dfrac{\partial^2 W_1(x,t)}{\partial x^2} + W_1(x,t) = \tilde{f}(x,t), \\ W_1(x,0) = 0, \quad (W_1)_t(x,0) = 0, \quad 0 \le x \le 1, \\ W_1(0,t) = 0, \quad W_1(1,t) = 0, \end{cases} \tag{4.3}$$

in which

$$\tilde{f}(x,t) = \sin(3\pi x)\big[\frac{\Gamma(3)}{\Gamma(3-\alpha)}t^{2-\alpha} + (1+9\pi^2)t^2\big].$$

Now, we are going to solve the corresponding homogeneous equation in (4.3) by the method of separating variables. With similar manner in subsection 3.1, we obtain a Sturm-Liouville problem and a linear ODE respect to x and t respectively. The eigenvalues and eigenfunctions of Sturm-Liouville are as

$$\lambda_n = n^2\pi^2, \quad X_n(x) = \sin(n\pi x), \quad n = 1, 2, \dots. \tag{4.4}$$

Then, we want to find a solution of the inhomogeneous problem in (4.3) which takes the form

$$W_1(x,t) = \sum_{n=1}^{\infty} B_n(t)\sin(n\pi x). \tag{4.5}$$

If we substitute this in (4.3) and use arguments in subsection 3.1 we have

$$D_t^{\alpha} B_n(t) + \left(1 + \frac{n^2\pi^2}{L^2}\right) B_n(t) = \tilde{f}_n(t), \tag{4.6}$$

in which

$$\tilde{f}_n(t) = \frac{2}{1}\int_0^1 \tilde{f}(x,t)\sin(n\pi x)dx = \begin{cases} H(t), & n = 3 \\ 0, & n \neq 3, \end{cases}$$

with

$$H(t) = \frac{\Gamma(3)}{\Gamma(3-\alpha)}t^{2-\alpha} + 9\pi^2 t^2 + t^2.$$

Since $W_1(x,t)$ satisfies the initial conditions in (4.3), we obtain

$$\sum_{n=1}^{\infty} B_n(0)\sin(n\pi x) = \sin(3\pi x), \tag{4.7}$$

which yields

$$B_n(0) = \frac{2}{1}\int_0^1 \sin(3\pi x)\sin(n\pi x)dx = \begin{cases} 1, & n = 3 \\ 0, & n \neq 3. \end{cases} \tag{4.8}$$

Hence lemma 2.1 implies that this problem has the solution as

$$B_n(t) = \begin{cases} \displaystyle\int_0^t \tau^{\alpha-1} E_{(\alpha,\alpha)}(-((n\pi)^2+1))\tau^{\alpha} H(t-\tau)d\tau, & n = 3 \\ 0, & n \neq 3. \end{cases} \tag{4.9}$$

Now, if we take the Laplace transform from both side of (4.9) we get

$$L[B_n(t)] = 0, \quad n \neq 3, \tag{4.10}$$

and

$$
\begin{aligned}
L[B_3(t)] &= L[\int_0^t \tau^{\alpha-1} E_{(\alpha,\alpha)}(-((3\pi)^2+1)\tau^\alpha) H(t-\tau) d\tau] \\
&= L[E_{(\alpha,\alpha)}(-((3\pi)^2+1)\tau^\alpha)] L[H(t)] \\
&= L\left[t^{\alpha-1} \sum_{k=0}^{\infty} \frac{(-1)^k[(3\pi)^2+1]^k t^{\alpha k}}{\Gamma(\alpha+k\alpha)} \right] L[H(t)] \\
&= L\left(\sum_{k=0}^{\infty} \frac{(-1)^k[(3\pi)^2+1]^k t^{\alpha k+\alpha-1}}{\Gamma(\alpha+k\alpha)} \right) L[H(t)] \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k[(3\pi)^2+1]^k}{s^{\alpha k+\alpha}} L[H(t)] \\
&= \left(\frac{1}{s^\alpha+(3\pi)^2+1} \right) \left(\frac{2}{s^{3-\alpha}+\frac{9\pi^2}{s^3}+\frac{1}{s^3}} \right) \\
&= \frac{2}{s^3}.
\end{aligned}
\tag{4.11}
$$

From (4.10) and (4.11), we obtain

$$
B_n(t) = \begin{cases} t^2, & n = 3, \\ 0, & n \neq 3. \end{cases}
$$

Therefore, the solution of (4.3) is

$$
W_1(x,t) = t^2 \sin(3\pi x).
$$

Then the exact solution of the fractional Klein-Gordon equation given in example 1 is

$$
u(x,t) = t^2 \sin(3\pi x) + t^2 x + t^3.
$$

Example 2. Consider the inhomogeneous fractional Klein-Gordon equation 1.1, with the initial condition and Neumann boundary conditions as follows:

$$
\begin{aligned}
&u(x,0) = 0, \quad 0 \le x \le 1, \\
&u_x(0,t) = t^3, \quad u_x(1,t) = 2t^4 + t^3, \quad t \ge 0,
\end{aligned}
\tag{4.12}
$$

in which

$$
\begin{aligned}
f(x,t) = {} & \frac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha}(\cos(2\pi x)+x) + \frac{\Gamma(5)}{\Gamma(5-\alpha)} t^{4-\alpha} x^2 + \\
& +4\pi^2 t^3 \cos(2\pi x) - 2t^4 + t^3 \cos(2\pi x) + x^2 t^4 + x t^3.
\end{aligned}
$$

As before, for solving the problem, we first transform it into a homogeneous boundary condition. Therefore let

$$u(x,t) = W_2(x,t) + V_2(x,t) = W_2(x,t) + t^4x^2 + t^3x,$$

where $W_2(x,t)$ is an unknown function and satisfies in problem with homogeneous boundary conditions as follows:

$$\begin{cases} D_t^\alpha W_2(x,t) - \dfrac{\partial^2 W_2(x,t)}{\partial x^2} + W_2(x,t) & = \tilde{f}(x,t), \\ W_2(x,0) = 0, \quad 0 \leq x \leq 1, \\ (W_2)_x(0,t) = 0, \quad (W_2)_x(1,t) = 0, \end{cases} \tag{4.13}$$

in which

$$\tilde{f}(x,t) = \cos(2\pi x)\left(\frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha} + (1+4\pi^2)t^3\right).$$

As before the eigenvalues and eigenfunctions of Sturm-Liouville are as follows:

$$\lambda_n = n^2\pi^2, \quad X_n(x) = \cos(n\pi x), \quad n = 1,2,\ldots. \tag{4.14}$$

Then, we obtain a solution of the inhomogeneous problem in (4.13) which takes the form

$$W_2(x,t) = \sum_{n=1}^{\infty} B_n(t)\cos(n\pi x). \tag{4.15}$$

If we substitute this in (4.13) and use arguments in subsection 3.2 we have

$$D_t^\alpha B_n(t) + \left(1 + n^2\pi^2\right) B_n(t) = \tilde{f}_n(t), \tag{4.16}$$

in which

$$f_n(t) = \frac{2}{1}\int_0^1 \tilde{f}(x,t)\cos(n\pi x)dx = \begin{cases} H(t), & n = 2 \\ 0, & n \neq 2, \end{cases}$$

with

$$H(t) = \frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha} + (1+4\pi^2)t^3.$$

Since $W_2(x,t)$ satisfies the initial conditions in (4.13), we obtain

$$\sum_{n=1}^{\infty} B_n(0)\cos(n\pi x) = \cos(2\pi x), \tag{4.17}$$

which gives

$$B_n(0) = \frac{2}{1}\int_0^1 \cos(2\pi x)\cos(n\pi x)dx = \begin{cases} 1, & n = 2 \\ 0, & n \neq 2. \end{cases} \tag{4.18}$$

Hence lemma 2.1 implies that this problem has the solution as

$$B_n(t) = \int_0^t \tau^{\alpha-1}E_{(\alpha,\alpha)}(-((n\pi)^2+1))\tau^\alpha \tilde{f}_n(t-\tau)d\tau. \tag{4.19}$$

By taking the Laplace transform from both side of (4.19), we get

$$L[B_n(t)] = 0, \quad n \neq 2, \tag{4.20}$$

and

$$\begin{aligned}
&L[B_2(t)] \\
&= L[\int_0^t \tau^{\alpha-1} E_{(\alpha,\alpha)}(-((3\pi)^2+1)\tau^{\alpha})H(t-\tau)d\tau] \\
&= L[E_{(\alpha,\alpha)}(-((2\pi)^2+1)\tau^{\alpha})]L[H(t)] \\
&= L[t^{\alpha-1}\sum_{k=0}^{\infty}\frac{(-1)^k[(2\pi)^2+1]^k t^{\alpha k}}{\Gamma(\alpha+k\alpha)}L[H(t)] \\
&= L\left(\sum_{k=0}^{\infty}\frac{(-1)^k[(2\pi)^2+1]^k t^{\alpha k+\alpha-1}}{\Gamma(\alpha+k\alpha)}\right)L[H(t)] \qquad (4.21)\\
&= \sum_{k=0}^{\infty}\frac{(-1)^k[(2\pi)^2+1]^k}{s^{\alpha k+\alpha}}L[H(t)] \\
&= \left(\frac{1}{s^{\alpha}+(2\pi)^2+1}\right)\left(\frac{6}{s^{4-\alpha}+\frac{24\pi^2}{s^4}+\frac{6}{s^6}}\right) \\
&= \frac{1}{s^4}.
\end{aligned}$$

From (4.20) and (4.21), we obtain

$$B_2(t) = \begin{cases} t^3, & n = 2, \\ 0, & n \neq 2. \end{cases} \tag{4.22}$$

Therefore, the solution of (4.13) is

$$W_2(x,t) = t^3\cos(2\pi x).$$

Then the exact solution of the fractional Klein-Gordon equation given in example 2 is

$$u(x,t) = t^3\cos(2\pi x) + t^4x^2 + t^3x.$$

Example 3. Consider the fractional Klein-Gordon problem

$$\left\{\begin{array}{l} D_t^{\alpha}u(x,t) + u_{xx}(x,t) + u(x,t) = f(x,t), \\ u(x,0) = 0, \quad u_t(x,0) = 0, \\ u(0,t) - \frac{1}{2\pi}u_x(0,t) = t^4 + t^3(1 - \frac{1}{2\pi}), \\ u(1,t) - \frac{1}{2\pi}u_x(1,t) = t^4 + t^3(2 - \frac{1}{2\pi}), \end{array}\right. \tag{4.23}$$

which

$$f(x,t) \quad = t^4 + \frac{\Gamma(5)}{\Gamma(5-\alpha)}t^{4-\alpha} + 4\pi^2 t^3(\sin(2n\pi) + \cos(2n\pi)) \tag{4.24}$$

$$+(\frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha} + t^3)(x + 1 + \sin 2n\pi + \cos 2n\pi), \tag{4.25}$$

so

$$V_3(x,t) = t^3(x+1) + t^4 \tag{4.26}$$

and from (3.37), we get

$$\tilde{f}(x,t) = (\sin 2\pi x + \cos 2\pi x)(\frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha} + t^3(4\pi^2 + 1)). \tag{4.27}$$

In this example, we have

$$\lambda_n = 2n\pi, \tag{4.28}$$

because of orthogonality of $X_n(x) = \sin 2\pi x + \cos 2\pi x$, $\tilde{f}_n(t)$, will be as follows:

$$\tilde{f}_n(t) = \left\{\begin{array}{ll} \frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha} + t^3(4\pi^2+1), & n = 2, \\ 0, & n \neq 2. \end{array}\right. \tag{4.29}$$

Now from Lemma 2.1, $B_2(t)$ is as

$$B_2(t) = \int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-(4\pi^2+1)\tau^{\alpha})\frac{\Gamma(4)}{\Gamma(4-\alpha)}(t-\tau)^{3-\alpha} \tag{4.30}$$

$$+ (t-\tau)^3(4\pi^2+1)d\tau. \tag{4.31}$$

By applying the Laplace transform for both side of the last equation just like what we did in example 1 for (4.11) we can show that

$$L(B_n(t)) = \left\{\begin{array}{ll} \frac{6}{s^4}, & n = 2, \\ 0, & n \neq 2. \end{array}\right. \tag{4.32}$$

Hence

$$B_2(t) = t^3 \tag{4.33}$$

and as a result, the solution of (4.23) is as follows:

$$u(x,t) = t^3(x + 1 + \sin 2\pi x + \cos 2\pi x). \tag{4.34}$$

5. Conclusion

In this paper, we obtained the analytical solution of inhomogeneous fractional Klein-Gordon equation with three types of boundary conditions by the method of separating variables. The fractional derivatives were considered in Caputo sense. We illustrated that the effectiveness of the method by solving three examples with different boundary condition.

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