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Impact of Fractional Order on Reaction Rates: Solutions to Kinetic Equations with Incomplete ℵ-Function

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Abstract

In this study, we investigate the significance of fractional kinetic equations in emerging a wide range of problems in science and engineering. Specifically, we derive a fractional kinetic equation solution involving the incomplete ℵ function using a well-established integral transform technique. To illustrate the impact of the fractional integral operator's order on reaction rates, we present several graphical results, highlighting the influence of fractional calculus on the system's dynamics.

Keywords. Laplace Transform, Riemann-Liouville Fractional Integral, Kinetic Equation, Incomplete ℵ-Function.

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1. Introduction

Over the years, many mathematicians have proposed and studied various special functions, such as the gamma, beta, and hypergeometric functions. These functions have been extensively researched due to their crucial roles in both theoretical mathematics and applications in engineering and the applied sciences [4, 5, 19]. The incomplete gamma function, for instance, was introduced by Legendre [11] and Schlömilch [17] in the 18th century and further developed by Tannery [23] in 1882 and Prym [14] in 1887. Following these foundational works, numerous mathematicians have conducted in-depth studies on the incomplete gamma function, expanding its applicability. In the 19th century, Sudland et al. [22] proposed and explored the ℵ-function, yielding significant results and uncovering its potential applications in fields such as applied mathematics and physics. This function has since attracted considerable attention due to its broad utility in addressing complex problems across various scientific disciplines.

The significance of fractional differential equations is enhanced by incorporating a convolution integral with a power-law memory kernel through fractional derivatives, effectively capturing the memory effects inherent in complex systems. In recent decades, fractional calculus has found innovative and impactful applications across numerous scientific disciplines, including engineering, economics, biology, and physics [13, 18, 20]. Notable applications include modeling diffusion processes, image processing, and the advection-dispersion of solutes in fractured media, highlighting the versatility and utility of fractional calculus in addressing diverse and complex phenomena.

Fractional kinetic equations have aroused great interest among researchers due to the applications of kinetic equations in mathematical physics, control systems, and astrophysics [12]. These equations are widely used to model various physical processes, such as reaction dynamics, diffusion in porous media, and relaxation mechanisms in complex structures [1]. Introducing incomplete special functions in a new fractional generalization of the kinetic equation (KE) brings an innovative perspective to this field. In particular, this approach incorporates incomplete H-functions, I-functions, ℵ-functions, and various polynomial families into the equation.

Solutions for these fractional kinetic equations are derived using the Laplace transform technique, which effectively handles the complexity introduced by fractional operators. Additionally, selected examples are explored to demonstrate

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specific cases of interest, accompanied by graphical representations to illustrate how varying parameters influence the system's behavior, offering more profound insights into the dynamics of fractional kinetic processes.

A fractional kinetic equation was established by Houbold and Mathai [9] for the following rates of change: $\delta(\mathfrak{N}_{\mathfrak{t}})$, the rate of growth $\mathfrak{p}(\mathfrak{N}_{\mathfrak{t}})$, and the rate of change of reaction $\mathfrak{N} = \mathfrak{N}(\mathfrak{t})$.

$$\frac{d\mathfrak{N}}{dt} = -\delta(\mathfrak{N}_{\mathfrak{t}}) + \mathfrak{p}(\mathfrak{N}_{\mathfrak{t}}), \tag{1.1}$$

where $\mathfrak{N}_{\mathfrak{t}}$ is determined by $\mathfrak{N}_t(t^*) = \mathfrak{N}(t - t^*),\ t^* > 0$.

Furthermore, when the term $\mathfrak{N}(t)$, representing the magnitude of spatial fluctuations, is neglected and defined as:

$$\frac{d\mathfrak{N}_k}{dt} = -c_k \mathfrak{N}_k(t), \tag{1.2}$$

then Houbold and Mathai [9] provide the limiting case of (1.1). Where $\mathfrak{N}_k(t) = \mathfrak{N}_0$ is the amount of density of species k, $c_k > 0$ at time $t = 0$. If the kinetic Equation (1.2) is integrated and index k is leave, we get

$$\mathfrak{N}(t) - \mathfrak{N}_0(t) = -c_0 \mathfrak{D}_t^{-1} \mathfrak{N}(t),$$

where ${}_0\mathfrak{D}_t^{-1}$ represents a specific case of the Riemann-Liouville fractional operator ${}_0\mathfrak{D}_t^{-v}$. The operator ${}_0\mathfrak{D}_t^{-v}$ is defined as follows:

$${}_0\mathfrak{D}_t^{-v} f(t) = \frac{1}{\Gamma(v)} \int_0^t (t-s)^{v-1} f(s) ds. \tag{1.3}$$

By taking into account the fractional derivative instead of the total derivative in (1.2). Houbold and Mathai [9] introduced the fractional concept to the classical kinetic equation.

$$\mathfrak{N}(t) - \mathfrak{N}_0 = -c^v {}_0\mathfrak{D}_t^{-v} \mathfrak{N}(t),$$

subsequently, a Mittag-Leffler function $E_v(\cdot)$ is the solution for $\mathfrak{N}(t)$.

$$\mathfrak{N}(t) = \mathfrak{N}_0 \sum_{r=0}^{\infty} \frac{(-1)^r (ct)^{vr}}{\Gamma(vr+1)} = \mathfrak{N}_0 E_v(-c^v t^v).$$

Saxena and Kalla [16] considered the fractional kinetic equation that follows:

$$\mathfrak{N}(t) - \mathfrak{N}_0 f(t) = -c^v {}_0\mathfrak{D}_t^{-v} \mathfrak{N}(t),$$

where $f(t) \in L(0, \infty)$.

The Laplace transform of Riemann-Liouville fractional integral (1.3) is defined as follows:

$$L[{}_0\mathfrak{D}_t^{-v} f(t) : s] = s^{-v} F(s),\ t > 0,\ \Re(v) > 0, \Re(s) > 0, \tag{1.4}$$

where $F(s)$ is the Laplace transform of the function $f(t)$ and defined as:

$$L[f(t) : s] = F(s) = \int_0^{\infty} f(t) e^{-st} dt,$$

where s is the complex frequency domain parameter.

The lower and upper incomplete gamma function $\gamma(\mathrm{r}, z)$ and $\Gamma(\mathrm{r}, z)$ respectively, which are usual incomplete gamma function are expressed as follows (see [6]):

$$\gamma(\mathrm{r}, z) = \int_0^z \theta^{\mathrm{r}-1}\, e^{-\theta}\, d\theta, \qquad \big(\Re(\mathrm{r}) > 0,\ z \geq 0\big),$$

$$\Gamma(\mathrm{r}, z) = \int_z^{\infty} \theta^{\mathrm{r}-1}\, e^{-\theta}\, d\theta, \qquad \big(\Re(\mathrm{r}) > 0,\ z \geq 0\big),$$

and adhere to the following decomposition rule:

$$\gamma(\mathrm{r}, z) + \Gamma(\mathrm{r}, z) = \Gamma(\mathrm{r}), \qquad \big(\Re(\mathrm{r}) > 0\big),$$

where $\Re(\mathrm{r})$ indicates real part of the parameter r. Furthermore, by taking $z = 0$, one obtains $\Gamma(\mathrm{r}, z) = \Gamma(\mathrm{r})$.

Inspired by the applications of the incomplete gamma function, Bansal et al. [2] introduced the incomplete $\aleph$-function, which is defined as follows:

$$
\begin{aligned}
{}^{\Gamma}\aleph^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},\,\mathrm{P_i},\,r}(Z) &= {}^{\Gamma}\aleph^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},\,\mathrm{P_i},\,r}\left[Z \,\middle|\, \begin{matrix} (\mathfrak{u}_1, \mathfrak{U}_1; x), (\mathfrak{u}_j, \mathfrak{U}_j)_{2,n}, [\mathrm{P_i}(\mathfrak{u}_{ji}, \mathfrak{U}_{ji})]_{n+1}, \mathrm{p_i} \\ (g_j, G_j)_{1,m}, [\mathrm{P_i}(g_{ji}, G_{ji})]_{m+1}, \mathrm{q_i} \end{matrix} \right] \\
&= \frac{1}{2\pi i} \int_{\$} K(\psi, x) Z^{-\psi} d\psi,
\end{aligned} \tag{1.5}
$$

$$
\begin{aligned}
{}^{\gamma}\aleph^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},\,\mathrm{P_i},\,r}(Z) &= {}^{\gamma}\aleph^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},\,\mathrm{P_i},\,r}\left[Z \,\middle|\, \begin{matrix} (\mathfrak{u}_1, \mathfrak{U}_1; x), (\mathfrak{u}_j, \mathfrak{U}_j)_{2,n}, [\mathrm{P_i}(\mathfrak{u}_{ji}, \mathfrak{U}_{ji})]_{n+1}, \mathrm{p_i} \\ (g_j, G_j)_{1,m}, [\mathrm{P_i}(g_{ji}, G_{ji})]_{m+1}, \mathrm{q_i} \end{matrix} \right] \\
&= \frac{1}{2\pi i} \int_{\$} M(\psi, x) Z^{-\psi} d\psi,
\end{aligned} \tag{1.6}
$$

where

$$
K(\psi, x) = \frac{\Gamma(1 - \mathfrak{u}_1 + \mathfrak{U}_1\psi, x) \prod_{j=1}^{m} \Gamma(g_j + G_j\psi) \prod_{j=2}^{n} \Gamma(1 - \mathfrak{u}_j - \mathfrak{U}_j\psi)}{\sum_{i=1}^{r} \mathrm{P}_i \left[\prod_{j=m+1}^{q_i} \Gamma(1 - g_{ji} - G_{ji}\psi) \prod_{j=n+1}^{p_i} \Gamma(\mathfrak{u}_{ji} + \mathfrak{U}_{ji}\psi) \right]}, \tag{1.7}
$$

and

$$
M(\psi, x) = \frac{\gamma(1 - \mathfrak{u}_1 + \mathfrak{U}_1\psi, x) \prod_{j=1}^{m} \Gamma(g_j + G_j\psi) \prod_{j=2}^{n} \Gamma(1 - \mathfrak{u}_j - \mathfrak{U}_j\psi)}{\sum_{i=1}^{r} \mathrm{P}_i \left[\prod_{j=m+1}^{q_i} \Gamma(1 - g_{ji} - G_{ji}\psi) \prod_{j=n+1}^{p_i} \Gamma(\mathfrak{u}_{ji} + \mathfrak{U}_{ji}\psi) \right]}.
$$

The incomplete $\aleph$-functions satisfy the following decomposition property:

$$
{}^{\gamma}\aleph^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},\,\mathrm{P_i},r}(Z) + {}^{\Gamma}\aleph^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},\,\mathrm{P_i},r}(Z) = \aleph^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},\,\mathrm{P_i},r}(Z).
$$

The general class of polynomials of index n, where $n = 0, 1, 2, \cdots$, has been introduced by Srivastava [21] as follows:

$$
S_n^m(z) = \sum_{k=1}^{[n/m]} \frac{A_{n,k}}{\Gamma(k+1)} (-n)_{mk}\, z^k, \tag{1.8}
$$

where $A_{n,k} \in \mathbb{R}$(or $\mathbb{C}$) are arbitrary constants, $[\cdot]$ denotes the largest integer function, and $(-n)_m$ represents the Pochhammer symbol. The number of known polynomials as special instances of the coefficients $A_{n,k}$ is given by Srivastava's polynomials.

2. Generalized solution of fractional kinetic equations

Theorem 2.1. *If $\mu, \alpha, \eta, \beta, \phi > 0$ are assumed, then the solution of*

$$
N(t) - N_0(t) t^{\mu-1} S_n^m[\alpha t^{\phi}]\, {}^{\Gamma}\aleph^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},\,\mathrm{r_i},\,s}(\beta t^{\eta}) = -c^{v}\, {}_0\mathfrak{D}_t^{-v} N(t), \tag{2.1}
$$

is given as

$$
\begin{aligned}
N(t) = N(t_0) t^{\mu-1} \sum_{i=1}^{\infty} (-c^v t^v)^i \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} (\alpha t^{\phi})^k}{\Gamma(k+1)} A_{n,k} \\
\times\, {}^{\Gamma}\aleph^{m,n+1}_{\mathrm{p_i}+1,\mathrm{q_i}+1,\mathrm{r_i},s}\left[\beta t^{n} \,\middle|\, \begin{matrix} (\mathfrak{u}_1, \mathfrak{U}_1; x), (\mathfrak{u}_j, \mathfrak{U}_j)_{2,n}, (1-\mu-\phi k, \eta), [\mathrm{r_i}(\mathfrak{u}_{ji}, \mathfrak{U}_{ji})]_{n+1}, \mathrm{p_i} \\ (g_j, G_j)_{1,m}, (1-\mu-\phi k - vi, \eta), [\mathrm{r_i}(g_{ji}, G_{ji})]_{m+1}, \mathrm{q_i} \end{matrix} \right].
\end{aligned}
$$

Proof. The Laplace transform method has been used to prove the result. Taking the Laplace transform of both sides of Equation (2.1), we obtain

$$L[N(t):w] - L[N_0(t)t^{\mu-1}S_n^m[\alpha t^{\phi}]^{\Gamma}\aleph^{m,\,n}_{p_i,\,q_i,\,r_i,\,s}(\beta t^{\eta}):w] = L[-c^{v}{}_0\mathfrak{D}_t^{-v}N(t):w].$$

Now using the Equations (1.4), (1.5), and (1.8), one obtains

$$N(w) + c^v w^{-v}N(w) = N_0\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}(\alpha)^k}{\Gamma(k+1)}A_{n,k}\times\frac{1}{2\pi i}\int_{\$}K(\psi,x)\beta^{-\psi}\frac{\Gamma(\mu+\phi k-\eta\psi)}{w^{\mu+\phi k-\eta\psi}}d\psi,$$

$$[1+c^v w^{-v}]N(w) = N_0\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}(\alpha)^k}{\Gamma(k+1)}A_{n,k}\times\frac{1}{2\pi i}\int_{\$}K(\psi,x)\beta^{-\psi}\frac{\Gamma(\mu+\phi k-\eta\psi)}{w^{\mu+\phi k-\eta\psi}}d\psi,$$

$$N(w) = N_0\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}(\alpha)^k}{\Gamma(k+1)}A_{n,k}\times\frac{1}{2\pi i}\int_{\$}K(\psi,x)\beta^{-\psi}\frac{\Gamma(\mu+\phi k-\eta\psi)}{w^{\mu+\phi k-\eta\psi}}(1+c^v w^{-v})^{-1}d\psi, \tag{2.2}$$

where, $N(w) = L[N(t):w]$ and $K(\psi,x)$ is defined as in Equation (1.7). By applying the binomial expansion $(1+y)^{-1} = \sum_{k=0}^{\infty}(-1)^k y^k$ to Equation (2.2), we obtain:

$$N(w) = N_0\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}(\alpha)^k}{\Gamma(k+1)}A_{n,k}\times\frac{1}{2\pi i}\int_{\$}K(\psi,x)\beta^{-\psi}\Gamma(\mu+\phi k-\eta\psi)d\psi$$
$$\times\sum_{i=0}^{\infty}(-c^v)^i w^{-(\mu+\phi k-\eta\psi+vi)},$$

applying the inverse Laplace transform on the above equation, we have

$$N(t) = N_0\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}(\alpha)^k}{\Gamma(k+1)}A_{n,k}\times\frac{1}{2\pi i}\int_{\$}K(\psi,x)\beta^{-\psi}\Gamma(\mu+\phi k-\eta\psi)d\psi$$
$$\times\sum_{i=0}^{\infty}(-c^v)^i\frac{t^{(\mu+\phi k-\eta\psi+vi-1)}}{\Gamma(\mu+\phi k-\eta\psi+vi)}.$$

Ultimately, we obtain the needed outcomes by utilizing (1.5) therein. □

Theorem 2.2. *If $\mu,\alpha,\eta,\beta,\ \phi>0$, are assumed, then the solution of*

$$N(t) - N_0(t)t^{\mu-1}S_n^m[\alpha t^{\phi}]^{\gamma}\aleph^{m,\,n}_{p_i,\,q_i,\,r_i,\,s}(\beta t^{\eta}) = -c^{v}{}_0\mathfrak{D}_t^{-v}N(t),$$

is given as

$$N(t) = N(t_0)t^{\mu-1}\sum_{i=1}^{\infty}(-c^v t^v)^i\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}(\alpha t^{\phi})^k}{\Gamma k+1}A_{n,k}$$
$$\times\,{}^{\gamma}\aleph^{m,n+1}_{p_i+1,q_i+1,r_i,s}\left[\beta t^n\left|\begin{array}{c}(\mathfrak{u}_1,\mathfrak{U}_1;x),(\mathfrak{u}_1,\mathfrak{U}_1)_{2,n},(1-\mu-\phi k,\eta),[r_i(\mathfrak{u}_{ji},\mathfrak{U}_{ji})]_{n+1},p_i\\ \\ (g_j,G_j)_{1,m},(1-\mu-\phi k-vi,\eta),[r_i(g_{ji},G_{ji})]_{m+1},q_i\end{array}\right.\right].$$

Proof. The proof follows directly from Equations (1.6) and (1.8) and proceeds analogously to Theorem 2.1. Consequently, the proof is omitted here. □

(i) ℵ-Function: When $x=0$, Equation (1.5) simplifies to the ℵ-function proposed by Sudland [22]:

$${}^{\Gamma}\aleph^{m,\,n}_{p_i,\,q_i,\,P_i,\,r}\left[Z\left|\begin{array}{c}(\mathfrak{u}_1,\mathfrak{U}_1;0),(\mathfrak{u}_j,\mathfrak{U}_j)_{2,n},[P_i(\mathfrak{u}_{ji},\mathfrak{U}_{ji})]_{n+1},p_i\\(g_j,G_j)_{1,m},[P_i(g_{ji},G_{ji})]_{m+1},q_i\end{array}\right.\right] = \aleph^{m,\,n}_{p_i,\,q_i,\,P_i,\,r}\left[Z\left|\begin{array}{c}(\mathfrak{u}_j,\mathfrak{U}_j)_{1,n},[P_i(\mathfrak{u}_{ji},\mathfrak{U}_{ji})]_{n+1},p_i\\(g_j,G_j)_{1,m},[P_i(g_{ji},G_{ji})]_{m+1},q_i\end{array}\right.\right],$$

Corollary 2.3. *If $\mu, \alpha, \eta, \beta,\ \phi > 0$, are assumed, then the solution of*

$$N(t) - N_0(t)t^{\mu-1}S_n^m[\alpha t^{\phi}]\aleph^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},\,\mathrm{r_i},\,s}(\beta t^{\eta}) = -c^{v}{}_0\mathfrak{D}_t^{-v}N(t),$$

is given as

$$N(t) = N(t_0)t^{\mu-1}\sum_{i=1}^{\infty}(-c^v t^v)^i \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}(\alpha t^{\phi})^k}{\Gamma(k+1)} A_{n,k} \times \aleph^{m,n+1}_{\mathrm{p_i}+1,\mathrm{q_i}+1,\mathrm{r_i},s}\left[\beta t^{n} \,\middle|\, \begin{matrix} (\mathfrak{u}_{\mathrm{j}},\mathfrak{U}_{\mathrm{j}})_{1,n}, (1-\mu-\phi k,\eta), [\mathrm{r_i}(\mathfrak{u}_{\mathrm{ji}},\mathfrak{U}_{\mathrm{ji}})]_{n+1}, \mathrm{p_i} \\ (g_j, g_j)_{1,m}, (1-\mu-\phi k - vi,\eta), [\mathrm{r_i}(g_{ji}, G_{ji})]_{m+1}, \mathrm{q_i} \end{matrix}\right].$$

(ii) Incomplete *I*-Function: Equations (1.5) and (1.6) reduce to the I-function proposed by Bansal and Kumar when $\mathrm{P_i} = 1$ (see [3]):

$${}^{\Gamma}\aleph^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},1,r}\left[Z \,\middle|\, \begin{matrix} (\mathfrak{u}_1,\mathfrak{U}_1;x), (\mathfrak{u}_{\mathrm{j}},\mathfrak{U}_{\mathrm{j}})_{2,n}, [1\cdot(\mathfrak{u}_{\mathrm{ji}},\mathfrak{U}_{\mathrm{ji}})]_{n+1}, \mathrm{p_i} \\ (g_j, G_j)_{1,m}, [1\cdot(g_{ji}, G_{ji})]_{m+1}, \mathrm{q_i} \end{matrix}\right] = {}^{\Gamma}I^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},r}\left[Z \,\middle|\, \begin{matrix} (\mathfrak{u}_1,\mathfrak{U}_1;x), (\mathfrak{u}_{\mathrm{j}},\mathfrak{U}_{\mathrm{j}})_{2,n}, (\mathfrak{u}_{\mathrm{ji}},\mathfrak{U}_{\mathrm{ji}})_{n+1}, \mathrm{p_i} \\ (g_j, G_j)_{1,m}, (g_{ji}, G_{ji})_{m+1}, \mathrm{q_i} \end{matrix}\right],$$

and

$${}^{\gamma}\aleph^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},1,r}\left[Z \,\middle|\, \begin{matrix} (\mathfrak{u}_1,\mathfrak{U}_1;x), (\mathfrak{u}_{\mathrm{j}},\mathfrak{U}_{\mathrm{j}})_{2,n}, [1\cdot(\mathfrak{u}_{\mathrm{ji}},\mathfrak{U}_{\mathrm{ji}})]_{n+1}, \mathrm{p_i} \\ (g_j, G_j)_{1,m}, [1\cdot(g_{ji}, G_{ji})]_{m+1}, \mathrm{q_i} \end{matrix}\right] = {}^{\gamma}I^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},r}\left[Z \,\middle|\, \begin{matrix} (\mathfrak{u}_1,\mathfrak{U}_1;x), (\mathfrak{u}_{\mathrm{j}},\mathfrak{U}_{\mathrm{j}})_{2,n}, (\mathfrak{u}_{\mathrm{ji}},\mathfrak{U}_{\mathrm{ji}})_{n+1}, \mathrm{p_i} \\ (g_j, G_j)_{1,m}, (g_{ji}, G_{ji})_{m+1}, \mathrm{q_i} \end{matrix}\right].$$

Corollary 2.4. *If $\mu, \alpha, \eta, \beta,\ \phi > 0$ are assumed, then the solution of*

$$N(t) - N_0(t)t^{\mu-1}S_n^m[\alpha t^{\phi}]^{\Gamma}I^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},s}(\beta t^{\eta}) = -c^{v}{}_0\mathfrak{D}_t^{-v}N(t), \tag{2.3}$$

is given as

$$N(t) = N(t_0)t^{\mu-1}\sum_{i=1}^{\infty}(-c^v t^v)^i \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}(\alpha t^{\phi})^k}{\Gamma(k+1)} A_{n,k} \times {}^{\Gamma}I^{m,n+1}_{\mathrm{p_i}+1,\mathrm{q_i}+1,s}\left[\beta t^{n} \,\middle|\, \begin{matrix} (\mathfrak{u}_1,\mathfrak{U}_1;x), (\mathfrak{u}_{\mathrm{j}},\mathfrak{U}_{\mathrm{j}})_{2,n}, (1-\mu-\phi k,\eta), (\mathfrak{u}_{\mathrm{ji}},\mathfrak{U}_{\mathrm{ji}})_{n+1}, \mathrm{p_i} \\ (g_j, G_j)_{1,m}, (1-\mu-\phi k - vi,\eta), (g_{ji}, G_{ji})_{m+1}, \mathrm{q_i} \end{matrix}\right].$$

(iii) *I*-Function: Equation (1.5) simplifies to the I-function proposed by Saxena when $x = 0$ and $\mathrm{P_i} = 1$ (see [15]):

$${}^{\Gamma}\aleph^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},1,r}\left[Z \,\middle|\, \begin{matrix} (\mathfrak{u}_1,\mathfrak{U}_1;0), (\mathfrak{u}_{\mathrm{j}},\mathfrak{U}_{\mathrm{j}})_{2,n}, [1\cdot(\mathfrak{u}_{\mathrm{ji}},\mathfrak{U}_{\mathrm{ji}})]_{n+1}, \mathrm{p_i} \\ (g_j, G_j)_{1,m}, [1\cdot(g_{ji}, G_{ji})]_{m+1}, \mathrm{q_i} \end{matrix}\right] = I^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},r}\left[Z \,\middle|\, \begin{matrix} (\mathfrak{u}_{\mathrm{j}},\mathfrak{U}_{\mathrm{j}})_{1,n}, (\mathfrak{u}_{\mathrm{ji}},\mathfrak{U}_{\mathrm{ji}})_{n+1}, \mathrm{p_i} \\ (g_j, G_j)_{1,m}, (g_{ji}, G_{ji})_{m+1}, \mathrm{q_i} \end{matrix}\right],$$

Corollary 2.5. *If $\mu, \alpha, \eta, \beta,\ \phi > 0$, are assumed, then the solution of*

$$N(t) - N_0(t)S_n^m[\alpha t^{\phi}]I^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},s}(\beta t^{\eta}) = -c^{v}{}_0\mathfrak{D}_t^{-v}N(t),$$

is given as

$$N(t) = N(t_0)t^{\mu-1}\sum_{i=1}^{\infty}(-c^v t^v)^i \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}(\alpha t^{\phi})^k}{\Gamma k+1} A_{n,k} \times I^{m,n+1}_{\mathrm{p_i}+1,\mathrm{q_i}+1,s}\left[\beta t^{n} \,\middle|\, \begin{matrix} (\mathfrak{u}_{\mathrm{j}},\mathfrak{U}_{\mathrm{j}})_{1,n}, (1-\mu-\phi k,\eta), (\mathfrak{u}_{\mathrm{ji}},\mathfrak{U}_{\mathrm{ji}})_{n+1}, \mathrm{p_i} \\ (g_j, G_j)_{1,m}, (1-\mu-\phi k - vi,\eta), (g_{ji}, G_{ji})_{m+1}, \mathrm{q_i} \end{matrix}\right].$$

3. Remark and applications

A few implications and uses of the findings above are addressed in this section. By properly specializing the Srivastava polynomial to produce a wide number of existing series, certain unique instances of the resultant discoveries can be developed. We look at the following instances to illustrate this:

Example 3.1. Prove that the solution of

$$N(t) - N_0(t)t^{\mu-1}\,{}^{\Gamma}\aleph^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},\,\mathrm{r_i},\,s}(\beta t^{\eta}) = -c^{v}\,{}_0\mathfrak{D}_t^{-v}N(t), \tag{3.1}$$

is given as

$$N(t) = N(t_0)t^{\mu-1}\sum_{i=1}^{\infty}(-c^v t^v)^i \times {}^{\Gamma}\aleph^{m,n+1}_{\mathrm{p_i}+1,\mathrm{q_i}+1,\mathrm{r_i},s}\left[\beta t^{n}\,\middle|\, \begin{array}{c} (\mathfrak{u}_1,\mathfrak{U}_1;x),(\mathfrak{u}_j,\mathfrak{U}_j)_{2,n},(1-\mu,\eta),[\mathrm{r_i}(\mathfrak{u}_{ji},\mathfrak{U}_{ji})]_{n+1},\mathrm{p_i} \\ (g_j,G_j)_{1,m},(1-\mu-vi,\eta),[\mathrm{r_i}(g_{ji},G_{ji})]_{m+1},\mathrm{q_i} \end{array}\right].$$

Solution. By setting $m=1$, $a=1$, $\phi=0$, and defining $A_{n,k}=\frac{k!}{(-n)_{mk}}$ for $k=0$ and $A_{n,k}=0$ for $k\neq 0$ in Equation (2.1), we derive the assertion (3.1) from the example that follows Theorem 2.1.

Example 3.2. Prove that the solution of

$$N(t) - N_0(t)t^{\mu-1}L_n^{\lambda}(\alpha t^{\phi})\,{}^{\Gamma}\aleph^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},\,\mathrm{r_i},\,s}(\beta t^{\eta}) = -c^{v}\,{}_0\mathfrak{D}_t^{-v}N(t) \tag{3.2}$$

is given as

$$N(t) = N(t_0)t^{\mu-1}\sum_{i=1}^{\infty}(-c^v t^v)^i\sum_{k=0}^{n}\begin{pmatrix} n+\lambda \\ n-k\end{pmatrix}\frac{(-\alpha t^{\phi})^k}{\Gamma(k+1)} \times {}^{\Gamma}\aleph^{m,n+1}_{\mathrm{p_i}+1,\mathrm{q_i}+1,\mathrm{r_i},s}\left[\beta t^{n}\,\middle|\, \begin{array}{c} (\mathfrak{u}_1,\mathfrak{U}_1;x),(\mathfrak{u}_j,\mathfrak{U}_j)_{2,n},(1-\mu-\phi k,\eta),[r_i(\mathfrak{u}_{ji},\mathfrak{U}_{ji})]_{n+1},\mathrm{p_i} \\ (g_j,G_j)_{1,m},(1-\mu-\phi k-vi,\eta),[\mathrm{r_i}(g_{ji},G_{ji})]_{m+1},\mathrm{q_i} \end{array}\right].$$

Solution. If we take $A_{n,k}=\begin{pmatrix} n+\lambda \\ n\end{pmatrix}\frac{1}{(\lambda+1)_n}$ and $m=1$ (i.e $S_n^m(z)=L_n^{\lambda}(z)$, where $L_n^{\lambda}(z)$ is Laguerre polynomial) in Equation (2.1). The assertion (3.2) pertains to the example of the theorem that follows Theorem 2.1.

Example 3.3. Prove that the solution of

$$N(t) - N_0(t)t^{\mu-1}H_n\Big(\frac{1}{2\sqrt{t}}\Big)\,{}^{\Gamma}\aleph^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},\,\mathrm{r_i},\,s}(\beta t^{\eta}) = -c^{v}\,{}_0\mathfrak{D}_t^{-v}N(t) \tag{3.3}$$

is given as

$$N(t) = N(t_0)t^{\mu-1}\sum_{i=1}^{\infty}(-c^v t^v)^i\sum_{k=0}^{[n/2]}\frac{(-1)^k}{k!}\cdot\frac{t^k}{(n-2k)!} \times {}^{\Gamma}\aleph^{m,n+1}_{\mathrm{p_i}+1,\mathrm{q_i}+1,\mathrm{r_i},s}\left[\beta t^{n}\,\middle|\, \begin{array}{c} (\mathfrak{u}_1,\mathfrak{U}_1;x),(\mathfrak{u}_j,\mathfrak{U}_j)_{2,n},(1-\mu-k,\eta),[\mathrm{r_i}(\mathfrak{u}_{ji},\mathfrak{U}_{ji})]_{n+1},\mathrm{p_i} \\ (g_j,G_j)_{1,m},(1-\mu-k-vi,\eta),[\mathrm{r_i}(g_{ji},G_{ji})]_{m+1},\mathrm{q_i} \end{array}\right].$$

$$N(t) - N_0(t)t^{\mu-1}H_n\Big(\frac{1}{2\sqrt{t}}\Big)\,{}^{\Gamma}I^{m,\,n}_{\mathrm{p_i},\,\mathrm{q_i},\,s}(\beta t^{\eta}) = -c^{v}\,{}_0D_t^{-v}N(t), \tag{3.4}$$

is given as

$$N(t) = N(t_0)t^{\mu-1}\sum_{i=1}^{\infty}(-c^v t^v)^i \sum_{k=0}^{[n/2]} \frac{(-1)^k t^k}{k!} \cdot \frac{1}{(n-2k)!}$$
$$\times\, {}^{\Gamma}I^{m,n+1}_{\mathrm{p_i}+1,\mathrm{q_i}+1,s}\left[\beta t^n \left| \begin{matrix} (\mathfrak{u}_1, \mathfrak{U}_1; x), (\mathfrak{u}_j, \mathfrak{U}_j)_{2,n}, (1-\mu-k,\eta), (\mathfrak{u}_{ji}, \mathfrak{U}_{ji})_{n+1}, \mathrm{p_i} \\ (g_j, G_j)_{1,m}, (1-\mu-k-vi,\eta), (g_{ji}, G_{ji})_{m+1}, \mathrm{q_i} \end{matrix} \right. \right].$$

Solution. If we take $a = 1$, $m = 2$, $\phi = 1$, and $A_{n,k} = (-1)^k$ (i.e., $S_n^2[t] = t^{n/2}H_n\left(\frac{1}{2\sqrt{2}}\right)$ in (2.1) and (2.3). The assertions (3.3) and (3.4) correspond to the examples of the theorem and corollary that follow Theorem 2.1 and Corollary 2.4, respectively.

Remark 3.4. The result reported by Jangid et al. is observed when the incomplete ℵ-function simplifies to an incomplete H-function in Equation (2.1) (see [10]).

Remark 3.5. The outcome reported by Choi and Kumar occurs when the incomplete ℵ-function simplifies to Fox's H-function in Equation (2.1) (see [7]).

4. Numerical results and discussion

In this section, we employ MATLAB to simulate the numerical results for the fractional kinetic equation across various parameter values, as depicted in Figures 1, 2, and 3. These figures illustrate that the value of $N(t)$ initially varies proportionally with the increase in v but ultimately adopts a different form.

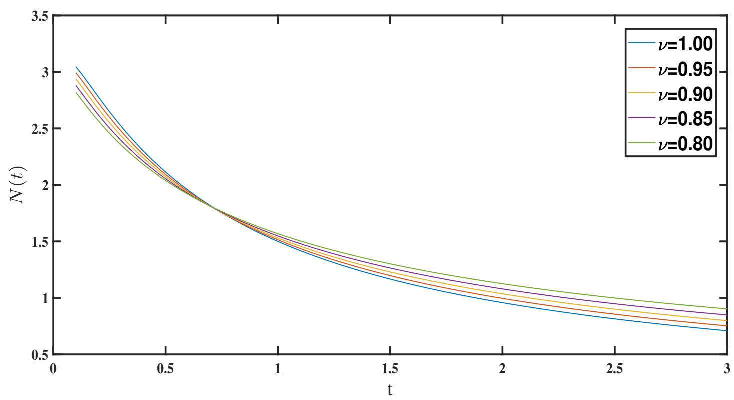

Figure 1. Plots of the fractional kinetic Equation (2.1) with $N_0(t) = 3, c = 2$.

Figures 1 to 3 offer insights into the temporal dynamics of the system by illustrating how the reaction rates evolve over time. Additionally, the graphs emphasize the influence of the order of the Riemann-Liouville integral operator, providing critical information about its impact on the system's behavior. The observed trends in reaction rates about the fractional parameter v underscore the system's sensitivity to this parameter, highlighting its significance in regulating the system's dynamics. Moreover, by identifying convergence regions, the graphs contribute to stability analysis, aiding in assessing the model's reliability. These findings enhance our predictive understanding of system behavior under varying conditions and suggest broad generalizations applicable across diverse scenarios, increasing their relevance in scientific and engineering contexts. Overall, the graphs serve as essential tools for comprehending the complexities of fractional kinetic equations, guiding further research, and informing engineering applications reliant on reaction kinetics.

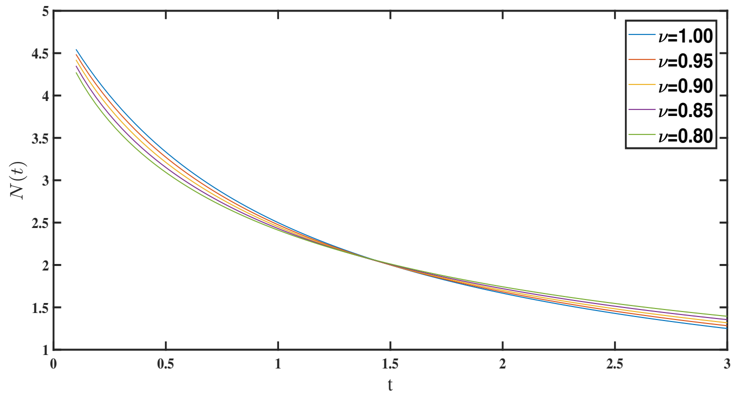

FIGURE 2. Plots of the fractional kinetic Equation (2.1) with $N_0(t) = 4.5, c = 0.8$.

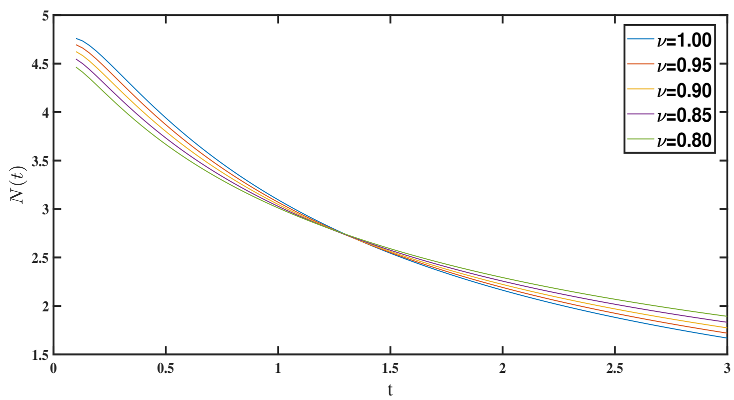

FIGURE 3. Plots of the fractional kinetic Equation (2.1) with $N_0(t) = 5, c = 1$.

5. CONCLUSION

This study presents generalized fractional kinetic equations associated with the incomplete ℵ-function, with solutions expressed in terms of this function. We propose a novel, computable extension of fractional kinetic equations through the incomplete ℵ-function, complemented by numerical results presented graphically, underscoring the novelty and significance of this work. The main findings are encapsulated in Theorems 2.1 and 2.2, and further specific cases are derived by assigning particular values to the parameters of the incomplete ℵ-function, yielding several well-known and essential results.

This study's outcomes present valuable implications for space research and astrophysics, where accurate modeling of fractional processes is essential. The insights gained from this work deepen our understanding of kinetic systems governed by fractional operators and establish a foundation for future studies exploring applications in physics, engineering, and beyond. The results lay a robust groundwork for continued research in fractional calculus applications, particularly for systems where conventional models may fail to capture intricate, memory-dependent dynamics.

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