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Classifications of different dimensional partial differential equations and their invariant solutions via symmetry reductions and optimal systems

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Abstract

The study analyzes the (1+2)-dimensional modified Breaking Soliton equation using classical symmetries and explores the (1+1)-dimensional heat equation and modified Boussinesq equation through Lie symmetry analysis and Lie subalgebras. Classical symmetries are derived from the solutions of nonlinear partial differential equations, and optimal systems are constructed using commutator relationships and adjoint representations. The research presents new invariant solutions and their graphical analyses, which are valuable for applied sciences and numerical simulations. Solutions explain phenomena such as circular membrane vibrations, heat conduction, and electromagnetic waves. Wave, contour, and patch contour solutions are used in sound, light, weather forecasting, medical imaging, and material science. This paper provides a comprehensive analysis of the generalized single and double reduction methods, highlighting the significance of inherited symmetries at each stage of the reduction process.

Keywords. Classifications, Symmetry reductions, Optimal systems, Invariant solutions.
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1. Introduction

Partial differential equations (PDEs) are crucial in applied science, modeling nonlinear phenomena. Most applied PDE models are nonlinear due to their complexity and broad scientific applications in physics, biology, and engineering [25]. Exact solutions of nonlinear PDEs are crucial for understanding physical phenomena. Various numerical and analytical methods exist, including the Hirota bilinear technique [34], invariant transformations [47], the homogeneous method [15], and Lie symmetry analysis [13, 24]. Effective techniques for explicit solutions include Bäcklund transformation and Hirota bilinear methods. Kakuli *et al.* [23] investigate the conservation laws and multi-reduction symmetries of the (2+1) dimensional Zakharov Kuznetsov (ZK) equation alongside a nonlinear wave equation. Using the double reduction theory, Naz *et al.* [36] derived several exact solutions for a class of nonlinear regularized long wave equations. Bokhari *et al.* [5], Sait *et al.* [43], and Muatjetjeja and Porogo [35] have also contributed to the applications of the generalized double reduction theory. The study by Azam *et al.* [2] examined nonlinear phenomena through different methodological approaches.

The heat equation, formulated by Joseph Fourier in 1822, models systems with multiple variables. Mathematicians have explored its kernels and implicit solutions, applying them to the Atiyah-Singer index theorem [4]. Joseph Valentin Boussinesq derived an equation for long wave propagation with small amplitude on water surfaces in the late 19^{th} century. Various generalizations exist, including the reformed, modified, and dispersive water wave equations. This study focuses on the modified Boussinesq (MB) equation. He [18, 19] applied different iteration methods for their solutions. The application of symmetry analysis significantly enhances the study of both ordinary differential equations (ODEs) and partial differential equations (PDEs). A highly general and effective method, this theory aids in finding analytical solutions for extensive classes of differential equations. A detailed examination of this method is available

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in [6, 21, 38, 41]. Lie analyzed the geometry of differential equations and developed their group properties. He also classified the solutions of the heat equation [33]. Mathematicians expanded on Lie's work in various directions, including group classification (Ibragimov [22], Ndogmo [37], Xu *et al.* [45], Ovsyannikov [42]), the non-classical method (Bluman and Cole [7]), nonlocal symmetries [32], approximate symmetries [3], and differential constraints [39]. Olver [38] and Bluman and Cole[8] studied similarity approaches and Lie group applications for nonlinear equations. Hu *et al.* [20] explored Lie point symmetries, the optimal system, and exact solutions of the Sakovich equation. This article analyzes the modified Breaking Soliton (MBS), modified Boussinesq (MB), and heat Equations (3.1), (3.44), and (3.71), developing optimal systems using subalgebra principles and constructing various invariant solutions.

A Lie point symmetry of a system is a local transformation group that maps each solution to another solution of the same system. The Lie symmetry method is used to analyze cancer networks modeled mathematically, addressing cancer cell expansion due to R checkpoint gene dysfunction. Symmetry generators create new similarity transformations for model reduction. Nadeem *et al.* [1] studied the thermodynamic behavior of radiative, chemically reactive flow in an induced MHD Sutterby nanofluid over a nonlinear stretching cylinder. We apply Lie point analysis to find analytic solutions for MHD flow and heat transfer, presenting results through tables and figures for clarity. Through the application of LPS, it is possible to derive solutions that maintain invariance under the group's transformations. Such solutions, known as invariant solutions, frequently uncover essential characteristics of the differential equation. Noether's theorem states that every continuous symmetry of a physical phenomenon corresponds to a conservation law. These phenomena can be mathematically modeled using PDEs [16, 17]. Various models and situations have benefited from the broad application of Lie symmetries. This theory enables the derivation of DE solutions through algorithms that are relatively standard and do not rely on estimates or guesses. Oliveri [40] and Sedov [44] applied dimensional similarity analysis to develop solutions for certain problems in applied physics. The application of LPS extends to ODEs, PDEs, nonlinear PDEs, and systems of ODEs and PDEs for obtaining exact solutions. The extensive use of (LPS) does not diminish its potential for further applications in contemporary and evolving physical problems.

Numerous studies [24, 26–31] and books [8, 38] address these topics. LPS analysis is widely used for PDE reduction, generating new solutions, and classifying DEs. The MB equation is analyzed via Lie symmetry, constructing LPS and reducing the model to ODEs using invariant transformations. This approach reduces the number of independent variables from the original PDEs. The equations are transformed into a nonlinear PDE with a single reduced variable. By further reducing one variable, the PDE becomes a nonlinear ODE that can be solved analytically to produce different solutions for the model. Druzhkov and Cheviakov [14] derive invariant reductions for partial differential equations. Chaudhry and Naz [11] explore closed-form solutions for a technology diffusion model through the application of Lie symmetries.

2. Preliminaries and Lie Symmetry Method

We overview of the notation and relevant findings in this section that were used in this article [9, 38]. Consider the PDEs of l^{th}-order system of m independent variables $z = (z^1, z^2, ..., z^m)$ and n dependent variables $v = (v^1, v^2, ..., v^n)$:

$$F_\alpha(z, v, v_1, ..., v_l) = 0, \qquad \alpha = 1, 2, ..., n, \tag{2.1}$$

where $v_1, ..., v_l$ indicate the all collections of 1^{st}, 2^{nd},..., l^{th}-order partial derivatives; that is, v^α_i=$D_i(v^\alpha)$, v^α_{ij}= D_iD_j,..., correlatively, using the operative of the total derivative with respect to z^j given by

$$D_i = \frac{\partial}{\partial z^j} + v^\alpha_j \frac{\partial}{\partial v^\alpha} + v^\alpha_{ji} \frac{\partial}{\partial v^\alpha_j} +, ... \qquad j = 1, 2, ..., m, \tag{2.2}$$

and operator of Lie-Bäcklund is

$$Z = \xi^i \frac{\partial}{\partial z^j} + \eta^\alpha \frac{\partial}{\partial v^\alpha}, \quad \xi^i, \eta^\alpha \in \mathfrak{A}, \tag{2.3}$$

where $\mathfrak{A}$ is the functions of space differential. The operator (2.3) is a shortened version of the infinite formal sum:

$$Z = \xi^i \frac{\partial}{\partial z^j} + \eta^\alpha \frac{\partial}{\partial v^\alpha} + \sum_{r\geq 1} \zeta^\alpha_{j_1,j_2,...,j_r} \frac{\partial}{\partial v^\alpha_{j_1,j_2,...,j_r}}, \tag{2.4}$$

where, by the prolongation formulae the additional coefficients are determined uniquely:

$$\begin{aligned} &\zeta_j^{\alpha} = D_j(Q^{\alpha}) + \xi^i v_{ji}^{\alpha}, \\ &\zeta_{j_1,\ldots,j_r} = D_{j1}, \ldots, D_{j_r}(Q^{\alpha}) + \xi^i v_{ij_1,\ldots,i_r}^{\alpha}, \qquad r \geq 1, \end{aligned} \tag{2.5}$$

in which (Q^{α}) is a function of Lie characteristic:

$$Q^{\alpha} = \eta^{\alpha} - \xi^i v_i^{\alpha}. \tag{2.6}$$

Now, we take the consecutive transformations of Lie group with independent variables t, x, y and dependent variable u:

$$\begin{aligned} u^* &= (x^*, x, y, t), \qquad t^* = (x^*, x, y, t), \\ x^* &= (x^*, x, y, t), \qquad y^* = (x^*, x, y, t). \end{aligned} \tag{2.7}$$

One parameter Lie transformation of group admits in the following form (a):

$$\begin{aligned} &\widetilde{x^*} \approx x^* + aT(x^*, x, y, t), \\ &\widetilde{x} \approx x + aT(x^*, x, y, t), \\ &\widetilde{y} \approx y + aT(x^*, x, y, t), \\ &\widetilde{t} \approx t + aT(x^*, x, y, t). \end{aligned} \tag{2.8}$$

For the Lie group transformations the infinitesimal generators can be conveyed in the consecutive form:

$$Z = \xi \frac{\partial}{\partial x} + \phi \frac{\partial}{\partial y} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}. \tag{2.9}$$

By solving the Lie equations [48], we are obtained the group transformations of $\widetilde{x^*}, \widetilde{x}, \widetilde{y}, \widetilde{t}$.

$$\begin{aligned} &\frac{d\widetilde{x^*}}{da} = T(x^*, x, y, t), \\ &\frac{d\widetilde{x}}{da} = Xx^*, x, y, t), \\ &\frac{d\widetilde{y}}{da} = Y(x^*, x, y, t), \\ &\frac{d\widetilde{z}}{da} = Z(x^*, x, y, t). \end{aligned} \tag{2.10}$$

Using conditions:

$$\widetilde{x^*}|_{a=0} = t, \widetilde{x}|_{a=0} = x, \widetilde{y}|_{a=0} = y, \widetilde{z}|_{a=0} = z.$$

Above Lie operators can be defined is as follows in the first extension:

$$H^{[1]} = H + X^{[t]} \frac{\partial}{\partial \widetilde{x^*}} + Y^{[t]} \frac{\partial}{\partial \widetilde{y}} + + Z^{[t]} \frac{\partial}{\partial \widetilde{z}}, \tag{2.11}$$

where,

$$\begin{aligned} X^{[T]} &= D_t(X) - \widetilde{x} D_t(T), \\ Y^{[T]} &= D_t(Y) - \widetilde{x} D_t(T), \\ Z^{[T]} &= D_t(Z) - \widetilde{x} D_t(T), \end{aligned} \tag{2.12}$$

with D_t equating the total differential operator, which can be explained in the following form:

$$D_t = \frac{\partial}{\partial t} + \widetilde{x} \frac{\partial}{\partial x} + \widetilde{y} \frac{\partial}{\partial y} + \widetilde{z} \frac{\partial}{\partial z} + \widetilde{\widetilde{x}} \frac{\partial}{\partial x} + + \widetilde{\widetilde{y}} \frac{\partial}{\partial y} + \widetilde{\widetilde{z}} \frac{\partial}{\partial z} + \ldots \tag{2.13}$$

The obtained infinitesimal transformation will be used to solve the equations:

$$\begin{aligned} &Ts_t + Xs_x + Ys_y + Zs_x = 0,\\ &Tr_t + Xr_x + Yr_y + Zr_x = 0,\\ &Tu_t + Xu_x + Yu_y + Zu_x = 0,\\ &Tv_t + Xv_x + Yv_y + Zv_x = 0. \end{aligned} \tag{2.14}$$

Above Equation (2.14) will yield a set of new dependent variables u, v and independent variable r, s which can be used to transform the non-linear system (3.1) into a linear system. In order to determine the Lie point symmetries of Equations (3.1), (3.44), and (3.71) the second- and fourth-order prolongation methods are utilized.

$$Pr^{[4]}X(u_{xt} - 4u_{xy}u_x - 2u_{xx}u_y - u_{xxxy} + f(u) = 0)|_{\Delta=0} = 0, \tag{2.15}$$

$$Pr^{[4]}X(u_{tt} - \delta u_{ttxx} - f_u u_{xx} - f_{uu}u_x^2 = 0)|_{\Delta=0} = 0, \tag{2.16}$$

$$Pr^{[2]}X(u_t - \alpha u_{xx} = 0)|_{\Delta=0} = 0. \tag{2.17}$$

3. Classifications, Lie Point Symmetries and Invariant Solutions of (1+2) Dimensional Modified Breaking Soliton Equation.

In this section, classical symmetries, and invariant solutions of (1+2) dimensional (MBS) equation are computed. The considered partial differential equation is

$$u_{xt} - 4u_{xy}u_x - 2u_{xx}u_y - u_{xxxy} + f(u) = 0. \tag{3.1}$$

Various packages in computer algebra systems exist implementing Lie symmetry calculations and related techniques. In this paper, we provide a package for differential equations symmetry analysis (SADE) in MAPLE, there are also some helpful packages for MAPLE.: PDE tools by Cheb-Terrab and Von Bulow [12], DESOLV by Vu *et al.* [46]. Applying the fourth prolongation Pr^4X to (3.1), following equations are determined of Equation (3.1):

$$\begin{aligned} &\xi^1_u = 0,\ \xi^1_x = 0,\ \xi^2_u = 0,\ \xi^2_y = 0,\ \xi^3_x = 0,\ \xi^3_u = 0,\ \xi^1_y = 0,\ \xi^1_{xy} = 0,\\ &\xi^1_{xu} = 0,\ \xi^2_{yu} = 0,\ \xi^3_{xu} = 0,\ \eta_{yu} = 0,\ \eta_{xu} = 0,\ \eta_u - \xi^1 t - \xi^2_x = 0,\\ &\eta_{tu} + 3\eta_{xxyu} = 0,\ 3\eta_{xyu} - \xi_{2t} = 0,\ \eta_{xxu} + \xi_{3t} = 0,\ \eta_u + \xi^1_t = 0,\\ &3\eta_{xxuu} = 0,\ -4\eta_{xu} = 0,\ 3\eta_{xyuu} - 4\eta_{yu} = 0,\ 3\eta f_u + \eta_{xxxy} + \eta_{tx} = 0,\\ &-4\eta_u + 4\xi^2_x + 2\xi^3_y + 3\eta_{xuu} = 0,\ -4\eta_x - \xi^2_t - \xi^2_{xxx} + 3\eta_{xxu} = 0. \end{aligned} \tag{3.2}$$

After solving the above determining equations, the remaining equation is

$$2f_{uu}^2 - f_u f_{uu} = 0. \tag{3.3}$$

Solving the Equation (3.3), lead to the following form of $f(u)$

$$f(u) = au + b, \tag{3.4}$$

$$f(u) = c_1 + c_2 e^{\frac{1}{2}u}. \tag{3.5}$$

Using value of (3.4), then the Equation (3.1) becomes

$$u_{xt} - 4u_{xy}u_x - 2u_{xx}u_y + u_{xxxy} + au(t,x,y) + b = 0. \tag{3.6}$$

Solving the Equation (3.6), we find that the symmetry of (3.6) is extended by the four vector fields:

$$Z_1 = \frac{\partial}{\partial t}, \quad Z_2 = \frac{\partial}{\partial x}, \quad Z_3 = \frac{\partial}{\partial y}, \tag{3.7}$$

$$Z_4 = (au+b)\frac{\partial}{\partial u} + at\frac{\partial}{\partial t} - ax\frac{\partial}{\partial x} + 3ay\frac{\partial}{\partial y}. \tag{3.8}$$

The varities of new different exact solutions of (3.1) for $f(u) = au + b$ are constructed by using the classical Lie point symmetries Z_1, Z_2 and Z_3 with the following similarity invariant transformations

$$s = x - \alpha t, u(t, x, y) = v(r, s), r = y - \beta t, \tag{3.9}$$

that yields the following simplification of Equation (3.6)

$$-\alpha v_{ss} - 4v_{rs}v_s - 2v_{ss}v_r + v_{sssr} + av(r, s) + b = 0. \tag{3.10}$$

Similarly for other reduction using the Lie point symmetries of Equation (3.10), the following similarity invariant transformations are

$$v(r, s) = p(m), m = y - \gamma r + s, n = r, \tag{3.11}$$

that yields the following simplification of Equation (3.11)

$$-\alpha p_{mm} + 6\gamma p_{mm}pm - \gamma p_{mmmm} + ap(m) + b. \tag{3.12}$$

In the next step the different invariant solutions are found of Equation (3.12).

Solution of (3.12) using $\gamma = 0, a = 0, c = 0$

$$-\alpha p_{mm} + b = 0, \tag{3.13}$$

This implies

$$p(m) = \frac{bm^2}{2\alpha} + c_1 m + c_2. \tag{3.14}$$

Now, transformation in original variables

$$u(t, x, y) = \frac{b(x + y - \alpha t)}{2\alpha} + c_1(x + y - \alpha t+) + c_2. \tag{3.15}$$

Equation (3.15) represent the exact solution of Equation (3.6) .

Solution of (3.12) using $\gamma = 0, c = 0$

$$-\alpha p_{mm} + a + b = 0. \tag{3.16}$$

This implies

$$p(m) = \frac{(a + b)m^2}{2\alpha} + c_1 m + c_2. \tag{3.17}$$

Now, transformation in original variables

$$u(t, x, y) = \frac{(a + b)(x + y - \alpha t)}{2\alpha} + c_1(x + y - \alpha t) + c_2. \tag{3.18}$$

Equation (3.18) represent the exact solution of Equation (3.6) .

Solution of (3.12) using $\gamma = 0, a = 0$

$$-\alpha p_{mm} + a + b = 0, \tag{3.19}$$

This implies

$$p\left(m\right) = \ln\left(-1/2c_1\left(-1 + \left(\tanh\left(1/2\frac{\sqrt{c\alpha c_1}\left(m + c_2\right)}{\alpha}\right)\right)^2\right)b^{-1}\right)c^{-1}. \tag{3.20}$$

Now, transformation in original variables

$$u\left(t, x, y\right) = \ln\left(-1/2c_1\left(-1 + \left(\tanh\left(1/2\frac{\sqrt{c\alpha c_1}\left(y - \alpha t + x + c_2\right)}{\alpha}\right)\right)^2\right)b^{-1}\right)c^{-1}. \tag{3.21}$$

Equation (3.21) represent the exact solution of Equation (3.6). The varities of new different exact solutions of (3.1) for $f(u) = au + b$ are constructed by using the classical LPS Z_4 with the following similarity invariant transformations are

$$\frac{au(t,x,y)+b}{at} = v(r,s), r = \frac{y}{t^3}, s = tx, \tag{3.22}$$

that yields the following simplification of Equation (3.12)

$$sv_{ss} - 4v_{sr}v_s - 2v_{ss}v_r + v_{sssr} + av(r,s) = 0. \tag{3.23}$$

Similarly for other reduction using the Lie point symmetries of Equation (3.23), the following similarity invariant transformations are

$$v(r,s) = p(m), m = s, n = r, \tag{3.24}$$

that yields the following simplification of Equation (3.24)

$$mp_{mm} + ap(m) = 0. \tag{3.25}$$

In this next step the different invariant solutions are found of Equation (3.25).

Solution of (3.25)

$$mp_{mm} + ap(m) = 0. \tag{3.26}$$

This implies

$$p(m) = c_1\sqrt{m}BesselJ(1, 2\sqrt{am}) + c_2\sqrt{m}BesselY(1, 2\sqrt{am}). \tag{3.27}$$

Now, transformation in original variables

$$u(t,x,y) = c_1\sqrt{tx}BesselJ(1, 2\sqrt{atx}) + c_2\sqrt{tx}BesselY(1, 2\sqrt{atx}). \tag{3.28}$$

Equation (3.28) represent the exact solution of Equation (3.6) .

On the same procedure using value of (3.5), then the Equation (3.1) becomes

$$u_{xt} - 4u_{xy}u_x - 2u_{xx}u_y + u_{xxxy} + c_1 + c_2e^{\frac{1}{2}u} = 0. \tag{3.29}$$

Solving the Equation (3.29), we find that the symmetry of (3.29) is extended by the three vector fields:

$$Z_1 = \frac{\partial}{\partial t}, \quad Z_2 = \frac{\partial}{\partial x}, \quad Z_3 = \frac{\partial}{\partial y}. \tag{3.30}$$

The varities of new different exact solutions of (3.29) for $f(u) = c_1 + c_2e^{\frac{1}{2}u}$ are constructed by using the classical Lie point symmetry Z_1, Z_2 and Z_3 with the following similarity invariant transformations

$$w = t, r = y - \beta t, u(t,x,y) = v(r,s), s = x - \alpha t, \tag{3.31}$$

that yields the following simplification of Equation (3.29)

$$c_1 + c_2e^{\frac{1}{2}v(r,s)} - \alpha v_{ss} - 2v_{ss}v_r - \beta v_{r,s} - 4v_sv_{rr} + v_{rsss} = 0. \tag{3.32}$$

Similarly for other reduction using the Lie point symmetries of Equation (3.32), the following similarity invariant transformations are

$$v(r,s) = p(m), m = s, n = r, \tag{3.33}$$

that yields the following simplification of Equation (3.33)

$$c_1 + c_2e^{p(m)} - \alpha p_{mm} = 0. \tag{3.34}$$

In this next step the different invariant solutions are found of Equation (3.34).

Solution of (3.34) using $\alpha = 1, c_1 = 0$

$$c_2e^{p(m)} - p_{mm} = 0. \tag{3.35}$$

This implies

$$p\left(m\right)=\ln\left(1/2\left(\left(\tan\left(1/2\frac{m+C_{2}}{C_{1}}\right)\right)^{2}+1\right){c_{2}}^{-1}{C_{1}}^{-2}\right). \tag{3.36}$$

Now, transformation in original variables

$$u\left(t,x,y\right)=\ln\left(1/2\left(\left(\tan\left(1/2\frac{(-t+x)+C_{2}}{C_{1}}\right)\right)^{2}+1\right){c_{2}}^{-1}{C_{1}}^{-2}\right). \tag{3.37}$$

Equation (3.37) represent the exact solution of Equation (3.29).

Solution of (3.34) using $c_1=0$

$$c_{2}e^{p(m)}-\alpha p_{mm}=0. \tag{3.38}$$

This implies

$$p\left(m\right)=\ln\left(1/2C_{1}\left(\left(\tan\left(1/2\frac{\sqrt{\alpha C_{1}}\left(m+C_{2}\right)}{\alpha}\right)\right)^{2}+1\right){c_{2}}^{-1}\right). \tag{3.39}$$

Now, transformation in original variables

$$u\left(t,x,y\right)=\ln\left(1/2C_{1}\left(\left(\tan\left(1/2\frac{\sqrt{\alpha C_{1}}\left((-\alpha t+x)+C_{2}\right)}{\alpha}\right)\right)^{2}+1\right){c_{2}}^{-1}\right). \tag{3.40}$$

Equation (3.40) represent the exact solution of Equation (3.29).

Solution of (3.34) using $c_1=0$

$$c_{2}e^{p(m)}-\alpha p_{mm}=0. \tag{3.41}$$

This implies

$$p\left(m\right)=C_{2}+1/2c_{1}m^{2}+C_{1}m. \tag{3.42}$$

Now, transformation in original variables

$$u\left(t,x,y\right)=1/2c_{1}(-\alpha t+x)^{2}+C_{1}(-\alpha t+x)+C_{2}. \tag{3.43}$$

Equation (3.43) represent the exact solution of Equation (3.29) .

3.1. **Lie Point Symmetries, Optimal System and Invariant Solutions of Heat Equation.** In this section, invariant solutions are evaluated of the heat equation by applying the classical symmetries through optimal Lie subalgebras. The considered partial differential equation is

$$u_t-\alpha u_{xx}=0. \tag{3.44}$$

The Lie point symmetry operator is defined as

$$Z=\zeta^{1}(t,\ x,\ u)\frac{\partial}{\partial t}+\zeta^{2}(t,\ x,\ u)\frac{\partial}{\partial x}+\eta(t,\ x,\ u)\frac{\partial}{\partial u}. \tag{3.45}$$

The second invariant formulation is used for generating the system of determining equations by using the invariant surface conditions

$$Z^{[2]}[u_t-\alpha u_{xx}]\mid_{(3.44)}=0, \tag{3.46}$$

where $Z^{[2]}$ is the second prolongation of Z and $|_{(3.44)}$ means evaluated on (3.46). In equation (3.46) Z_2 in determinant form

$$Z^{[k]}=Z+\eta_{i}(t,x,u,u_{(1)})\frac{\partial}{\partial u_{i}}+...+\eta_{i_{1}...i_{k}}(t,x,u,u_{(1)},...,u_{(k)})\frac{\partial}{\partial u_{i_{1}...i_{k}}}, \tag{3.47}$$

in which η's are given by

$$\begin{aligned} \eta_i^\alpha &= D_i(\phi^\alpha) - u_j^\alpha D_i(\xi^j), \\ \eta_{ij}^\alpha &= D_j(\eta_i^\alpha) - u_{il}^\alpha D_j(\xi^l), \\ &\vdots \\ \eta_{i_1\ldots i_k}^\alpha &= D_{i_k}(\eta_{i_1\ldots i_{k-1}}^\alpha) - u_{li_1\ldots i_{k-1}}^\alpha D_{i_k}(\xi^l), \end{aligned} \tag{3.48}$$

where D_{i_k} is represented the total derivative . The system of determining equations is solved the construct the following Lie point symmetries.

$$Z_1 = \frac{\partial}{\partial t}, \quad Z_2 = \frac{\partial}{\partial x}, \quad Z_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}. \tag{3.49}$$

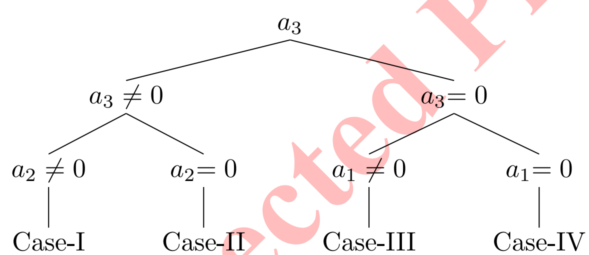
The following nonzero commutation relations are used to construct the optimal system

$$[Z_1, Z_3] = 2Z_1, \quad [Z_2, Z_3] = Z_2, \quad [Z_3, Z_1] = -2Z_1, \quad [Z_3, Z_2] = -Z_2. \tag{3.50}$$

Thus, on a general element $Z \in L_3$ implementing the adjoint operator. We find an optimal system of one-dimensional subalgebras by using the adjoint table, for Lie point symmetries which is as follows.

$$Z = a_1Z_1 + a_2Z_2 + a_3Z_3. \tag{3.51}$$

There is the following tree diagram and there non-similar symmetry generators



,

$$\begin{aligned} Z^1 &= Z_3 \pm Z_2, \\ Z^2 &= Z_3, \\ Z^3 &= a_1Z_1 \pm Z_2, \\ Z^4 &= \pm Z_2. \end{aligned}$$

There are following four possible cases

Case-I: $a_3 \neq 0, a_2 \neq 0$: In this case the adjoint operation for $\epsilon = a$ on Z is

$$\acute{Z} = Ad(e^{aZ_2})Z = a_2Z_2 + Z_3, \tag{3.52}$$

$$Z^1 = Ad(e^{aZ_3})\acute{Z} = Z_3 \pm Z_2, \tag{3.53}$$

where a=ln$|\frac{\pm 1}{a_2}|$ when $a_2 > 0$ or $a_2 < 0$.

Case-II: $a_3 \neq 0, a_2 = 0$: The adjoint operation for $\epsilon = a$ on Z, for the second case

$$\acute{Z} = Ad(e^{aZ_2})Z = a_2Z_2 + Z_3, \tag{3.54}$$

$$Z^2 = Z_3. \tag{3.55}$$

Case-III: $a_3 = 0, a_1 \neq 0$: The adjoint operation for $\epsilon = a$ on Z is presented as

$$Z^3 = Ad(e^{aZ_2})Z = a_1Z_1 \pm Z_2. \tag{3.56}$$

Case-IV: $a_3 = 0, a_1 = 0$: The process for the solutions after applying the adjoint operation for $\epsilon = a$ on Z, yields

$$Z^4 = Ad(e^{aZ_2})Z = \pm Z_2. \tag{3.57}$$

The following optimal system is developed by using Equations (3.52)-(3.57),

$$Z_3 \pm Z_{Z_2}, \quad Z_3, \quad a_1 Z_1 + Z_2, \quad a_1 \neq 0, \quad \pm Z_2. \tag{3.58}$$

In this next step the different invariant solutions are found by using the constructed optimal systems of Equation (3.44).

Solution of (3.44) using $Z_3 \pm Z_2$

Similarity transformations and reduced form are

$$r = \frac{x \pm 1}{\sqrt{t}}, \quad s = \frac{1}{2} \ln t, \quad v(r) = u(t,x), \tag{3.59}$$

$$r v_r + 2\alpha v_{rr} = 0. \tag{3.60}$$

The special invariant solution of the above reduction is constructed as

$$v(r) = C_1 + erf(\frac{r}{2\sqrt{\alpha}})C_2. \tag{3.61}$$

The invariant solution of Equation (3.44)

$$u(t,x) = C_1 + erf(\frac{x \pm 1}{2\sqrt{\alpha t}})C_2. \tag{3.62}$$

Solution of (3.44) using $Z = a_1 Z_1 + Z_2$

The similarity transformations and reduced form are

$$r = \frac{-t + a_1 x}{a_1}, \quad s = \frac{t}{a_1}, \quad v(r) = u(t,x), \tag{3.63}$$

$$v_r + \alpha a_1 v_{rr} = 0. \tag{3.64}$$

This implies

$$v(r) = C_1 + C_2 e^{\frac{-r}{\alpha a_1}}. \tag{3.65}$$

The invariant solution of Equation (3.44)

$$u(t,x) = C_1 + C_2 e^{\frac{t - a_1 x}{\alpha a_1^2}}. \tag{3.66}$$

Solution of (3.44) using $Z = Z_3$

The similarity transformations and reduced form are

$$r = \frac{x}{\sqrt{t}}, \quad s = \frac{1}{2} \ln t, \quad v(r) = u(t,x), \tag{3.67}$$

$$r v_r + 2\alpha v_{rr} = 0. \tag{3.68}$$

The invariant solution of Equation (3.44) is

$$v(r) = C_1 + erf(\frac{r}{2\sqrt{\alpha}})C_2. \tag{3.69}$$

$$u(t,x) = C_1 + erf(\frac{x}{2\sqrt{\alpha t}})C_2. \tag{3.70}$$

Equation (3.70) represent the exact solution of Equation (3.44).

Solution of (3.44) using $Z = \pm Z_2$

The solutions for $Z = \pm Z_2$ is constant.

3.2. **Lie Point Symmetries, Optimal System and Invariant Solutions of Modified Boussinesq Equation.** In this section, classical symmetries, optimal system and invariant solutions of modified Boussinesq equation are computed. The considered partial differential equation is

$$u_{tt} - \delta u_{ttxx} - f_u u_{xx} - f_{uu} u_x^2 = 0. \tag{3.71}$$

Various packages in computer algebra systems exist implementing Lie symmetry calculations and related techniques. In this paper, we provide a package for differential equations symmetry analysis (SADE) in MAPLE, there are also some helpful packages for MAPLE.: PDE tools by Cheb-Terrab and Von Bulow[12], which is distributed since Release 11, DESOLV by Carminati and Vu [10] and Vu *et al.* [46]. From Equation (3.71) determining equations are

$$\begin{aligned}
&\xi^1_u = 0, \quad \xi^1_x = 0, \quad \xi^2_u = 0, \quad \xi^2_t = 0, \quad \xi^2_{xx} = 0, \quad \xi^1_{tt} - 2\phi_{tu} = 0,\\
&\phi_{uu} = 0, \quad \phi_{tt} = 0, \quad \phi_x = 0, \quad 2(\delta-1)\xi^1_t + (1-\delta)\phi_u + 2\delta\xi^2_x = 0,\\
&2(1-\delta)\xi^2_x f_u + (\delta-1)\phi_u f_u - \phi f_{uu} - 2\delta\xi^1_t f_u = 0,\\
&2(1-\delta)\xi^2_x f_{uu} + (\delta-2)\phi_u f_{uu} - \phi f_{uuu} - 2\delta\xi^1_t f_{uu} = 0.
\end{aligned} \tag{3.72}$$

After solving the above determining equations, the remaining equation is

$$f_u f_{uu} f_{uuuu} + f_{uu}^2 - 2 f_u f_{uuu}^2 = 0. \tag{3.73}$$

Solving the Equation (3.73), lead to the following form of $f(u)$

$$f(u) = ae^{bu} + c. \tag{3.74}$$

If $f(u) = ae^{bu} + c$, then Equation (3.71) becomes

$$u_{tt} - \delta u_{ttxx} - abe^{bu} u_{xx} - ab^2 e^{bu} u_x^2 = 0. \tag{3.75}$$

The Lie point symmetry operator is defined in Equation (3.45). The system of nonlinear partial differential equation is developed by using invariant surface condition defined in Equation (3.46). The solution of above system of determining equations take the following three Lie point symmetries.

$$Z_1 = \frac{\partial}{\partial t}, \quad Z_2 = \frac{\partial}{\partial x}, \quad Z_3 = \frac{\partial}{\partial u} - \beta t \frac{\partial}{\partial t}. \tag{3.76}$$

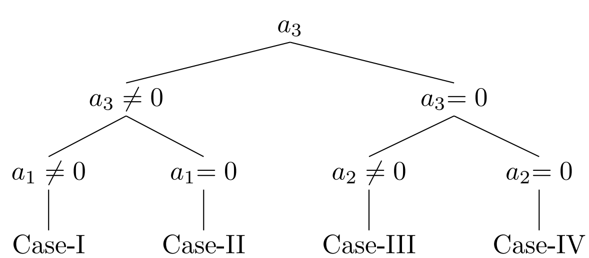
For the nonzero commutators the principle algebras are

$$[Z_1, Z_3] = -\beta Z_1, \quad [Z_3, Z_1] = \beta Z_1. \tag{3.77}$$

The general element $Z \in L_3$ is defined for the construction of optimal system with the help of nonzero commutation relations and their adjoint representations. Examine a general element $Z \in L_3$. We have

$$Z = a_1 Z_1 + a_2 Z_2 + a_3 Z_3. \tag{3.78}$$

We find an optimal system of one-dimensional subalgebras by using the adjoint table, for Lie point symmetries which is as follows.



Non-similar symmetry generators:

$$Z^1 = a_1 Z_1 + a_2 Z_2,$$
$$Z^2 = a_2 Z_2,$$
$$Z^3 = a_2 Z_2 + Z_3,$$
$$Z^4 = Z_3.$$

There are following four possible cases.

Case-I: $a_3 \neq 0, a_2 \neq 0$: In this case the adjoint operation for for $\epsilon = a$ on Z is

$$\acute{Z} = Ad(e^{aZ_2})Z = a_2 Z_2 + Z_3, \tag{3.79}$$
$$Z^1 = Ad(e^{aZ_3})\acute{Z} = Z_3 \pm Z_2, \tag{3.80}$$

where a=ln$|\frac{\pm 1}{a_2}|$ when $a_2 > 0$ or $a_2 < 0$.

Case-II: $a_3 \neq 0, a_2 \neq 0$:The adjoint operation for $\epsilon = a$ on Z, for the second case

$$\acute{Z} = Ad(e^{aZ_1})Z = a_2 Z_2 + a_3 Z_3, \tag{3.81}$$
$$Z^2 = a_2 Z_2 + Z_3. \tag{3.82}$$

Case-III: $a_3 = 0, a_1 \neq 0$: The adjoint operation for $\epsilon = a$ on Z, yields

$$Z^3 = Ad(e^{aZ_2})Z = a_1 Z_1 \pm Z_2. \tag{3.83}$$

Case-IV: $a_3 = 0, a_1 = 0$: In process of the adjoint operation for $\epsilon = a$ on Z, we get

$$Z^4 = Ad(e^{aZ_2})Z = \pm Z_2. \tag{3.84}$$

According to Equations (3.79)-(3.84) one dimensional subalgebra acknowledge the optimal system by Equation (3.75) are given in the following

$$a_1 Z_1 + a_2 Z_2, \quad a_2 Z_2, \quad a_2 Z_2 + Z_3, \quad Z_3. \tag{3.85}$$

In this next step the different invariant solutions are found by using the constructed optimal systems of Equation (3.75).

Solution of (3.75) using $Z = a_1 Z_1 + a_2 Z_2$

The similarity variables for operator $a_1 Z_1 + a_2 Z_2$ are

$$r = \frac{-a_2 t + x a_1}{a_1}, \quad s = \frac{t}{a_1}, \quad u(t,x) = v(r), \tag{3.86}$$

which grant the following simplification of Equation (3.75)

$$-v_{rr} + a_1^3 e^{a_2 v(r)} v_r^2 + a_1^3 e^{a_2 v(r)} v_{rr} + a_2 \delta v_{rrrr} = 0. \tag{3.87}$$

Equation (3.87) cannot provide the exact solution.

Solution of (3.75) using $Z = a_2\ Z_2$

The variables similarity for the operator $a_2 Z_2$ are

$$r = x, \quad s = \frac{x}{a_2}, \quad v(r) = u(t,x), \tag{3.88}$$

which grant the following simplification of Equation (3.75)

$$b v_r^2 + v_{rr} = 0. \tag{3.89}$$

This implies

$$v(r) = \frac{ln(C_1 r b + C_2 b)}{b}. \tag{3.90}$$

Now, in form of $u(t,x)$

$$u(t,x) = \frac{ln(C_1 tb + C_2 b)}{b}. \tag{3.91}$$

Equation (3.91) represent the exact solution of Equation (3.75).

Solution of (3.75) using $Z = a_2 Z_2 + Z_3$

The operator of similarity variables $a_2 Z_2 + Z_3$ are

$$r = \frac{a_2 ln(t) + x\beta}{\sqrt{\beta}}, \quad s = \frac{-ln(t)}{\beta}, \quad v(r) = u(t,x), \tag{3.92}$$

which grant the following simplification of Equation (3.75)

$$\beta v_r - a_2 v_{rr} + b^2\beta^2 v_r^2 e^{bv(r)+2s\beta} + bv_{rr}e^{bv(r)+2s\beta} - \beta\delta v_{rrr} + a_2\delta v_{rrrr} - \frac{\beta}{a_2} = 0. \tag{3.93}$$

Equation (3.93) cannot provide the exact solution.

Solution of (3.75) using $Z = Z_3$

The similarity variables for operator Z_3 are

$$r = x, \quad s = \frac{-ln(t)}{\beta}, \quad v(r) - \frac{ln(t)}{\beta} = u(t,x), \tag{3.94}$$

which grant following simplification of Equation (3.75)

$$v_r = e^{2s\beta} - ab^2 e^{b(v(r)+s)} v_r^2 - abe^{b(v(r)+s)} v_{rr}. \tag{3.95}$$

This implies

$$v(r) = v_r = \frac{ln(\frac{e^{-s}(-2\beta+b)r^2 - 2rC_1 ab + 2C_2 ab}{a})}{b}. \tag{3.96}$$

Now, in form of $u(t,x)$

$$u(t,x) = \frac{-ln2 + ln(t^{\frac{-2\beta+b}{\beta}} - 2ab(xC_1 - C_2) - ln(a))}{b} - \frac{ln(t)}{\beta}. \tag{3.97}$$

Equation (3.97) represent the exact solution of Equation (3.75) .

4. Figures

Invariant solutions are graphically represented through methods like wave profile solutions, contour plots, and patch contours. A wave profile shows how a wave evolves over time and space, often as a continuous curve. Contour plots connect points of equal value, revealing steady-state distributions, while patch contours fill regions between contour lines with colors, providing a clearer view of solution variations. These representations help visualize how invariant solutions behave across different contexts. Specifically, Figures 1–11) illustrate the invariant solutions for the equations referenced as (3.15), (3.18), (3.21), (3.28), (3.37), (3.40), (3.43), (3.62), (3.66), (3.70), (3.91).

5. Tables

Using Equation (3.49), we construct Tables 1 and 3, while Tables 2 and 4 are constructed using Equation (3.76). With the help of these tables, we then derive the optimal systems for equations (3.44) and (3.75). The commutator table of commutation relations between symmetry generators by using the Lie brackets defined as $[Z_i, Z_j] = Z_i Z_j - Z_j Z_i$, for $1 \leq i, j \leq 3$

The adjoint presentation is defined as

$$Ad(exp(\epsilon Z_i)X_j) = Z_j - \epsilon[Z_i, X_j] + \frac{1}{2}\epsilon^2[Z_i, [Z_i, Z_j]] - .... \tag{5.1}$$

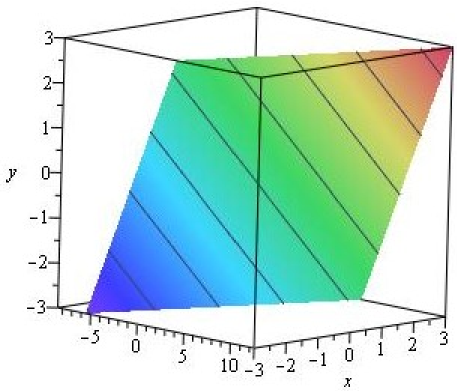


FIGURE 1. Three-dimensional graphical representation of wave profiles for Eq. (3.15), when $t = -1$, $c_1 = 1$, $c_2 = 2$, $b = 1$, $\alpha = 1$.

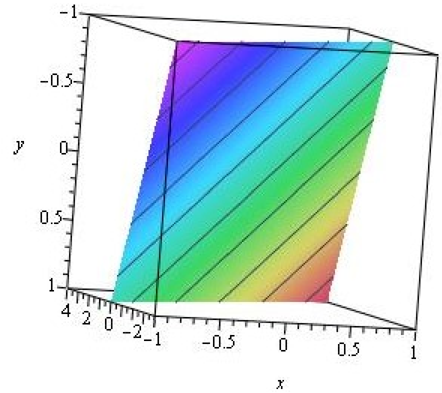


FIGURE 2. Three-dimensional graphical representation of Eq. (3.18), for $t = 1$, $c_1 = 1$, $c_2 = 2$, $b = 1$, $a = 1$, $\alpha = 1$.

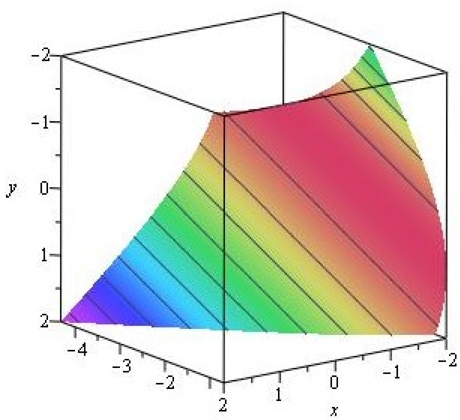


FIGURE 3. Three-dimensional graphical representation of Eq. (3.21), when $t = 1$, $c_1 = 1$, $c_2 = 2$, $b = 1$, $c = 1$, $\alpha = 1$.

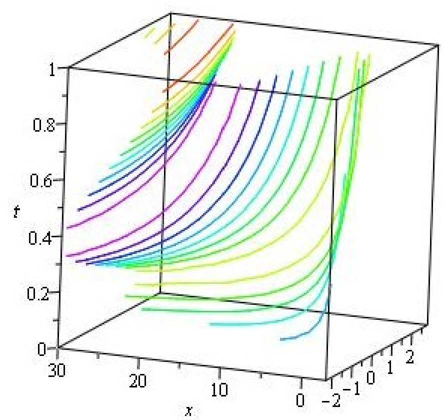


FIGURE 4. Contourplot 3d: Graphical representation of Eq. (3.28), when $c_1 = 1$, $c_2 = 2$, $a = 1$.

TABLE 1. Commutator Table of Principle Lie Algebras.

$[Z_i, Z_j]$	Z_1	Z_2	Z_3
Z_1	0	0	$2Z_1$
Z_2	0	0	Z_2
Z_3	$-2Z_1$	$-Z_2$	0

TABLE 2. Commutator Table of Principle Lie Algebras.

$[Z_i, Z_j]$	Z_1	Z_2	Z_3
Z_1	0	0	$-\beta Z_1$
Z_2	0	0	0
Z_3	βZ_1	0	0

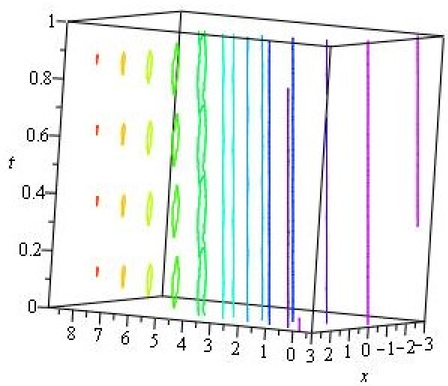


Figure 5. Contourplot3d: Patch contour graphical representation for Equation (3.37), when $C_1 = 1$, $C_2 = 2$, $c_2 = 1$.

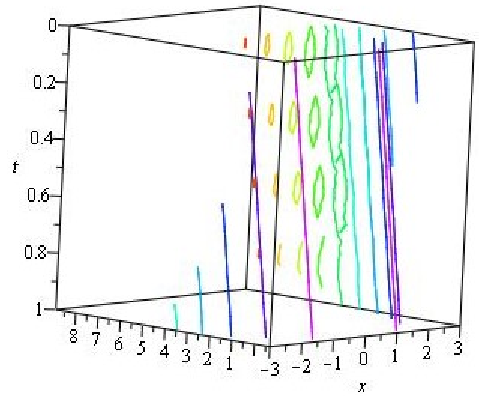


Figure 6. Contourplot3d: Patch contour graphical representation for Equation (3.40), when $c_2 = 2$, $C_1 = 1$, $C_2 = 1$, $\alpha = 1$.

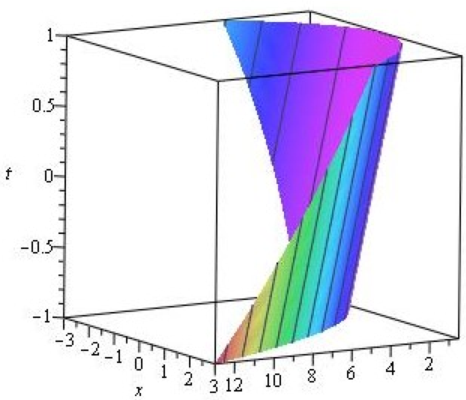


Figure 7. Plot3d: Patch contour graphical representation for Equation (3.43), when $c_2 = 1$, $C_1 = 1$, $C_2 = 1$, $\alpha = 1$.

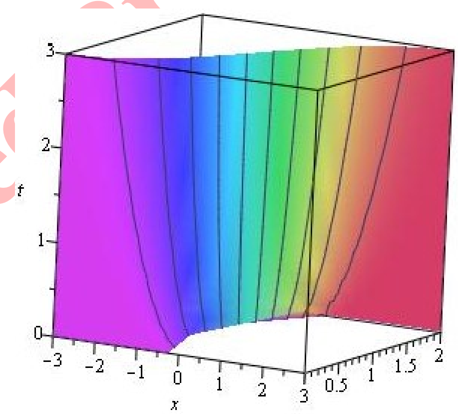


Figure 8. The solution of u(x,t) for Equation (3.62) when $C_1 = 1$, $C_2 = 1$, $\alpha = 1$.

Table 3. Adjoint Representations of Lie Algebra.

Ad	Z_1	Z_2	Z_3
Z_1	Z_1	Z_2	$Z_3 - 2\epsilon Z_1$
Z_2	Z_1	Z_2	$Z_3 - \epsilon Z_2$
Z_3	$e^{2\epsilon} Z_1$	$Z_2 e^{\epsilon}$	Z_3

Table 4. Adjoint Representations of Lie Algebra.

Ad	Z_1	Z_2	Z_3
Z_1	Z_1	Z_2	$Z_3 + \beta Z_1$
Z_2	Z_1	Z_2	Z_3
Z_3	$e^{\beta\epsilon} Z_1$	Z_2	Z_3

6. Conclusion

In this article, we explore the Lie symmetries, similarity reduction, and invariant solutions of the (1+2)-dimensional modified Breaking Soliton equation. We find that Olver's Lie-group analysis technique effectively examines the symmetries and invariant solutions of differential equations. We classified two forms of the involved unknown functions

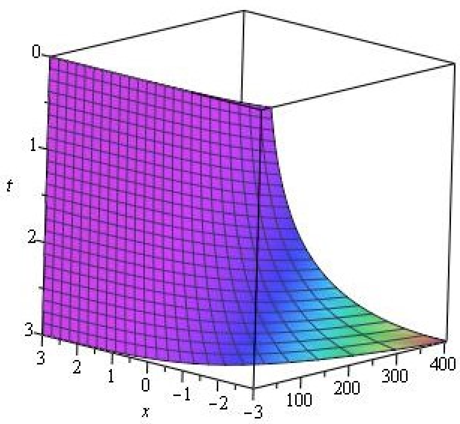


FIGURE 9. The solution of u(x,t) for Equation (3.66) when $C_1 = 1, C_2 = 1, \alpha = 1, a_1 = 1$.

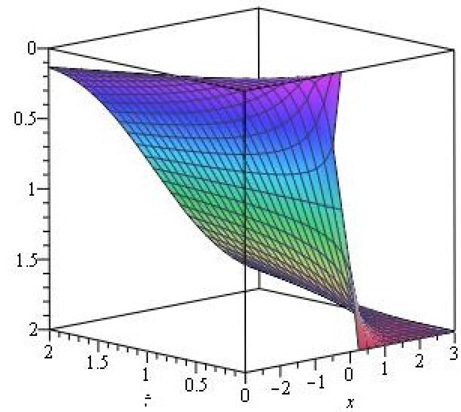


FIGURE 10. The solution of u(x,t) for Equation (3.70) when $C_1 = 1, C_2 = 1, \alpha = 1$.

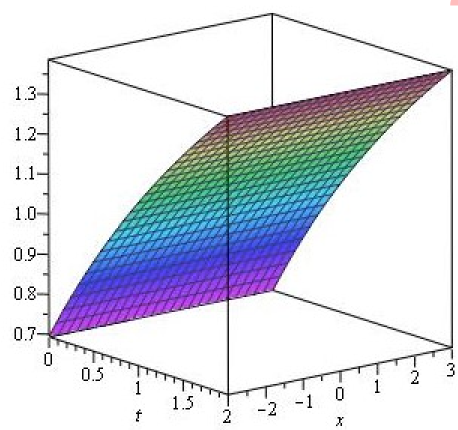


FIGURE 11. The solution of u(x,t) for Equation (3.91) when $C_1 = 1, C_2 = 1, b = 1$.

and constructed four classical symmetries for the modified Breaking Soliton equation. Using these symmetries and similarity transformations, we generated various new invariant solutions. We derived several Lie-group transformations from the symmetry generators and obtained invariant solutions via characteristic equations.

Additionally, we analyzed Lie point symmetries, optimal systems, and similarity reductions for the heat equation and modified Boussinesq equation using Lie symmetry analysis. We developed optimal systems for both models based on the commutation relations of constructed Lie symmetry operators and their adjoint representations, leading to different classes of new invariant solutions. A graphical analysis of the calculated solutions was also presented. In this paper, we introduce Lie transformations that transform previous solutions into new ones, enabling more comprehensive solutions.

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