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Power series solutions of fractional Lotka-Volterra equation

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Abstract

In this paper, the power series method is applied to fractional Lotka-Volterra equation, which is one of the most famous competition models emerging in demography and economics. We obtain some power series solutions of the governing equation and prove their convergence. In addition, we analyze the various types of competitive roles depicted by this model through the truncated graphs of these power series solutions. From the graphs, we can find that the fractional order affects the speed of population growth or decrease, and this effect can be seen as continuous with respect to the order.

Keywords. Power series method, Fractional differential equations, Lotka-Volterra model.
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1. Introduction

Nonlinear differential equations are important tools for describing complex phenomena in fields such as physics, engineering, biology, and economics. One of them, the well known population model namely the logistic equation is given by

$$u'(t) = Au(t) - Bu^2(t), \tag{1.1}$$

which is introduced by Verhulst in 1838 [22]. Due to the introduction of a nonlinear term, Eq. (1.1) effectively quenches the unbounded growth in human population model proposed by Malthus [14] and alleviates the pessimistic prediction of human fate brought about by Malthus' prophecy [1, 23]. In addition, the predator-prey population model is used to describe the growth rate of competing biological species. The most famous competition models are given by Lotka and Volterra, who introduce two coupling constants C_1 and C_2 to describe the interaction between two competitors and transform the simple logistic equation into the following Volterra-Lotka coupled differential equations:

$$\begin{aligned} u' &= A_1u - B_1u^2 + C_1uv, \\ v' &= A_2v - B_2v^2 + C_2vu, \end{aligned} \tag{1.2}$$

where u and v are the population of two different biological species. This system of equations contains all fundamental parameters that impact the rate of growth. Among them, A_i represents the reproductive capacity of each species, B_i the niche capacity limitations related to the niche size, and C_i the interaction with other species. The signs of C_1 and C_2 determine the type of competitive roles shown in Table 1 [16].

Fractional differential equations (FDEs), due to nonlocal and memory effects of fractional derivative and integral [7, 12, 19, 20], are successfully used in various fields of natural science and engineering [6, 24–29]. Moreover, in recent years, various forms of fractional logistic and Lotka-Volterra equations have been studied by many scholars using different analytical and numerical methods [4, 5, 8, 11, 13, 15, 17, 18, 21]. Especially, Pareek et al. [18] obtained the analytical approximate solution in the form of convergent infinite series for fractional deterministic Lotka-Volterra

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TABLE 1. Signs of C_1 and C_2 determine the type of competitive roles

$C_1 C_2$	Type	Explanation
$-\ -$	Pure competition	Occurs when both species suffer from each other's existence
$+\ -$	Predator-prey	Occurs when one of them serves as direct food to the other
$+\ +$	Mutualism	Occurs in case of symbiosis or a win-win situation
$+\ 0$	Commensalism	Occurs when one benefits from the existence of the other, who nevertheless remains unaffected
$-\ 0$	Amensalism	Occurs when one suffers from the existence of the other, who is impervious to what is happening
$0\ \ 0$	Neutralism	Occurs if there is no interaction whatsoever

model by homotopy analysis method. Manohar et al. [15] used fractional reduced differential transform method to solve two cases of the Lotka–Volterra model with the Caputo fractional order derivatives numerically. In this paper, we study the following coupled fractional Volterra-Lotka equations:

$$\begin{aligned} D^\alpha u &= A_1 u - B_1 u^2 + C_1 uv, \\ D^\alpha v &= A_2 v - B_2 v^2 + C_2 vu, \end{aligned} \tag{1.3}$$

by the power series method which is applied by Area and Nieto to study fractional logistic and Allee logistic equations [2, 3]. We extended their work by applying this method to the coupled fractional Volterra-Lotka equations, and obtained some novel analytical solutions that differ from the results in [4, 5, 8, 11, 13, 15, 17, 18, 21]. In addition, we used these solutions with different fractional order to comparatively analyze all the types of competitive roles in Table 1.

It should be noted that D^α denotes the following Caputo fractional derivative for an absolutely continuous function $f(t)$:

$$D^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} \mathrm{d}s, \quad 0 < t, \quad 0 < \alpha < 1. \tag{1.4}$$

From the definition, we can get the Caputo fractional derivative of a constant is zero, and that of a power function is

$$D^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, \quad \gamma > -1. \tag{1.5}$$

For other properties of the Caputo fractional derivative, we can refer to the monographs [7, 12, 19, 20]. The Gamma function $\Gamma(z) = \int_a^\infty e^{-z} t^{z-1} \mathrm{d}t$ is extensively used in this paper, and we can refer to some other important special functions and their applications in [9].

The motivation of this paper is to extend the classical Volterra-Lotka coupled differential equations to their corresponding fractional order forms, which can more accurately characterize the genetic effects and long-range dependencies of population density changes. The main contributions of our work is the first application of power series method to fractional Volterra-Lotka coupled differential equations and obtaining some analytical solutions with potential application value. The rest of this paper is organized as follows. In Section 2, we apply the power series method to obtain some convergent power series solutions and analyze the dynamic properties of these solutions graphically for the six types of competitive role in Table 1. In Section 3, the conclutions and outlooks are presented.

2. Methods and Results

In this section, we obtained some convergent power series solutions of Eq. (1.3) using the power series method, and discussed all types of competitive roles listed in Table 1 through the graphs of these solutions.

Let us assume

$$u(t)=\sum_{n=0}^{\infty}a_n(t^{\alpha})^n,\quad v(t)=\sum_{n=0}^{\infty}b_n(t^{\alpha})^n, \tag{2.1}$$

then

$$u^2=\sum_{n=0}^{\infty}\Big(\sum_{j=0}^{n}a_ja_{n-j}\Big)(t^{\alpha})^n,\quad v^2=\sum_{n=0}^{\infty}\Big(\sum_{j=0}^{n}b_jb_{n-j}\Big)(t^{\alpha})^n, \tag{2.2}$$

$$uv=\sum_{n=0}^{\infty}\Big(\sum_{j=0}^{n}a_jb_{n-j}\Big)(t^{\alpha})^n,\quad vu=\sum_{n=0}^{\infty}\Big(\sum_{j=0}^{n}b_ja_{n-j}\Big)(t^{\alpha})^n. \tag{2.3}$$

From Eq. (1.5), we can get

$$D^{\alpha}u=\sum_{n=0}^{\infty}\frac{\Gamma\big((n+1)\alpha+1\big)}{\Gamma(n\alpha+1)}a_{n+1}(t^{\alpha})^n,\quad D^{\alpha}v=\sum_{n=0}^{\infty}\frac{\Gamma\big((n+1)\alpha+1\big)}{\Gamma(n\alpha+1)}b_{n+1}(t^{\alpha})^n. \tag{2.4}$$

Substituting (2.1)–(2.4) into Eq. (1.3) and equating the coefficients of different powers of t^{α} yields that

$$\begin{aligned}&\frac{\Gamma\big((n+1)\alpha+1\big)}{\Gamma(n\alpha+1)}a_{n+1}=A_1a_n-B_1\sum_{j=0}^{n}a_ja_{n-j}+C_1\sum_{j=0}^{n}a_jb_{n-j},\\&\frac{\Gamma\big((n+1)\alpha+1\big)}{\Gamma(n\alpha+1)}b_{n+1}=A_2b_n-B_2\sum_{j=0}^{n}b_jb_{n-j}+C_2\sum_{j=0}^{n}b_ja_{n-j},\end{aligned} \tag{2.5}$$

from which, we can obtain the explicit expressions of a_n and b_n. For $k\geq 0$, we have

$$\begin{aligned}a_{n+1}&=\frac{\Gamma(n\alpha+1)}{\Gamma\big((n+1)\alpha+1\big)}\Big(A_1a_n-B_1\sum_{j=0}^{n}a_ja_{n-j}+C_1\sum_{j=0}^{n}a_jb_{n-j}\Big),\\b_{n+1}&=\frac{\Gamma(n\alpha+1)}{\Gamma\big((n+1)\alpha+1\big)}\Big(A_2b_n-B_2\sum_{j=0}^{n}b_jb_{n-j}+C_2\sum_{j=0}^{n}b_ja_{n-j}\Big),\end{aligned} \tag{2.6}$$

with the initial conditions $a_0=u(0)$ and $b_0=v(0)$. Therefore, the power series solutions of Eq. (1.3) are

$$\begin{aligned}u(t)&=a_0+\sum_{n=0}^{\infty}\frac{\Gamma(n\alpha+1)(t^{\alpha})^{n+1}}{\Gamma\big((n+1)\alpha+1\big)}\Big(A_1a_n-B_1\sum_{j=0}^{n}a_ja_{n-j}+C_1\sum_{j=0}^{n}a_jb_{n-j}\Big),\\v(t)&=b_0+\sum_{n=0}^{\infty}\frac{\Gamma(n\alpha+1)(t^{\alpha})^{n+1}}{\Gamma\big((n+1)\alpha+1\big)}\Big(A_2b_n-B_2\sum_{j=0}^{n}b_jb_{n-j}+C_2\sum_{j=0}^{n}b_ja_{n-j}\Big).\end{aligned} \tag{2.7}$$

Theorem 2.1. *The power series solutions (2.7) are convergent in a neighborhood of $t=0$.*

Proof. Considering Eqs. (2.6), we can obtain

$$\begin{aligned}|a_{n+1}|&\leq\frac{\Gamma(n\alpha+1)}{\Gamma\big((n+1)\alpha+1\big)}\Big(|A_1||a_n|+|B_1|\sum_{j=0}^{n}|a_j||a_{n-j}|+|C_1|\sum_{j=0}^{n}|a_j||b_{n-j}|\Big),\\|b_{n+1}|&\leq\frac{\Gamma(n\alpha+1)}{\Gamma\big((n+1)\alpha+1\big)}\Big(|A_2||b_n|+|B_2|\sum_{j=0}^{n}|b_j||b_{n-j}|+|C_2|\sum_{j=0}^{n}|b_j||a_{n-j}|\Big).\end{aligned} \tag{2.8}$$

From the properties of Gamma function, $\Gamma(\xi)$ is an increasing function for $\xi\geq 2$. That is to say, if a positive integer N_{α} satisfies $N_{\alpha}\alpha+1\geq 2$ and $(N_{\alpha}-1)\alpha+1<2$, then for $n\geq N_{\alpha}$,

$$\Gamma\big((n+1)\alpha+1\big)>\Gamma(n\alpha+1)>\Gamma(2)=1. \tag{2.9}$$

So $\frac{\Gamma(n\alpha+1)}{\Gamma\big((n+1)\alpha+1\big)} < 1$ for arbitrary $n \geq N_\alpha$. When $n < N_\alpha$, the inequality $0.5 < \Gamma(n\alpha+1) < 1$ holds, that is, $\frac{\Gamma(n\alpha+1)}{\Gamma\big((n+1)\alpha+1\big)} < 2$. Therefore, Eqs. (2.8) can be written as

$$\begin{aligned} |a_{n+1}| &\leq M\Big(|a_n| + \sum_{j=0}^{n}|a_j||a_{n-j}| + \sum_{j=0}^{n}|a_j||b_{n-j}|\Big),\\ |b_{n+1}| &\leq M\Big(|b_n| + \sum_{j=0}^{n}|b_j||b_{n-j}| + \sum_{j=0}^{n}|b_j||a_{n-j}|\Big), \end{aligned} \tag{2.10}$$

where $M = \max\{2|A_i|, 2|B_i|, 2|C_i|\}$.

Consider another power series

$$P(t) = \sum_{n=0}^{\infty} p_n (t^\alpha)^n, \quad Q(t) = \sum_{n=0}^{\infty} q_n (t^\alpha)^n, \tag{2.11}$$

where $p_0 = |a_0|$, $q_0 = |b_0|$, and

$$p_{n+1} = M\Big(p_n + \sum_{j=0}^{n} p_j p_{n-j} + \sum_{j=0}^{n} p_j q_{n-j}\Big), \quad n \geq 0, \tag{2.12}$$

$$q_{n+1} = M\Big(q_n + \sum_{j=0}^{n} q_j q_{n-j} + \sum_{j=0}^{n} q_j p_{n-j}\Big), \quad n \geq 0. \tag{2.13}$$

Therefore, it is easily seen that $|a_n| \leq p_n$ and $|b_n| \leq q_n$ for $n = 0, 1, 2, \ldots$, that is, the power series (2.11) are the majorant series of (2.1). We next show that the power series (2.11) are convergent. By simple calculation, we can get

$$P(t) = p_0 + Mt^\alpha\big(P(t) + P^2(t) + P(t)Q(t)\big), \tag{2.14}$$

$$Q(t) = q_0 + Mt^\alpha\big(Q(t) + Q^2(t) + Q(t)P(t)\big). \tag{2.15}$$

Consider the following implicit function with respect to the independent variable t:

$$F(t, P, Q) = P - p_0 - Mt^\alpha(P + P^2 + PQ), \tag{2.16}$$

$$G(t, P, Q) = Q - q_0 - Mt^\alpha\big(Q + Q^2 + QP\big), \tag{2.17}$$

which are analytic in a neighborhood of $(0, p_0, q_0)$, and $F(0, p_0, q_0) = 0$, $G(0, p_0, q_0) = 0$, $\frac{\partial(F,G)}{\partial(P,Q)}|_{(0,p_0,q_0)} = 1 \neq 0$. Therefore, by implicit function theorem, the power series (2.11) are analytic in a neighborhood of $t = 0$. It implies that the power series solution (2.1) are convergent in a neighborhood of $t = 0$. This completes the proof. □

What follows is an analysis of the types of competitive role listed in Table 1 based on the obtained power series solutions. In order to draw some truncated graphs of the power series solutions (2.7), the first few terms of them are given below:

$$\begin{aligned} a_1 &= \frac{1}{\Gamma(\alpha+1)}\Big(A_1 a_0 - B_1 a_0^2 + C_1 a_0 b_0\Big),\\ b_1 &= \frac{1}{\Gamma(\alpha+1)}\Big(A_2 b_0 - B_2 b_0^2 + C_2 b_0 a_0\Big), \end{aligned} \tag{2.18}$$

$$\begin{aligned} a_2 &= \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\Big(A_1 a_1 - 2B_1 a_0 a_1 + C_1(a_0 b_1 + b_0 a_1)\Big),\\ b_2 &= \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\Big(A_2 b_1 - 2B_2 b_0 b_1 + C_2(b_0 a_1 + a_0 b_1)\Big), \end{aligned} \tag{2.19}$$

$$a_3 = \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\Big(A_1 a_2 - B_1(2a_0a_2 + a_1^2) + C_1(a_0b_2 + a_1b_1 + a_2b_0)\Big),$$
$$b_3 = \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}\Big(A_2 b_2 - B_2(2b_0b_2 + b_1^2) + C_2(b_0a_2 + b_1a_1 + b_2a_0)\Big), \tag{2.20}$$

$$a_4 = \frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}\Big(A_1 a_3 - 2B_1(a_0a_3 + a_1a_2) + C_1(a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)\Big),$$
$$b_4 = \frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}\Big(A_2 b_3 - 2B_2(b_0b_3 + b_1b_2) + C_2(b_0a_3 + b_1a_2 + b_2a_1 + b_3a_0)\Big), \tag{2.21}$$

$$a_5 = \frac{\Gamma(4\alpha+1)}{\Gamma(5\alpha+1)}\Big(A_1 a_4 - B_1(2a_0a_4 + 2a_1a_3 + a_2^2) + C_1(a_0b_4 + a_1b_3 + a_2b_2 + a_3b_1 + a_4b_0)\Big),$$
$$b_5 = \frac{\Gamma(4\alpha+1)}{\Gamma(5\alpha+1)}\Big(A_2 b_4 - B_2(2b_0b_4 + 2b_1b_3 + b_2^2) + C_2(b_0a_4 + b_1a_3 + b_2a_2 + b_3a_1 + b_4a_0)\Big). \tag{2.22}$$

Assuming that the initial values and related parameters are fixed, we mainly focus on how the signs of parameters C_i determine the interaction between two species. The following graphs represent the corresponding types of competitive roles in Table 1. They are truncated graphs of the first six terms ($n = 0, \ldots, 5$) of power series solutions (2.7) in these types. Figures 1 and 2 illustrate the dynamic behavior for the power series solution (2.7) for mutualism and pure competition between two species with the same initial and parameters values, respectively. It can be seen that the fractional order affects the speed of population growth or decrease, and this effect can be seen as continuous with respect to order α. Therefore, we can choose an appropriate fractional order model based on real-world observation data, rather than just having a single first-order model. Figures 3 to 6 reflect the trend of population changes for two species with different initial and parameters values under the remaining four types in Table 1. From these graphs, it can be seen that although the given initial and parameter values are not actual data, they roughly reflect the trend of changes in the population of two species for the six types of competitive role in Table 1.

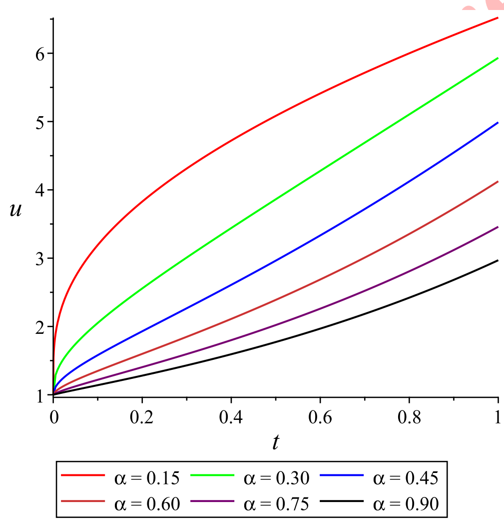

FIGURE 1. Mutualism between two species with $a_0 = b_0 = 1$, $A_i = B_i = C_i = 1$ for different fractional orders.

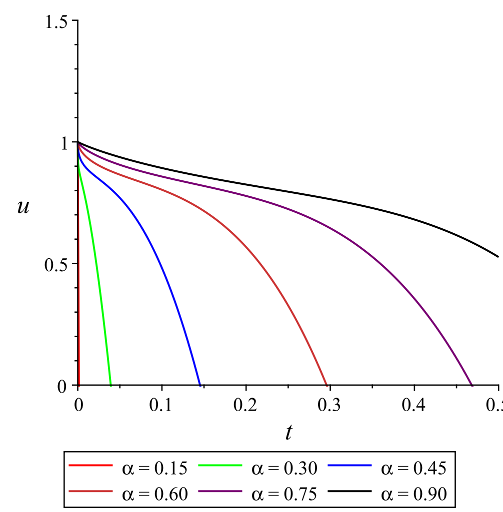

FIGURE 2. Pure competition between two species with $a_0 = b_0 = 1$, $A_i = B_i = 1$, $C_i = -1$ for different fractional orders.

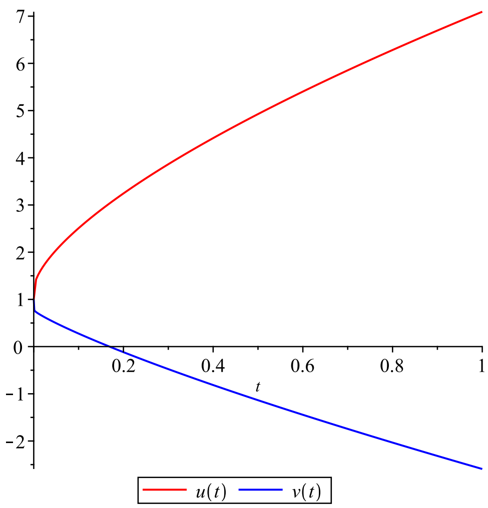

FIGURE 3. Predator-prey between two species with $a_0 = b_0 = 1$, $A_i = B_i = 1$, $C_1 = 1$, $C_2 = -1$ for $\alpha = 0.15$.

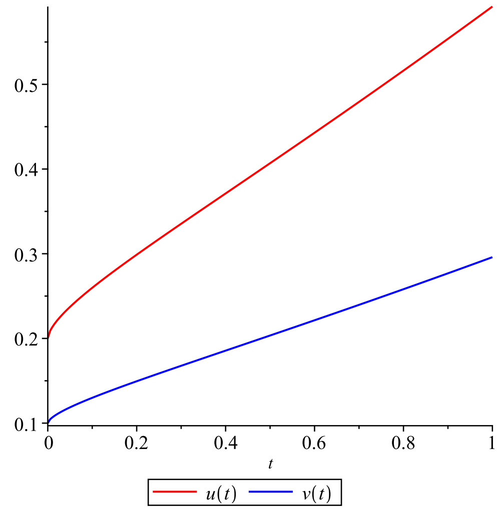

FIGURE 4. Commensalism between two species with $a_0 = 0.2$, $b_0 = 0.1$, $A_i = B_i = 1$, $C_1 = 1$, $C_2 = 0$ for $\alpha = 0.60$.

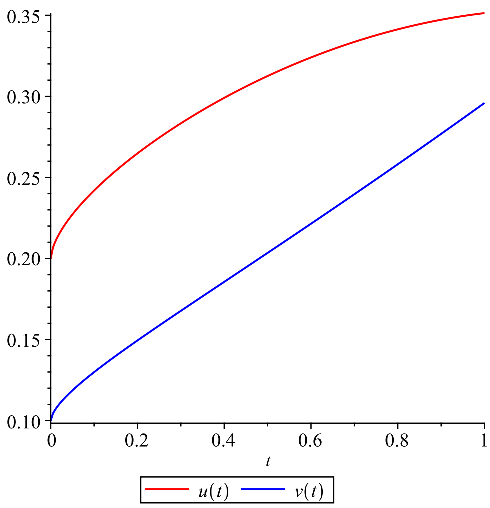

FIGURE 5. Amensalism between two species with $a_0 = 0.2$, $b_0 = 0.1$, $A_i = B_i = 1$, $C_1 = -1$, $C_2 = 0$ for $\alpha = 0.60$.

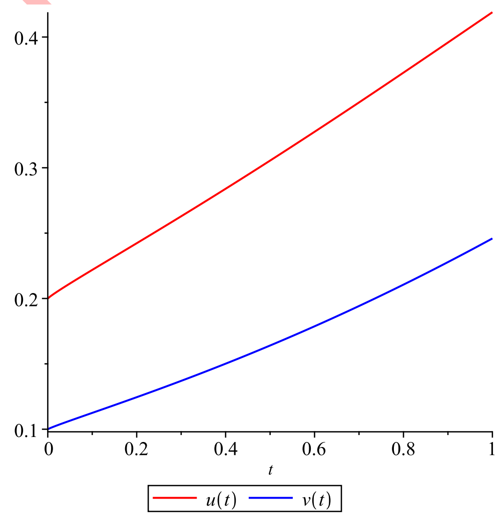

FIGURE 6. Neutralism between two species with $a_0 = 0.2$, $b_0 = 0.1$, $A_i = B_i = 1$, $C_i = 0$ for $\alpha = 0.90$.

3. CONCLUSION

This paper extends the power series method to coupled nonlinear fractional ordinary differential equations. We obtained the power series solution of fractional Lotka-Volterra equation and proved its convergence. All types of competitive roles represented by the equation were analyzed through graphics. The introduction of fractional order greatly enriches the discussion of competitive role for Lotka-Volterra equation. Although the signs of parameters C_i for the interaction between species determine the types of competitive roles shown in Table 1, this degree of determination is also influenced by the fractional order. For the given two species with the fixed initial values ($u(0)$ and $v(0)$) and parameters (A_i, B_i and C_i), fractional order can adjust the degree of conformity between the model and real data. In addition, we believe that the present study will help us to further generalize the power series method to study more nonlinear FDEs. Excitingly, there is a latest paper this year that systematically introduces the power series method [10].

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