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On the numerical solution of Bagley-Torvik equation using the Müntz-Legendre wavelet collocation method

Anber Abraheem Shlash Mohammad[1], **Suleiman Ibrahim Mohammad**[2,3,*], **Asokan Vasudevan**[4], **Muhammad Turki Alshurideh**[5], **and Ding Nan**[6]

[1]Digital Marketing Department, Faculty of Administrative and Financial Sciences, Petra University, Jordan.

[2]Electronic Marketing and Social Media, Economic and Administrative Sciences Zarqa University, Jordan.

[3]Research follower, INTI International University, 71800 Negeri Sembilan, Malaysia.

[4]Faculty of Business and Communications, INTI International University, 71800 Negeri Sembilan, Malaysia.

[5]Department of Marketing, School of Business, The University of Jordan, Amman, Jordan.

[6]Faculty of Liberal Arts, Shinawatra University, Thailand.

Abstract

Solving the Bagley–Torvik (BT) equation using an effective scheme and finding its numerical solution are the main goals of this work. The scheme benefits the collocation method based on the Müntz-Legendre (ML) wavelets. To apply the method, after approximating the unknown solution by mapping it in the wavelet space, we replace it in the desired equation and then obtain the residual using the operational matrices of the derivative and Caputo fractional derivative (CFD). Applying the collocation method results in a linear algebraic system. To implement the collocation method, either Chebyshev or Legendre roots are used as collocation points or uniformly spaced grids are selected. The error analysis is investigated, and some numerical examples are presented to show the accuracy and effectiveness of the scheme. Thanks to the flexibility of ML wavelets, and the method's structure, we can even obtain the exact solution in some cases.

Keywords. Fractional differential equation, Wavelet collocation method, Müntz-Legendre wavelets, Bagley–Torvik equation.

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1. Introduction

Engineering and other disciplines have seen a rise in the utilization of fractional calculus for explaining various physical phenomena. Various equations involving fractional derivatives have been extensively studied, prompting the emergence of various mathematical algorithms to solve them, including the wavelet spectral element [5], the wavelet method [40, 41], Adomian decomposition [8], the finite element-meshfree method [21], multi-step methods [11], implicit integration factor method [44], cubic Hermit spline method ghasem, Müntz-Legendre Petrov-Galerkin method [33] etc.

In [5], the authors have introduced the necessary and sufficient conditions for equation

$$\begin{aligned} &{}^{c}\mathcal{D}^{\alpha}_{a^+}(x)(t) = f[t, x(t), {}^{c}\mathcal{D}^{\alpha_1}_{a^+}(x)(t), \cdots, {}^{c}\mathcal{D}^{\alpha_\sigma}_{a^+}(x)(t)], \quad t \in [a, b], \\ &x^{(\kappa)}(a) = b_\kappa, \quad b_\kappa \in \mathbb{R}, \;\; \kappa = 0, 1, \cdots, n-1, \end{aligned}$$

to have a unique solution. Also, interestingly, an algorithm has been presented to reduce the computational cost using the wavelet properties [5]. Bin Jebreen et al. [13] successfully solved the fractional Cauchy problem using the wavelet collocation technique. The study of multi-order fractional differential equations (FDEs) has been conducted in [16]. This research utilized the Galerkin method, employing fractional-order Legendre functions as the bases. In

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∗ Corresponding author. Email: dr_sliman@yahoo.com.

[9], the multi-order FDEs are studied through the fractional-Lucas optimization method. To further studies about the numerical method for FDEs refer to [1, 2, 38].

Our objective in this paper is to introduce an effective algorithm for solving a famous fractional differential equation, called BT equation

$$ax''(t) + b\,{}^{c}\mathcal{D}_0^{\beta}(x)(t) + cx(t) = f(t), \quad \beta = 3/2, \quad 0 \le t \le 1, \tag{1.1}$$

with initial conditions

$$x(0) = x_0, \quad x'(0) = x_1, \tag{1.2}$$

in which $a \neq 0$, b and c are constant, and ${}^{c}\mathcal{D}_0^{\beta}$ indicates the Caputo fractional derivative

$${}^{c}\mathcal{D}_0^{\beta}(x)(t) = \frac{1}{\Gamma(\kappa-\beta)} \int_0^t \frac{x^{(\kappa)}(z)dz}{(t-z)^{\beta-\kappa+1}}, \quad \kappa = -[-\beta]. \tag{1.3}$$

This equation arises in modeling a rigid plate bounded by a Newtonian fluid that the first introduced by Torvik and Bagley [42]. They achieved very interesting results that showed the interior oscillations of a rigid plate immersed in a Newtonian fluid do not create a relation between a retarding force and the velocity. Instead, there is a proportional relationship between the retarding force and the fractional derivative of order 3/2 of the displacement. They concluded that the concept of fractional derivative arises naturally while studying the behavior of real materials. So, there is reason to suspect that using fractional derivatives in constitutive relationships to describe real materials is not coincidental. In [27], we can find a thoroughly discusion about the existing unique solution of this equation under considering the homogeneous initial conditions. In this work, the fractional derivative is considered of the Riemann–Liouville types, while homogeneous initial conditions equip the equation. To consider the nonhomogeneous initial conditions, Diethelm et al. [10] assumed this equation with CFD and the nonhomogeneous initial conditions. They solved this equation by fractional linear multi-step method under a reformulation of it to a system of fractional differential equations. The existence and uniqueness of the solution are investigated in [22], and the analytical solution is also introduced in [27] as follows.

$$x(t) = \int_0^t K(t-z)f(z)dz, \tag{1.4}$$

in which

$$K(t) = 1/a \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \left(\frac{a}{c}\right)^i t^{2i+1} E^{(i)}_{1/2,2+3i/2}\left(\frac{-b}{a}\sqrt{t}\right),$$

and $E_{k,l}(t)$ indicates the Mittag–Leffler function. It is interested to noting that for general functions f, it is not convenient to evaluate the aforementioned integral, so using the numerical schemes can be a good solution to overcome this defect. Yüzbaşi, [43] used the Bessel collocation method to solve the desired equation with the boundary conditions. In [7], authors proposed a new algorithm using generalized Taylor collocation method. After reformulating the BT equation to a system of ordinary differential equations related to the inverse forms of Abel-integral equations, Leszczynsk et al. [20] proposed a numerical scheme to solve the problem. Considering the source function f as the Heaviside function, the Adomian decomposition method has been utilized to solve the BT equation [31]. More related papers that deal with finding the solution of the BT equation may be found in references [15, 18, 24, 25].

Recent studies show that wavelets effectively solve differential equations and represent various operators [4]. In the literature on wavelet theory, two main families exist, including scalar wavelets, and multi-wavelets. In contrast to scalar wavelets, which rely on a single generator to construct wavelet spaces, multiwavelets employ multiple generators within the framework of multi-resolution analysis [14]. Consequently, multiwavelets offer several advantages over scalar wavelets in various applications. When discussing these features, let us not forget symmetry, a closed form, high vanishing moments, and orthogonality. The Alpert multi-wavelet is a well-known multi-wavelet with various applications in image processing and numerical analysis [4, 34–37]. The ML wavelets are a type of multiwavelets that has been recently utilized in numerical work, such as solving fractional optimal control problems [28], multi-order fractional differential equations [13], and fractional pantograph equation [30].

After this brief introduction, the organization of the remaining sections will be as follows: Section 2 introduces the ML wavelets and their properties. Solving the Bagley-Torvik equation using the wavelet collocation method is the objective of section 3. Additionally, we shall present the error analysis for the method. We conduct several numerical experiments to demonstrate the accuracy and usefulness of the method, in section 4. Finally, brief conclusions are included in section 5.

2. Müntz-Legendre wavelets

Suppose that the space $F(\mathcal{B})$ is spanned by a sequence of functions $\{t^{\beta_n}\}_{n=0}^{\infty}$

$$F(\mathcal{B}) := \bigcup_{n=0}^{\infty} F_n(\mathcal{B}) = span\{t^{\beta_n}, n = 0, 1, \dots\}. \tag{2.1}$$

in which $F_n(\mathcal{B}) := span\{t^{\beta_0}, t^{\beta_1}, \dots, t^{\beta_n}\}$ and $\mathcal{B} = \{0 = \beta_0 < \beta_1 < \dots\}$ is an increasing sequence [3]. To verify $\overline{F(\mathcal{B})} = C[0,1]$, the condition

$$\sum_{\beta_n > 0} \frac{1 + \log \beta_n}{\beta_n} = \infty, \tag{2.2}$$

is sufficient, and condition

$$\lim_{n \to \infty} \frac{\beta_n}{n \log n} = 0, \tag{2.3}$$

is necessary. So it follows that $F(\mathcal{B})$ is dense in $C[0,1]$ (The space of continuous functions on $[0,1]$). These criteria are introduced by S. N. Bernstein. It is also worth mentioning that he proposed the conditions of existence and uniqueness

$$\sum_{n=1}^{\infty} \frac{1}{\beta_n} = \infty, \tag{2.4}$$

where $\mathcal{B} = \{0 = \beta_0 < \beta_1 < \dots\}$. This conjecture was verified two years later by Müntz [26]. Note that the sufficient and necessary conditions to verify that $\overline{F(\mathcal{B})} = L^2(0,1)$ are specified in [39].

It is crucial to comprehend that using the functions $\{t^{\beta_n}\}_{n=0}^{\infty}$ as bases is not advisable. Consequently, the subsequent definitions of the ML functions are structured to ensure orthogonality and facilitate straightforward evaluation.

Assuming that χ represents a simple contour that encircles all the zeros of the integrand's denominator, the ML polynomials can be articulated in the following closed form [6, 39]

$$ML_n(t; \mathcal{B}) := \frac{1}{2\pi i} \int_{\chi} \prod_{k=1}^{n-1} \frac{z + \beta_k + 1}{z - \beta_k} \frac{t^k}{z - \beta_n} dz, \tag{2.5}$$

Now, one can determine the ML polynomials as:

$$L_n(t; \mathcal{B}) = \sum_{k=0}^{n} c_{k,n} t^{\beta_k}, \quad t \in [0,1], \tag{2.6}$$

where the constant β_k are detrmined by $\beta_k := \{k\eta : \eta \in \mathbb{R}, k = 0, \dots n\}$, and the sequence $\{\beta_k\}_{k=1}^{\infty}$ is ascending. Furthermore, the coefficients $c_{k,n}$ are specified by [6]

$$c_{k,n} := \frac{\prod_{i=0}^{n-1}(\beta_k + \beta_i + 1)}{\prod_{i=0, i \neq k}^{n}(\beta_k - \beta_i)}. \tag{2.7}$$

Motivated by Theorem 2.4 in [6], It is an accepted fact that the ML polynomials are orthogonal and one can obtain

$$\int_0^1 L_{n'}(t) L_{n''}(t) dt = \delta_{n',n''} / (\beta_n + \beta_{n''} + 1), \tag{2.8}$$

where $L_n(t) := L_n(t;\mathcal{B})$.

Given the multiplicity parameter $\nu \in \mathbb{N}$ and refinement level parameter $s \in \mathbb{N}_0$. A sequence of nested subspaces $\{V_s\}_{s\in\mathbb{N}_0} \subset L^2([0,1])$ exists that are spanned by $\phi^n_{s,a}$ (see, multi-resolution analysis (MRA) [23]), i,e.,

$$V_s = span\{\phi^n_{s,a} := \phi^n(2^s - a) : \quad a \in \mathcal{M},\ n \in \mathcal{V}\}, \tag{2.9}$$

in which $\mathcal{V} := \{0, 1, \ldots, \nu - 1\}$ and $\mathcal{M} := \{0, 1, \ldots, 2^s - 1\}$.

Using ML polynomials $L_n(t)$, the ML wavelets are specified by [13]

$$\phi^n_{s,a} = \begin{cases} 2^{s/2}\sqrt{2\beta_n + 1}L_n(2^s t - a), & \frac{a}{2^s} \le t \le \frac{a+1}{2^s}, \\ 0, & otherwise. \end{cases} \tag{2.10}$$

To approximate $x \in L^2[0,1]$, one introduces the projection operator $\mathcal{P}_s : L^2[0,1] \to V_s$, viz

$$x(t) \approx \mathcal{P}_s(x)(t) = \sum_{a=0}^{2^s-1}\sum_{n=0}^{\nu-1} x_{a,n}\phi^n_{s,a}(t) = X^T \mathrm{F}(t) \in V_s, \tag{2.11}$$

in which $[\mathrm{F}(t)]_{a\nu+n+1,1} := \phi^n_{s,a}(t)$, and

$$x_{a,n} = \langle x, \phi^n_{s,a}\rangle = \int_0^1 x(t)\phi^n_{s,a}(t)dt. \tag{2.12}$$

The following lemma provides an estimate of the approximation bound (2.11) (see e.g., [29]).

Lemma 2.1. *Let $\nu \in \mathbb{N}$ and $s \in \mathbb{N}_0$. Assume that for any $\omega < \nu$, $x \in H^\omega[0,1]$. Then*

$$\|x - \mathcal{P}_s(x)\|_2 \le c(2^{s-1})^{-\omega}(\nu - 1)^{-\omega}\|x^{(\omega)}\|_2, \tag{2.13}$$

and when $\omega' \ge 1$, we get

$$\|x - \mathcal{P}_s(x)\|_{H^{\omega'}([0,1])} \le c(\nu - 1)^{2\omega' - \frac{1}{2} - \omega}(2^{s-1})^{\omega' - \omega}\|x^{(\omega)}\|_2. \tag{2.14}$$

Here $H^\omega([0,1])$ indicates the Sobolev space endowed with norm

$$\|x\|_{H^\omega([0,1])} = \left(\sum_{i=0}^{\omega}\|x^{(i)}\|_2^2\right)^{1/2}. \tag{2.15}$$

2.1. **Matrix representation of fractional integration.** The main object of this subsection is to find a square matrix I_β to help us represent the fractional integral operator (FIO) via it. To determine the elements of I_β, the operation of FIO on $\mathrm{F}(t)$ must be estimated using ML wavelets, viz.,

$$\mathcal{P}_s(\mathcal{I}^\beta_0)(\mathrm{F}(t)) \approx I_\beta(t)\mathrm{F}(t), \tag{2.16}$$

where $\mathcal{I}^\beta_0$ states the FIO, i.e.,

$$\mathcal{I}^\beta_0(x)(t) := \frac{1}{\Gamma(\beta)}\int_0^t (t - z)^{\beta - 1}x(z)dz, \quad \beta \in \mathbb{R}^+, \quad 0 \le t \le 1, \tag{2.17}$$

where $\Gamma(\beta)$ specifies the Gamma function.

In the sequel, the piecewise Taylor functions of fractional order (FPTFs) will be introduced to help us calculate the entries of I_β, i.e.,

$$p^n_{s,a}(t) = \begin{cases} t^{\beta_n}, & \frac{a}{2^s} \le t \le \frac{a+1}{2^s}, \\ 0, & otherwise, \end{cases} \quad a \in \mathcal{M}, n \in \mathcal{V}, s \in \mathbb{N}_0. \tag{2.18}$$

Considering $P(t)$ as a vector function in which the $(a\nu + n + 1)$-th element is set to $p^n_{s,a}(t)$, it can be shown a relationship between the ML wavelets $\phi^n_{s,a}(t)$ and FPTFs i.e.,

$$\mathrm{F}(t) = \Upsilon^{-1}P(t), \tag{2.19}$$

where

$$\Upsilon_{i,j} = \langle P_j(t), \mathrm{F}_i(t)\rangle = \int_0^1 \mathrm{F}_i(t)P_j(t)dt, \quad i,j=1:N, \qquad N=2^s\nu. \tag{2.20}$$

Considering W as a ν-dimensional vector whose elements are $\{t^{\beta_n}\}_{n=1}^{\nu}$, one can confirm that

$$P(t) = [W,\dots,W]^T. \tag{2.21}$$

Taking into account the definition of FIO, it can be verify that [17]

$$\mathcal{I}_0^\beta(t^\alpha) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta}. \tag{2.22}$$

Thus, it follows from (2.18) and (2.22) that

$$\mathcal{I}_0^\beta(P_i)(t) = \frac{\Gamma(\beta_i+1)}{\Gamma(\beta_i+\beta+1)}x^{\beta_i+\beta}, \quad i=1:N. \tag{2.23}$$

Now one can introduce the matrix $I_{P,\beta}(t)$ that satisfies

$$\mathcal{I}_0^\beta(P)(t) = I_{P,\beta}(t)P(t), \tag{2.24}$$

with

$$I_{P,\beta}(t) = \begin{pmatrix} B_\beta(t) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_\beta(t) \end{pmatrix} \tag{2.25}$$

where $B_\beta(t) := t^\beta K$, $(\mathcal{I}_0^\beta(W)(t) = B_\beta(t)W(t))$ and

$$\begin{cases} (K)_{i,j} = \Gamma(\beta_i+1)/\Gamma(\beta_i+\beta+1), & i=j, \\ (K)_{i,j}=0, & i\neq j, \\ i,j=1:N. \end{cases} \tag{2.26}$$

To specify $I_\beta(t)$, we have

$$\begin{aligned} \mathcal{P}_s(\mathcal{I}_0^\beta)(\mathrm{F}(t)) &= \mathcal{P}_s(\mathcal{I}_0^\beta)(\Upsilon^{-1}P(t)) \\ &= \Upsilon^{-1}I_{P,\beta}(t)P(t) \\ &= \Upsilon^{-1}I_{P,\beta}(t)\Upsilon\mathrm{F}(t). \end{aligned} \tag{2.27}$$

Therefore, one can obtain

$$I_\beta(t) := \Upsilon^{-1}I_{P,\beta}(t)\Upsilon. \tag{2.28}$$

2.2. **Matrix representation of CFD.** Given $\beta\in\mathbb{R}^+$, assume that $AC^\beta([0,1])$ specifies the space of functions such that

$$AC^\beta[0,1] = \{x:[0,1]\to\mathbb{C}, \quad \& \quad \mathcal{D}^{(\beta-1)}(x)\in AC[0,1]\},$$

where $\mathcal{D}=\frac{d}{dt}$ indicates the derivative operator.

Recall that for any $x(t)\in AC^\beta([0,1])$, we have

$${}^c\mathcal{D}_0^\beta(x)(t) = \frac{1}{\Gamma(\kappa-\beta)}\int_0^t \frac{x^{(\kappa)}(z)dz}{(t-z)^{\beta-\kappa+1}} =: \mathcal{I}_0^{\kappa-\beta}\mathcal{D}^\kappa(x)(t), \tag{2.29}$$

in which $\kappa=-[-\beta]$. Consequently, one can prove that

$${}^c\mathcal{D}_0^\beta(z^{\alpha-1})(t) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta}, \quad (\alpha>\kappa). \tag{2.30}$$

The main objective of this section is to introduce a square matrix D_β that fulfills

$${}^c\mathcal{D}_0^\beta(\mathrm{F}(t)) \approx D_\beta\mathrm{F}(t). \tag{2.31}$$

To calculus the elements of D_β, we use the relation ${}^c\mathcal{D}_0^\beta = \mathcal{I}_0^{\kappa-\beta}\mathcal{D}^\kappa$ (due to the CFD definition) as follows:

$$\begin{aligned}{}^c\mathcal{D}_0^\beta(\mathrm{F}(t)) = \mathcal{I}_0^{\kappa-\beta}\mathcal{D}^\kappa(\mathrm{F}(t)) &\approx \mathcal{I}_0^{\kappa-\beta}(D^\kappa \mathrm{F}(t)) \\ &= D^\kappa \mathcal{I}_0^{\kappa-\beta}(\mathrm{F}(t)) \approx D^\kappa I_{\kappa-\beta}(\mathrm{F}(t)).\end{aligned}$$

Here the square matrix D refers to the matrix representation of operator $\mathcal{D}$ (see, e.g., [33]). Thus, one can obtain

$$D_\beta := D^\kappa I_{\kappa-\beta}. \tag{2.32}$$

Introducing this matrix leads to a reduction in computational cost by avoiding computing the operation of CFD on the ML wavelets.

3. Wavelet collocation method

Recall that the main subject of this paper is introducing an effective algorithm based on the wavelet collocation method for solving BT equation

$$ax''(t) + b{}^c\mathcal{D}_0^{3/2}(x)(t) + cx(t) = f(t), \quad 0 \le t \le 1, \tag{3.1}$$

with initial conditions

$$x(0) = x_0, \quad x'(0) = x_1. \tag{3.2}$$

To apply the collocation method, the unknown solution is first mapped into the approximation space V_s by using the operator $\mathcal{P}_s$, i,e.,

$$x(t) \approx \mathcal{P}_s(x)(t) = X^T\mathrm{F}(t) := x_N(t). \tag{3.3}$$

The N-dimensional vector X holds the unknowns that require specification.

Inserting $x_N(t)$ into the Equation (3.1) gives rise to equation

$$a{x''}_N(t) + b{}^c\mathcal{D}_0^{3/2}(x_N)(t) + cx_N(t) = f(t), \tag{3.4}$$

Using operational matrices $D_{3/2}$ and D, we can approximate the functions ${x''}_N(t)$, ${}^c\mathcal{D}_0^{3/2}(x_N)(t)$ and $f(t)$ using the projection operator $\mathcal{P}_s$ as follows

$$\begin{aligned}&{x''}_N(t) \approx \mathcal{P}_s({x''}_N)(t) = X^T D^2 \mathrm{F}(t),\\ &{}^c\mathcal{D}_0^{3/2}(x_N)(t) \approx \mathcal{P}_s({}^c\mathcal{D}_0^{3/2}(x_N)(z))(t) = X^T D_{3/2}\mathrm{F}(t),\\ &f(t) \approx \mathcal{P}_s(f)(t) = F^T\mathrm{F}(t).\end{aligned} \tag{3.5}$$

Putting (3.5) back into (3.3), one can introduce the residual

$$r_s(t) := \left(aX^TD^2 + bX^TD_{3/2} + cX^T - F^T\right)\mathrm{F}(t). \tag{3.6}$$

The collocation method's goal is to bring $r_s(t)$ closer to zero. Selecting the collocation points $\{t_n\}_{n=1}^N \in [0,1]$, the method leads to a linear system that satisfies $r_s(t_n) = 0$. This is equivalent to solving the system

$$X^T\left(aD^2 + bD_{3/2} + cI\right) = F^T, \tag{3.7}$$

and finding the unknown coefficients X. In other words, there is a linear system

$$AX = F, \tag{3.8}$$

with $A := (aD^2 + bD_{3/2} + cI)^T$, that must be solved to gain the unknowns. To use the initial conditions, we replace the first and the second elements of $r_s(t_i)$ by

$$r_s(t_1) := X^T\mathrm{F}(0) - x_0, \quad r_s(t_2) := X^T D\mathrm{F}(0) - x_1,$$

respectively.

It is to be noted here that in this study, the collocation points are either chosen as the roots of Chebyshev or Legendre polynomials, or they are chosen as uniform points from $[0, 1]$.

In more abstract terms, to map $C([0,1])$ to V_s, there is a projection operator Q_N. To be more precise, the projection $Q_N(x)$ maps w into V_S such that interpolates it at the points $\{t_N\}_{N=1}^N \in [0,1]$. As a result, $Q_N r_s = 0$ can be used instead of $r_s(t_n)$. Equivalently, one can write

$$Q_N\left(\left(aX^TD^2 + bX^TD_{3/2} + cX^T\right)\mathrm{F}(t)\right) = Q_N\left(F^T\mathrm{F}(t)\right). \tag{3.9}$$

3.1. **Error analysis.**

Lemma 3.1. *([17]). Given $\beta > 0$ and $1 \le q \le \infty$, The operator $\mathcal{I}_0^\beta$ is bounded, viz*

$$\|\mathcal{I}_0^\beta(x)\|_q \le \frac{1}{\Gamma(\beta+1)}\|x\|_q. \tag{3.10}$$

Theorem 3.2. *Assuming r_s as the residual introduced in (3.6), the error of presented method applied to solve the BT Equation (3.1) can be obtained as*

$$\|x - x_N\| \le C\|r_s^{(N+1)}(t)\|, \tag{3.11}$$

where x_N is the approximate solution and C is a constant.

Proof. Subtracting (3.9) from (3.1) leads to

$$\begin{aligned} &ax''(t) - aQ_N(x''_N)(t) + b^c\mathcal{D}_0^{3/2}(x)(t) - bQ_N(^c\mathcal{D}_0^{3/2}(x_N))(t) + cx(t) - cQ_N(x_N)(t) \\ &\quad = f(t) - Q_N(f)(t). \end{aligned} \tag{3.12}$$

Equivalently, one can write

$$\begin{aligned} &ax''(t) - ax''_N + ax''_N - aQ_N(x''_N)(t) + b^c\mathcal{D}_0^{3/2}(x)(t) - b^c\mathcal{D}_0^{3/2}(x_N)(t) + b^c\mathcal{D}_0^{3/2}(x_N)(t) \\ &\quad - bQ_N(^c\mathcal{D}_0^{3/2}(x_N))(t) + cx(t) - cx_N(t) + cx_N(t) - cQ_N(x_N)(t) = f(t) - Q_N(f)(t). \end{aligned} \tag{3.13}$$

Given $e_N = x - x_N$, it is easy to gain the following relation

$$\begin{aligned} &ae''_N(t) + a\left(I - Q_N\right)(e''_N)(t) + b^c\mathcal{D}_0^{3/2}(e_N)(t) + b\left(I - Q_N\right)(^c\mathcal{D}_0^{3/2}(e_N))(t) + ce_N(t) \\ &\qquad + c\left(I - Q_N\right)(e_N)(t) = \left(I - Q_N\right)(f)(t). \end{aligned} \tag{3.14}$$

Considering

$$r_s(x) = ax_N(t) + b^c\mathcal{D}_0^{3/2}(x_N)(t) + cx_N(t) - f(t), \tag{3.15}$$

and making some simplifications in Equation (3.14), it is straightforward to reach the relation

$$ae''_N(t) + b^c\mathcal{D}_0^{3/2}(e_N)(t) + ce_N(t) = \left(I - Q_N\right)(r_s)(t). \tag{3.16}$$

It follows from (3.16) that

$$e''_N(t) = \frac{1}{a}\left(I - Q_N\right)(r_s)(t) - \frac{b}{a}{}^c\mathcal{D}_0^{3/2}(e_N)(t) - \frac{c}{a}e_N(t). \tag{3.17}$$

Taking norm and using Lemma 3.1, we get

$$\|e''_N(t)\| \le \frac{c}{a}\|e_N(t)\| + \frac{1}{a}\|\left(I - Q_N\right)(r_s)(t)\| + \frac{b}{a\Gamma(3/2)}\|e''_N(t)\|. \tag{3.18}$$

Let $\rho := 1 - \frac{b}{a\Gamma(3/2)} \ge 0$, we can rewrite (3.18) as follows;

$$\|e''_N(t)\| \le \frac{c}{a\rho}\|e_N(t)\| + \frac{1}{a\rho}\|\left(I - Q_N\right)(r_s)(t)\|. \tag{3.19}$$

On the other hand, using the Cauchy formula for repeated integration, we have

$$e_N(t) = \int_0^t (t - x)e''_N(x)dx. \tag{3.20}$$

Taking the norm, one can show that

$$\|e_N(t)\| \le M\|e''{}_N(x)\|. \tag{3.21}$$

Substituting (3.19) into (3.21), leads to

$$\|e_N(t)\| \le \frac{cM}{a\rho}\|e_N(t)\| + \frac{M}{a\rho}\|\left(I - Q_N\right)(r_s)(t)\| \tag{3.22}$$

Assuming $\sigma := 1 - \frac{cM}{a\rho} \ge 0$ and using the interpolation error, one can obtain

$$\|e_N(t)\| \le C\|r_s^{(N+1)}(t)\|, \tag{3.23}$$

where $C = \frac{M}{a\rho\sigma}$. □

4. Numerical simulations and results

This section includes several illustrative examples to demonstrate the efficacy of the presented scheme. To provide a comprehensive overview of the method's effectiveness, absolute errors

$$e_N = |x(t) - x_N(t)|,$$

and L_2 error

$$L^2 - error = \left(\int_0^1 |x(t) - x_N(t)|^2\right)^{1/2}.$$

may be reported in tables or figures.

To achieve greater accuracy, we elevate precision above 50 digits. All examples were run using Maple and Matlab softwares (version 2022).

Example 4.1. First, we implement the present method for Equation (3.1) with $a = 1$, $b = 8/17$, $c = 13/51$, $x_0 = 0$, and $x_1 = 27/125$. Also, we have

$$f(t) = \frac{t^{-1/2}}{89250\sqrt{\pi}}\left(48u(t) + 7\sqrt{\pi t}v(t)\right), \quad 0 < t \le 1,$$

in which

$$v(t) = 3250t^5 - 9425t^4 + 264880t^3 - 448107t^2 + 233262t - 34578,$$
$$u(t) = 16000t^4 - 32480t^3 + 21280t^2 - 4746t.$$

The exact solution mentioned in [18] is equal to

$$x(t) = x^5 - \frac{29}{10}t^4 + \frac{76}{25}t^3 - \frac{339}{250}t^2 + \frac{27}{125}t.$$

To solve this example using the presented method, we can obtain the exact solution by selecting $\eta = 1$, $N = 6$, and all three collocation points, including Chebyshev roots, Legendre roots, and uniform grids. Figure 1 is plotted to show the accuracy of the present method, taking $\eta = 1$, $N = 6$ and Legendre roots. To have a comparison between the existing methods and the presented method in this work, the absolute errors at different times are tabulated in Table 1. The scheme presented offers a highly accurate approximate solution when compared to other options.

Example 4.2. Considering the Bagley-Torvik Equation (3.1), the second example is presented as

$$x''(t) + \frac{1}{2}{}^c\mathcal{D}_0^{3/2}(x)(t) + \frac{1}{2}x(t) = \begin{cases} 8, & 0 \le t \le 1, \\ 0, & t > 1, \end{cases} \quad 0 \le t \le 1, \tag{4.1}$$

with conditions

$$x(0) = 1, \quad x'(0) = 0. \tag{4.2}$$

TABLE 1. The results obtained for Example 4.1 compared to other methods.

	Proposed method	[43]	[32]	
$t\backslash N$	6	8	8	16
0.1	4.00×10^{-52}	1.08×10^{-02}	3.60×10^{-04}	9.13×10^{-05}
0.2	3.16×10^{-51}	8.96×10^{-03}	1.58×10^{-03}	1.33×10^{-04}
0.3	8.70×10^{-51}	3.78×10^{-03}	1.79×10^{-03}	1.22×10^{-04}
0.4	1.58×10^{-50}	1.44×10^{-07}	1.63×10^{-03}	8.99×10^{-04}
0.5	2.16×10^{-50}	1.00×10^{-03}	1.16×10^{-03}	1.77×10^{-05}
0.6	2.36×10^{-50}	6.62×10^{-08}	5.84×10^{-04}	5.36×10^{-05}
0.7	2.03×10^{-50}	1.26×10^{-03}	1.27×10^{-04}	2.15×10^{-05}
0.8	1.20×10^{-50}	1.28×10^{-03}	1.20×10^{-04}	1.65×10^{-06}
0.9	2.60×10^{-51}	2.07×10^{-08}	5.54×10^{-04}	1.42×10^{-06}

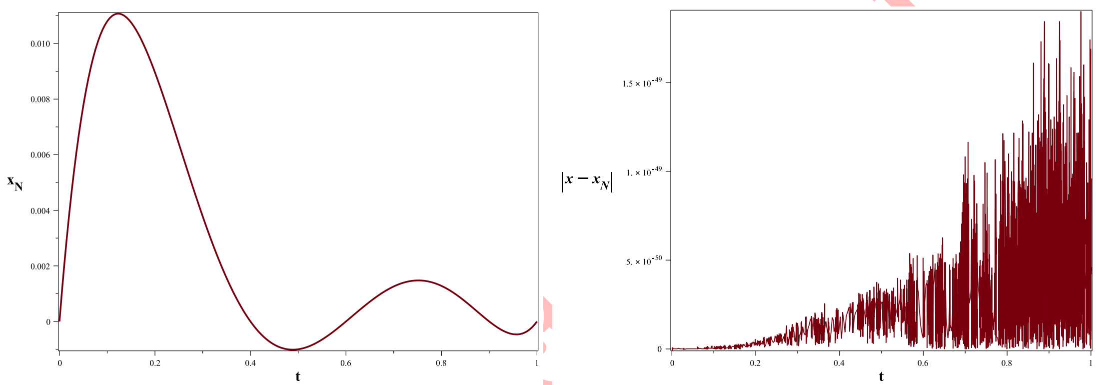

FIGURE 1. The obtained numerical solution and corresponding absolute error for Example 4.1.

For this example, the exact solution can be obtained via (1.4).

To compare results from this method and others, Table 2 shows absolute errors at different times. As you see, our results are better than the results of others. Figure 2 is plotted to show the accuracy of the method. It is noted that when N increases, the error reduces. The same results can be seen in Table 3, where the CPU time is also reported.

5. Conclusions

In this paper, a well-known fractional equation named the BT equation is solved using the collocation method. To implement the scheme, first, the unknown solution is mapped to wavelet space by ML wavelets. Then, using the operational matrices of CFD and derivative, the problem reduces to a linear system of algebraic equations via the collocation method. Various collocation points are considered, and the problem is solved effectively and accurately. The illustrated examples demonstrate that when the parameter eta is selected correctly, the method gives the exact solution. Otherwise, the expected results can be obtained by increasing the parameter N. Due to the flexibility of ML wavelets in selecting their basis power and the presented algorithm's structure, this method has a high potential to solve various fractional equations.

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TABLE 2. Assessing the accuracy of the method compared to other methods (Example 4.2).

	Presented method	[18]	[7]
t	$\eta = 0.5, N = 16$	$\eta = 0.5, N = 16$	$N = 16$
0.1	2.74×10^{-12}	7.60×10^{-10}	1.93×10^{-06}
0.2	4.16×10^{-12}	1.00×10^{-10}	4.90×10^{-06}
0.3	4.98×10^{-12}	1.00×10^{-10}	8.40×10^{-06}
0.4	5.27×10^{-12}	2.00×10^{-10}	1.28×10^{-05}
0.5	5.16×10^{-12}	7.00×10^{-10}	2.13×10^{-05}
0.6	4.63×10^{-12}	6.00×10^{-09}	3.16×10^{-05}
0.7	3.83×10^{-12}	1.70×10^{-08}	4.42×10^{-05}
0.8	2.77×10^{-12}	4.30×10^{-08}	5.43×10^{-05}
0.9	1.49×10^{-12}	1.01×10^{-07}	1.22×10^{-04}

TABLE 3. The L^2-error for $\eta = 0.5$ (Example 4.2).

		$N = 8$	$N = 10$	$N = 12$	$N = 14$	$N = 16$
Chebyshev nodes	L^2-error	3.02×10^{-05}	2.64×10^{-06}	3.51×10^{-08}	1.47×10^{-10}	5.28×10^{-13}
	CPU time	2.531	3.391	4.547	8.718	22.016
Legendre nodes	L^2-error	6.68×10^{-06}	1.07×10^{-06}	1.18×10^{-07}	4.73×10^{-11}	3.88×10^{-12}
	CPU time	2.547	3.046	4.547	9.125	19.016
Uniform meshes	L^2-error	4.00×10^{-05}	2.15×10^{-06}	2.07×10^{-07}	8.60×10^{-10}	1.54×10^{-10}
	CPU time	2.688	3.110	4.609	8.172	21.062

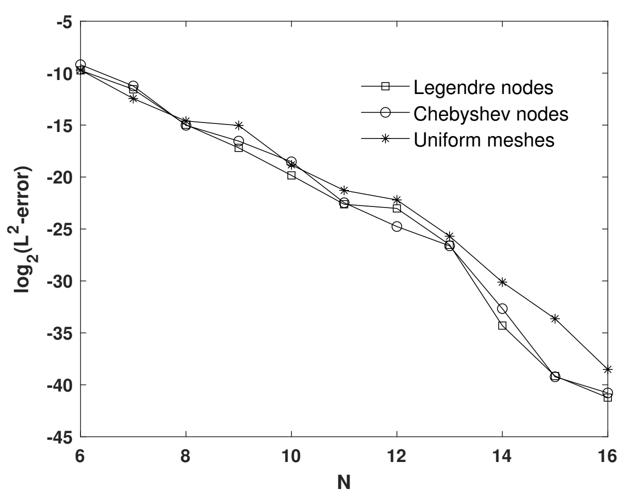

FIGURE 2. The obtained L^2–error via different values of N for Example 4.2.

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