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Multi-Soliton Solutions to the K-P Equation of Tenth Order

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Abstract

K-P equation is an important (2+1) - dimensional nonlinear PDE which has not only multi-solitons but also has complete integrability. In order to describe the long waves that propagates with weak dispersion in the direction of additional spatial variable y, Kadomstev and Petviashili formulated this model. In the literature, many researchers are interested to propose and work on higher order nonlinear PDE's possessing multi-solitons. Two powerful methods employed by researchers are Hirota's method to obtain multi-solitons and $\tanh - \coth$ method to obtain one soliton solutions. In our work, K-P equation of order ten is derived and using Hirota's method, its multi solitons are worked out. The derived equation is also treated with the tanh method. This article emphasizes few bounded solutions to the equation in context. The main aim of the paper is to demonstrate the generalization of the K-P equation using Hirota operators and to study corresponding multi-solitons. We discuss few open problems for the proposed tenth order K-P equation.

Keywords. Higher order KP equation, The Hirota bilinear method, The tanh method.
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1. Introduction

The K-P equation is the (2+1) dimensional extension of KdV equation with one more spatial variable y which has not only multi-solitons but also has complete integrability. Kadomstev and Petviashili formulated this model to describe the long waves that propagates with weak dispersion in the direction of additional spatial variable y. And in their work, they studied the stability of the soliton solution [16, 17]. Depending on the negative coefficient (-1) and positive coefficient (+1) of u_{yy} the K-P equation is classified as KP-I and KP-II equation. KP-I equation governs the physical system possessing high surface tension whereas KP-II is used to describe the physical system with weak surface tension. The K-P equation plays significant role in the allied fields like fluid dynamics, shallow water waves, plasma physics, astrophysics and so on. Both KP-I and KP-II equations exhibits line soliton which are generalized soliton solutions of KdV equation. KP-I exhibits lump soliton solution and KP-II possesses periodic solution expressing optical solitons [1–4, 11]. In the vast literature, many methods substantiate such soliton solutions to the nonlinear PDE to name a few : inverse scattering method, perturbation method, the tanh method, Adomian decomposition method, the Hirota's method, G'/G method, perturbation homotopy method, collocation method, Lie symmetry method and many recent novel methods [5–10, 12, 13, 15, 18–26, 30].

Prominent researchers R. Hirota, J. Hietarinta, W. Malfliet, R. S. Johnson, A. M. Wazwaz and many more have contributed to develop mainly Hirota's method and $\tanh - \coth$ method for multi-solitons through their works [13, 15, 16, 28, 30]. The Hirota's method is one of the efficient method to deduce multi-solitons. And that can be achieved by expressing the given differential equation in its bilinear form using Hirota operators [14, 15, 26]. The tanh method is also one of the popular methods to find soliton solutions to the nonlinear PDE's by expressing them as a finite series expansion in terms of tanh [22–24].

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This article mainly focuses on computing certain bounded solutions to the tenth order K-P (tK-P) equation, in particular soliton solutions. As K-P equation is integrable it admits multi solitons. In present work, we just give the necessary condition of integrability by finding multi-solitons of the derived equation. We also employed tanh method to tK-P equation to ensure the one soliton solution of Hirota's direct method. The work is carried out sequentially as follows : In section 2, we derive tK-P using Hirota Operators. In section 3, we obtain its multi-solitons using the Hirota's Direct method and Hirota's one-soliton solution is reconfirmed through the tanh method. In section 4, few 2d and 3d plots of the derived solutions are plotted and in the concluding section, open problems are discussed.

2. Derivation of the Generalized Tenth Order K-P Equation

In this section we derive K-P equation of tenth order (tK-P).

The K-P equation reads as [17, 30]

$$(u_t + 6uu_x + u_{xxx})_x \pm u_{yy} = 0. \tag{2.1}$$

By substituting $u = w_x$ in (2.1) and integrating twice with respect to x it can be reduced into Hirota bilinear form

$$\left(D_x D_t + D_x^4 \pm D_y^2\right)(f \cdot f) = 0, \tag{2.2}$$

where u and f are related by

$$u = 2\frac{\partial^2}{\partial x^2} \log f(x,t,y). \tag{2.3}$$

We note that order of the K-P Equation (2.1) is 4 and its corresponding bilinear form also has order 4. In order to study the effect of increasing order of bilinearity on multi-soliton, we generalize (2.2) to 10^{th} order given by:

$$P(D)(f \cdot f) = \left(D_x D_t + D_x^4 + \alpha D_x^6 + \beta D_x^8 + \gamma D_x^{10} \pm D_y^2\right)(f \cdot f) = 0, \tag{2.4}$$

where α, β and γ are real constants.

By Hirota operators [15] (2.4) is equivalent to

$$\begin{aligned}
&[ff_{xt} - f_x f_t] + [ff_{4x} - 4f_x f_{3x} + 3f_{xx}f_{xx}] + \alpha\big[f_{6x}f - 6f_{5x}f_x + 15f_{4x}f_{2x} - 10f_{xxx}f_{xxx}\big] \\
&+ \beta\big[ff_{8x} - 8f_{7x}f_x + 28f_{6x}f_{xx} - 56f_{5x}f_{3x} + 35f_{4x}f_{4x}\big] + \gamma\big[ff_{10x} - 10f_{9x}f_x + 45f_{8x}f_{2x} \\
&- 120f_{7x}f_{3x} + 210f_{6x}f_{4x} - 126f_{5x}f_{5x}\big] \pm \big[f_{yy}f - f_y f_y\big] = 0.
\end{aligned} \tag{2.5}$$

Using the relation (2.3) and the Hirota's operators [15, 26] in (2.4) gives us the desired tK-P equation as

$$\begin{aligned}
&u_{xt} + u_{4x} + 6(uu_{xx} + u_x^2) + \alpha\big[u_{6x} + 15(uu_{4x} + 2u_x u_{3x} + u_{xx}^2) + 45(2uu_x^2 + u^2 u_{xx})\big] \\
&+ \beta\big[u_{8x} + 28(uu_{6x} + 2u_{5x}u_x) + 98u_{xx}u_{4x} + 70u_{3x}^2 + 210(u^2 u_{4x} + 4uu_x u_{3x}) \\
&+ 420(uu_{xx}^2 + u^3 u_{xx} + 3u^2 u_x^2 + u_{2x}u_x^2)\big] + \gamma\big[u_{10x} + 45(uu_{8x} + 2u_{7x}u_x) + 255u_{6x}u_{xx} \\
&+ 210(2u_{5x}u_{3x} + u_{4x}^2) + 4410uu_{xx}u_{4x} + 1575u_{xx}^3 + 630(u^2 u_{6x} + 4uu_x u_{5x} + 2u_{4x}u_x^2) \\
&+ 3150(2u_x u_{xx}u_{3x} + uu_{3x}^2 + u^3 u_{4x} + 6u^2 u_x u_{3x} + 3u^2 u_{xx}^2 + 6uu_x^2 u_{xx}) + 4725(4u^3 u_x^2 + u^4 u_{xx})\big] \pm u_{yy} = 0.
\end{aligned} \tag{2.6}$$

As we are focusing on bounded solutions to (2.6), in the ensuing section we compute multi-soliton solutions and reconfirm the one soliton solution by the tanh method.

3. Solution by Hirota's method and the tanh method

In this section, we deduce that multi-solitons exists for the derived tK-P Equation (2.5) using Hirota's direct method. In order to deduce the soliton solution, the unknown function f has to be determined, where the solution $u(x,t,y)$ and $f(x,t,y)$ are related by the transformation, $u = 2\dfrac{\partial^2}{\partial x^2} \log f(x,t,y)$.

And we assume that $f = 1 + \sum_{n=1}^{\infty} \epsilon^n f_n$, where $f_1, f_2 \ldots$, are yet to be found. For more details please see [12–15, 30].

3.1. **One soliton solution.** For one soliton solution , consider the bilinear form (2.4) and observing the fact that for one soliton solution $f_k = 0, k \geq 2$.

We obtain $f(x,t,y) = 1 + \epsilon f_1 = 1 + \epsilon e^{\theta}$ where, $\theta = kx + my - ct, \epsilon, k,$ c and m are real constants. Using this $f(x,t,y)$ in (2.5), we have

$$\epsilon[-kc + k^4 + \alpha k^6 + \beta k^8 + \gamma k^{10} \pm m^2]e^{\theta} = 0.$$

$$\text{This implies, } c = \frac{k^4 + \alpha k^6 + \beta k^8 + \gamma k^{10} \pm m^2}{k}.$$

Hence the one soliton solution of (2.6) is

$$\begin{aligned} u &= 2\frac{\partial^2}{\partial x^2}\log f(x,t,y) \\ &= \frac{k^2}{2}\operatorname{sech}^2\left(\frac{kx + my - ct}{2}\right). \end{aligned} \tag{3.1}$$

3.2. **Two Soliton Solution.** For two soliton solution, consider the bilinear form (2.4) and using the fact that for two soliton solution $f_k = 0, k \geq 3$.

We obtain, $f(x,t,y) = 1 + \epsilon f_1 + \epsilon^2 f_2$.

$$\begin{aligned} \text{where } f_1 &= e^{\theta_1} + e^{\theta_2}, \\ f_2 &= a_{12}e^{\theta_1+\theta_2}, \theta_i = k_i x + m_i y - c_i t, (i = 1,2), \end{aligned}$$

k_i, c_i and m_i are real constants and the coupling constant a_{12} to be determined.

By considering $P(D)(f \cdot f) = 0$ and equating the the coefficients of ϵ^2, we obtain

$$\left(D_x D_t + D_x^4 + \alpha D_x^6 + \beta D_x^8 + \gamma D_x^{10} \pm D_y^2\right)(1 \cdot f_2 + f_1 \cdot f_1 + f_2 \cdot 1) = 0.$$

$$\text{Implies, } \left(D_x D_t + D_x^4 + \alpha D_x^6 + \beta D_x^8 + \gamma D_x^{10} \pm D_y^2\right)(2(f_2 \cdot 1) + (f_1 \cdot f_1)) = 0.$$

Which results in ,

$$a_{12} = -\frac{\left[-(k_1 - k_2)(c_1 - c_2) + (k_1 - k_2)^4 + \alpha(k_1 - k_2)^6 + \beta(k_1 - k_2)^8 + \gamma(k_1 - k_2)^{10} \pm (m_1 - m_2)^2\right]}{\left[-(k_1 + k_2)(c_1 + c_2) + (k_1 + k_2)^4 + \alpha(k_1 + k_2)^6 + \beta(k_1 + k_2)^8 + \gamma(k_1 + k_2)^{10} \pm (m_1 + m_2)^2\right]} \tag{3.2}$$

Therefore, f can be expressed as

$$f = 1 + \epsilon\left(e^{\theta_1} + e^{\theta_2}\right) + \epsilon^2 a_{12}e^{\theta_1+\theta_2}.$$

Using the above f in $u = 2\dfrac{\partial^2}{\partial x^2}\log f(x,t,y)$ is the two soliton solution of (2.6).

3.3. **Three Soliton solution.** In this subsection, we examine the existence of three soliton solution of (2.4) For that, we consider the auxilliary function

$$f = 1 + \epsilon f_1 + \epsilon^2 f_2 + \epsilon^3 f_3 \tag{3.3}$$

$$\begin{aligned} \text{where, } f_1 &= e^{\theta_1} + e^{\theta_2} + e^{\theta_3}, \\ f_2 &= a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3}, \\ f_3 &= b_{123}\, e^{\theta_1+\theta_2+\theta_3}, \theta_i = k_i x + m_i y - c_i t, i = 1,2,3. \end{aligned} \tag{3.4}$$

and b_{123} is a coupling constant.

In the existence of multi-solitons the constant b_{123} plays a crucial role. If it can be expressed as product in terms of a_{12}, a_{23} and a_{13} satisfying the three soliton condition, then one can conclude that the nonlinear partial differential equation possesses multi-soliton solutions [12, 14, 15, 30].

We compute,

$$a_{ij} = -\frac{\left[-(k_i-k_j)(c_i-c_j)+(k_i-k_j)^4+\alpha(k_i-k_j)^6+\beta(k_i-k_j)^8+\gamma(k_i-k_j)^{10}\pm(m_i-m_j)^2\right]}{\left[-(k_i+k_j)(c_i+c_j)+(k_i+k_j)^4+\alpha(k_i+k_j)^6+\beta(k_i+k_j)^8+\gamma(k_i+k_j)^{10}\pm(m_i+m_j)^2\right]}, \tag{3.5}$$

where $1 \le i < j \le 3$, k_i, c_i and m_i are real constants.

By considering $P(D)(f \cdot f) = 0$ and equating the coefficients of ϵ^3, we obtain,

$$\left(D_x D_t + D_x^4 + \alpha D_x^6 + \beta D_x^8 + \gamma D_x^{10} \pm D_y^2\right)(1 \cdot f_3 + f_1 \cdot f_2 + f_2 \cdot f_1 + f_3 \cdot 1) = 0. \tag{3.6}$$

After substituting the values of f_1, f_2 and f_3 from (3.4) and (3.5) into (3.6) results in,

$$b_{123} = -\frac{a_{12}P(k_3-k_1-k_2)+a_{13}P(k_2-k_1-k_3)+a_{23}P(k_1-k_2-k_3)}{P(k_1+k_2+k_3)} \tag{3.7}$$

Now, computing $P(D)(f_1 \cdot f_3 + f_3 \cdot f_1 + f_2 \cdot f_2) = 0$, with the condition $f_n = 0, n \ge 4$, we obtain,

$$b_{123} = a_{12}a_{23}a_{31}. \tag{3.8}$$

From (3.7) and (3.8), it follows that,

$$-\frac{a_{12}P(k_3-k_1-k_2)+a_{13}P(k_2-k_1-k_3)+a_{23}P(k_1-k_2-k_3)}{P(k_1+k_2+k_3)} = a_{12}a_{23}a_{31}. \tag{3.9}$$

Further, simplifying the above Equation (3.9), results in the three soliton condition, given by

$$\begin{aligned} &P(k_1-k_2)P(k_1+k_3)P(k_2+k_3)P(k_3-k_1-k_2) \\ &+ P(k_1-k_3)P(k_1+k_2)P(k_2+k_3)P(k_2-k_1-k_3) \\ &+ P(k_2-k_3)P(k_1+k_2)P(k_1+k_3)P(k_1-k_2-k_3) \\ &= P(k_1-k_2)P(k_2-k_3)P(k_1-k_3)P(k_1+k_2+k_3). \end{aligned} \tag{3.10}$$

Hence,

$$f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + a_{12}e^{\theta_1+\theta_2} + a_{13}e^{\theta_1+\theta_3} + a_{23}e^{\theta_2+\theta_3} + a_{12}a_{13}a_{23}\, e^{\theta_1+\theta_2+\theta_3}.$$

Using the above f in $u = 2\dfrac{\partial^2}{\partial x^2}\log f(x,t,y)$ is the three soliton solution to (2.6).

The higher order soliton solutions can be obtained in analogous way. Thus, we arrive at the conclusion that the proposed tK-P equation admits multi-solitons.

3.4. **The Tanh method.** This sub section is devoted to the tanh method which reconstructs the one soliton solution that is obtained from the Hirota's method. The basic idea of tanh method is to express the given PDE into an ODE by suppressing multi-variables by a single variable. And expressing the solution by the powers of tanh. As it exhibits auto truncation of the series, it is handy to treat the complicated nonlinear PDE by this method. [22–24, 29, 30].

By introducing the variable $z = kx + my - ct$ and denoting $u(x,t,y) = U(z)$ then the PDE (2.6) can be transformed to the following ODE;

$$\begin{aligned} &(-kc \pm m^2)U_{zz} + k^4U_{4z} + 6k^2(UU_{zz} + U_z^2) + \alpha[k^6U_{6z} + 15k^4(UU_{4z} + 2U_zU_{3z} + U_{zz}^2) \\ &+ 45k^2(U^2U_{zz} + 2UU_z^2)] + \beta[k^8U_{8z} + 28k^6(UU_{6z} + 2U_zU_{5z}) + 98k^6U_{zz}U_{4z} + 70k^6U_{3z}^2 \\ &+ 210k^4(U^2U_{4z} + 4UU_zU_{3z}) + 420(k^4UU_{zz} + k^2U_{zz}U^3 + 3k^2U^2U_z^2 + k^2U_{zz}U_z^2)] \\ &+ \gamma[k^{10}U_{10z} + 45k^8(UU_{8z} + 2U_zU_{7z}) + 255k^8U_{zz}U_{6z} + 210k^8(2U_{3z}U_{5z} + U_{4z}^2) + 4410k^6UU_{zz}U_{4z} \\ &+ 1575k^6U_{zz}^3 + 630(k^6U^2U_{6z} + 2k^6U_z^2U_{4z} + 4k^6UU_zU_{5z}) + 4725k^2(U^4U_{zz} + 4U^3U_z^2) \\ &+ 3150(2k^6U_zU_{zz}U_{zzz} + k^6UU_{3z}^2 + k^4U^3U_{4z} + 6k^4U^2U_zU_{3z} + 3k^4U^2U_{zz}^2 + 6k^4UU_z^2U_{zz})] = 0, \end{aligned} \tag{3.11}$$

where $U_z = D_zU$.

Now we solve (3.11) using the tanh method. The stage of auto truncation can be determined by equating the exponents of higher derivative and the highest power of nonlinear terms of the differential equation.

So, by balancing the highest order term U_{10z} and the highest power nonlinear term $U_{zz}U_{6z}$

$$M + 10 = 2M + 8.$$

We obtain

$$M = 2.$$

Hence, we consider

$$U(Y) = \sum_{k=0}^{2} a_k Y^k, \text{ where } Y = \tanh \frac{z}{2}.$$

Using the above in the ODE (3.11) we get $a_1 = 0$ and by fixing $a_0 = \frac{k^2}{2}$ and $a_2 = -\frac{k^2}{2}$ the solution is

$$\begin{aligned} U &= \frac{k^2}{2}\left[1 - \tanh^2\left(\frac{z}{2}\right)\right] \\ &= \frac{k^2}{2}\,(1 - Y^2). \end{aligned}$$

Now, we deduce that the velocity c of the soliton solution to (3.11) is same as the velocity c in Hirota's one soliton. For that we simplify the terms of the ODE (3.11) as follows ;

- $(-kc \pm m^2)U_{zz} + k^4 U_{4z} + 6k^2(UU_{zz} + U_z^2) = \frac{k^2}{4}[-kc \pm m^2 + k^4](3Y^2 - 1)(1 - Y^2);$

- $k^6 U_{6z} + 15k^4(UU_{4z} + 2U_z U_{3z} + U_{zz}^2) + 45k^2(U^2 U_{zz} + 2UU_z^2) = \frac{k^8}{4}(3Y^2 - 1)(1 - Y^2);$

- $k^8 U_{8z} + 28k^6(UU_{6z} + 2U_z U_{5z}) + 98k^6 U_{zz}U_{4z} + 70k^6 U_{3z}^2 + 210k^4(U^2 U_{4z} + 4UU_z U_{3z})$
$+ 420(k^4 UU_{zz} + k^2 U_{zz}U^3 + 3k^2 U^2 U_z^2 + k^2 U_{zz} U_z^2) = \frac{k^{10}}{4}(3Y^2 - 1)(1 - Y^2);$

- $k^{10}U_{10z} + 45k^8(UU_{8z} + 2U_z U_{7z}) + 255k^8 U_{zz}U_{6z} + 210k^8(2U_{3z}U_{5z} + U_{4z}^2) + 4410k^6 UU_{zz}U_{4z}$
$+ 1575k^6 U_{zz}^3 + 630(k^6 U^2 U_{6z} + 2k^6 U_z^2 U_{4z} + 4k^6 UU_z U_{5z}) + 4725k^2(U^4 U_{zz} + 4U^3 U_z^2)$
$+ 3150(2k^6 U_z U_{zz} U_{zzz} + k^6 UU_{3z}^2 + k^4 U^3 U_{4z} + 6k^4 U^2 U_z U_{3z} + 3k^4 U^2 U_{zz}^2 + 6k^4 UU_z^2 U_{zz})$
$= \frac{k^{12}}{4}(3Y^2 - 1)(1 - Y^2).$

Using this simplified terms in the ODE (3.11), we obtain

$$c = \frac{k^4 + \alpha k^6 + \beta k^8 + \gamma k^{10} \pm m^2}{k}.$$

Hence the solution to (2.6) is

$$u = \frac{k^2}{2}\operatorname{sech}^2\left(\frac{kx - ct + my}{2}\right)$$

with

$$c = \frac{k^4 + \alpha k^6 + \beta k^8 + \gamma k^{10} \pm m^2}{k};$$

which agrees with the one-soliton solution in section (3.1). Thus, we described the soliton solution to the tK-P equation using two different methods namely the Hirota's method and the tanh method.

4. Plots of solutions

In this section, we plot few graphs of the solutions that are given in the previous sections for different values and we conclude by highlighting few open problems.

(1) **Plots for one soliton solutions (1ss) of Equation (2.4) by Hirota's method.**
For 1ss of (2.4), we have chosen the particular values as :
$\epsilon = 1, k = 1, \alpha = \beta = \gamma = 1$ and $m = 1$ then $f = 1 + exp(x + y - t)$.

Case (i) : For t=0 .

Maple code :
$> plot3d((1/2) * sech((x + y) * (1/2))^2, y = -10..10, x = -10..10)$

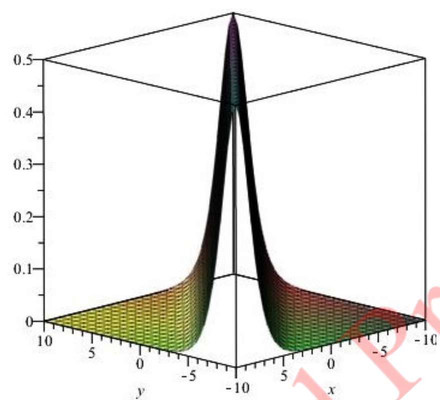

FIGURE 1. One soliton solution with the particular choices $\epsilon = 1, k = 1, \alpha = \beta = \gamma = 1$ and $m = 1$, $t = 0$.

Case (ii) : For y=0 .

Maple code :
$> plot3d((1/2) * sech((x - 3 * t) * (1/2))^2, t = -10..10, x = -10..10)$

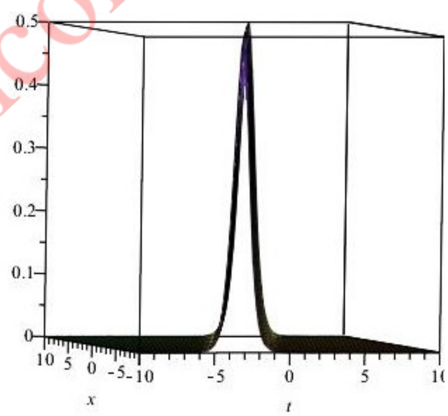

FIGURE 2. One soliton solution with the particular choices $\epsilon = 1, k = 1, \alpha = \beta = \gamma = 1$ and $m = 1$, $y = 0$, $c = 3$.

Case (iii) : For y=0, t=0 .

Maple code :

$> plot((1/2)*sech((1/2)*x)^2, x = -10..10)$

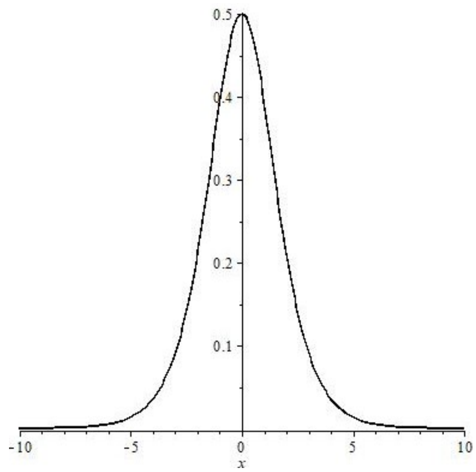

FIGURE 3. One soliton solution with the particular choices $\epsilon = 1, k = 1, \alpha = \beta = \gamma = 1$ and $m = 1$, $y = 0$ and $t = 0$.

(2) **Plots for two soliton solutions (2ss) of Equation (2.4) by Hirota's method.**
For the following particular choices :
$\alpha = 1, \beta = 1, \gamma = 1, k_1 = 1, k_2 = 2, m_1 = 1$ and $m_2 = 2$ we have $c_1 = 3,\ c_2 = 678$ and $a_{12} \approx 0.0104$.

Case (i) : y=0

Maple code :
$> plot3d(2*(exp(x-3*t)+4*exp(2*x-678*t)+(0.104e-1*9)*exp(3*x-681*t))/(1+exp(x-3*t)+exp(2*x-678*t)+0.104e-1*exp(3*x-681*t))-2*(exp(x-3*t)+2*exp(2*x-678*t)+(0.104e-1*3)*exp(3*x-681*t))^2/(1+exp(x-3*t)+exp(2*x-678*t)+0.104e-1*exp(3*x-681*t))^2, x = -6..6, t = -2..5).$

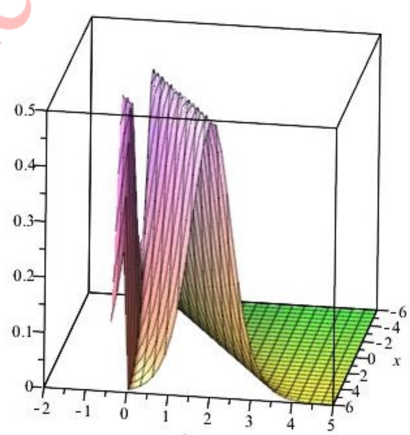

FIGURE 4. Two soliton solution with $\alpha = 1, \beta = 1, \gamma = 1, k_1 = 1, k_2 = 2, m_1 = 1, y = 0$ and $m_2 = 2$, $c_1 = 3,\ c_2 = 678$ and $a_{12} \approx 0.0104$.

Case (ii) : t=0 .

Maple code :

$> plot3d(2*(exp(x+y)+4*exp(2*x+2*y)+(0.104e-1*9)*exp(3*x+3*y))/(1+exp(x+y)+exp(2*x+2*y)+0.104e-1*exp(3*x+3*y))-2*(exp(x+y)+2*exp(2*x+2*y)+(0.104e-1*3)*exp(3*x+3*y))^2/(1+exp(x+y)+exp(2*x+2*y)+0.104e-1*exp(3*x+3*y))^2, x=-8..8, y=-1..5.5)$.

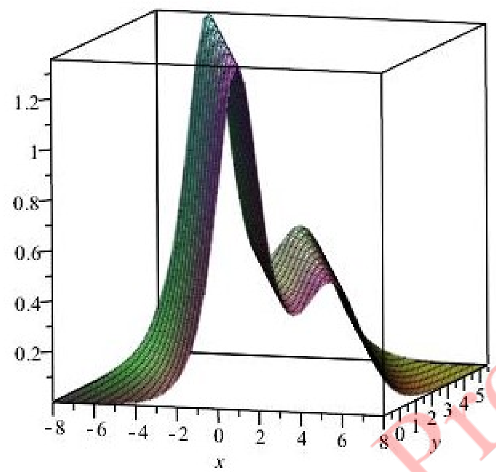

FIGURE 5. Two soliton solution with $\alpha=1, \beta=1, \gamma=1, k_1=1, k_2=2, m_1=1, t=0$ and $m_2=2$, $c_1=3,\ c_2=678$ and $a_{12}\approx 0.0104$.

Case (iii) : t=0, y=0 .

Maple code :

$> plot(2*(exp(x)+4*exp(2*x)+(0.104e-1*9)*exp(3*x))/(1+exp(x)+exp(2*x)+0.104e-1*exp(3*x))-2*(exp(x)+2*exp(2*x)+(0.104e-1*3)*exp(3*x))^2/(1+exp(x)+exp(2*x)+0.104e-1*exp(3*x))^2, x=-8..8)$.

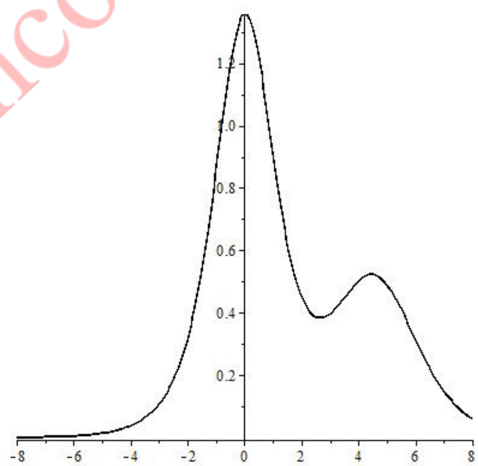

FIGURE 6. Two soliton solution with $\alpha=1, \beta=1, \gamma=1, k_1=1, k_2=2, m_1=1, t=0, y=0$ and $m_2=2,\ c_1=3,\ c_2=678$ and $a_{12}\approx 0.0104$.

5. Conclusion

Summing up, we derived tenth order K-P equation and discussed its *multi-solitons* by applying Hirota's Direct method. Also, we have employed the tanh method to obtain *soliton solution* which agreed with the one-soliton of Hirota's method.

In addition to the multi-solitons that are computed, it will be interesting to workout other types of solutions such as rational, singular, shock wave and periodic wave solutions to tK-P equation. Studying tK-P equation in a coupled system by applying complex transform, namely, $u(x, y, t) = p(x, y, t) + iq(x, y, t)$ and computing their solutions is also open. The present work establishes the existence of multi-solitons to tK-P equation which is only a necessary condition for its integrability [27]. So, whether tK-P equation is integrable or not will be one more open problem.

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