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Hopf Bifurcation and Chaotic Attractors in Two Special Jerk System Cases

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Abstract

This paper is focused on studying the Hopf bifurcation with self-excited and hidden chaotic attractors in special types of chaotic jerk systems. The stability of the equilibrium point and Hopf bifurcation is rigorously investigated for the proposed systems. It is remarkable to analyze the Hopf bifurcation using focus quantity techniques. These bifurcations may be either supercritical or subcritical, depending on the control parameters. To investigate the dynamic behavior of the systems, an analysis of self-excited chaotic attractors and hidden chaotic attractors was performed. Additionally, bifurcation analysis and evaluation of Lyapunov exponents revealed complex transitions among periodic, self-excited chaotic and hidden chaotic attractors as the system parameters varied. It was found that the systems exhibit both self-excited and hidden attractors, as demonstrated by the bifurcation diagrams, Lyapunov exponents and cross sections. All of the results provided in this study were acquired applying the Maple and Matlab software.

Keywords. Jerk system, Hopf bifurcation, Chaotic, Self-excited attractors and Hidden attractors.

2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

In the modern era, chaos is a highly fascinating and intricate nonlinear phenomenon that has received extensive research attention. Ever since Lorenz constructed the first chaotic system in 1963, numerous researchers have unveiled new chaotic systems, such as Chen's attractor [7], Rossler attractor [34], Lu attractor [22], and the logistic map [26]. There are two types of attractors in chaotic systems, as classified by Kuznetsov et al. [20]: self-excited attractors and hidden attractors [1]. A self-excited attractor is characterized by its basin of attraction being connected to an unstable equilibrium, whereas a hidden attractor is defined by its basin of attraction not intersecting with small neighborhoods of any equilibrium [27, 44]. There exist various types of hidden chaotic attractors, including those with stable equilibrium points [29], surfaces of equilibrium points [15], lines or curves of equilibrium points [4, 16], or no equilibrium points at all [10]. In physics, a jerk differential equation can be expressed through the following third-order dynamic equation:

$$\dddot{x} = g(x, \dot{y}, \ddot{x}), \tag{1.1}$$

where x, $\dot{x}$, $\ddot{x}$ and $\dddot{x}$ represent the displacement, velocity, acceleration and jerk, respectively. By defining $y = \dot{x}$ and $z = \dot{y} = \ddot{x}$, differential equation (1.1) can be reformulated as:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = g(x, y, z). \end{cases} \tag{1.2}$$

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Chaotic attractors in jerk systems have diverse applications, including circuits[6, 45], memristors [41], encryption[12, 33], and more. Sang et al. [35] investigated the chaotic mechanisms in some jerk systems by utilizing bifurcation diagrams, Poincare cross sections, phase portraits, and Lyapunov exponents to analyze the underlying mechanisms of chaos. They also examined local bifurcations for both pitchfork and Hopf bifurcations. Kengne et al. [19] conducted a study on a novel jerk system incorporating sine hyperbolic nonlinear terms. Their work revealed that the system exhibits a period-doubling route to chaos, symmetry recovery crises, and multi-stability.

Vijayakumar et al. in [42] focused on the dynamics of a new 4D chaotic hyper-jerk system, demonstrating its ability to exhibit various hidden behaviors such as hidden point attractors, hidden periodic attractors, and hidden chaotic attractors. The outcomes revealed interesting properties in the proposed system, including period-doubling transitions, asymmetric bubbles, and the coexistence of attractors. Bonny et al. in [5] investigated a new 3D jerk system with three nonlinearity terms. They showcased the dynamical properties of the jerk system through the utilization of phase portraits, bifurcation diagrams, Lyapunov exponents, coexisting attractors, and multistability. The authors in [23] presented a 3-dimensional jerk system with a sinusoidal nonlinear term. They conducted analyses on various dynamic behaviors, including the stability of equilibrium points, parameter and initial value bifurcations, phase diagrams, and basins of attraction. Additionally, the system generated complex dynamics, such as single-scroll chaotic attractors, double-scroll chaotic attractors, and multi-scroll attractors.

Joshi and Ranjan [18] studied the jerk system with sine hyperbolic nonlinearity. In their work, they investigated the appearance of a hidden attractor in the system by applying numerical simulations, along with exploring several fundamental properties of the system. Rajagopal et al. [32] studied the jerk equation given by:

$$\dddot{x} - ax + b\dot{x} + \ddot{x} + \cosh(x) = 0.$$

They demonstrated various dynamic behaviors by organizing the system's parameters, including the presence of a one-scroll chaotic attractor,the coexistence of chaotic, and a periodic attractors. Additionally, the stability of equilibrium points and Hopf bifurcation were analyzed. Ozbal et al. [28] conducted an investigation into the behavior of the dynamic jerk system using bifurcation diagrams, Poincare maps, Lyapunov exponents, and spectrum analysis. They also provided system trajectories to examine the chaotic dissipative behavior of the system and successfully generated a double-scroll chaotic attractor. For the following jerk equation

$$\dddot{x} = -\alpha[\ddot{x} + \dot{x} + x - (\tanh(ax - b) + c)]. \tag{1.3}$$

Wannaboon et al. [46] investigated the dynamic behavior of the system through the using a bifurcation diagram, Lyapunov exponents, chaotic attractor analysis, waveform analysis in the time domain, and circuit analysis. The authors in [43] presented a new chaotic jerk system with three cubic nonlinear terms. The numerical investigation of the proposed jerk system explores bifurcation structures, revealing occurrences such as period-doubling, periodic windows, and coexisting bifurcations. Yan et al. [48] presented three-dimensional Jerk chaotic system with line equilibrium points. The system has various dynamical behaviors including coexisting attractors, multistability of various attractor and transient chaos. In reference [24], the authors investigated stability and some types of bifurcations for a Class of Inversion Invariant Jerk Equations. Furthermore, they establish integrable deformations that stabilize equilibrium points.

A Hopf bifurcation is a type of local bifurcation in which the stability of an equilibrium point in a dynamical system changes as a parameter varies. It occurs when the equilibrium has a pair of purely imaginary eigenvalues, along with specific transversality conditions and no zero eigenvalues. The bifurcation can be classified as supercritical or subcritical, resulting in the formation of either a stable or unstable limit cycle within an invariant two-dimensional manifold, respectively. Salih and Mohammed [36] studied a Modified Sprott C system. They investigated the stability of equilibrium points and the appearance of a Hopf bifurcation in the system at a bifurcation point. Using normal form theory, they revealed that the bifurcating periodic solutions are unstable and that the bifurcation is of the subcritical type, with increasing periods of these solutions. The authors in [8] presented a predatorprey model with Allee effect and double time delays. This research analyzes the model's dynamics, emphasizing positivity, existence, stability and Hopf bifurcations. Using normal form theory and the center manifold theorem, it clarifies the stability of periodic solutions and the direction of the Hopf bifurcation. Numerical simulations confirm the theoretical analysis, indicating that a weak Allee effect delay can enhance stability in the model, shifting it from instability to stability.

Husien and Amen in [13] investigated a quadratic chaotic system modeling with three nonlinear terms. They established the existence of two limit cycles that bifurcate due to the characteristics of the system as an electronic circuits model. Particularly, the first and second Lyapunov coefficients are used to show the bifurcation of two limit cycles from an equilibrium point near to a Hopf critical point. Additionally, first-order averaging theory is applied to confirm the presence of unstable periodic orbits arising from the zero-Hopf equilibrium. Also in [31], the authors studied local bifurcations in a special type of chaotic jerk system. In their work, the various bifurcations are investigated including saddle-node, transcritical, zero-Hopf, Hopf and pitchfork at the origin and characterizes parameters leading to a zero-Hopf equilibrium point. It demonstrates that using first-order averaging theory results in a single periodic solution bifurcate from the origin equilibrium, while the focus quantities method is applied to analyze the periodicity of the cubic part of the system. Under certain conditions, it shown that three periodic solutions can bifurcate from the origin of the system. The mathematical model delves into the impact of dissolved oxygen depletion on plankton populations by examining potential equilibrium points, revealing three distinct equilibria. The study further explores Hopf bifurcation by utilizing the oxygen production rate from phytoplankton photosynthesis as a parameter to determine conditions for stable limit cycles [2]. Additionally, the research investigates the stability and local bifurcations, such as saddle-node, transcritical and Hopf bifurcation, in the phytoplankton-zooplankton model with depleted oxygen and strong Allee effects [3]. The study also focuses on analyzing the local stability of equilibria and local bifurcations within the cancer immune chemotherapy vitamins model (CICV) as detailed in [17]. Furthermore, the research explores the occurrence of bifurcations, including those of transcritical or Hopf types.

This paper is organized as follows: Section two presents the proposed two chaotic jerk systems and provides an analysis of their stability. In Section three, Hopf bifurcations are investigated using focus quantity techniques. The final section of this work explores the self-excited and hidden chaotic attractors of the proposed jerk systems by utilizing bifurcation diagrams, Lyapunov exponents and phase portraits.

2. The Proposed Chaotic Jerk Systems and Stability Analysis

Consider the following two chaotic jerk systems

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -ax - by - cz - xy + d\ z\ \mathrm{erf}(z), \end{cases} \tag{2.1}$$

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -ax - by - cz - xy + d\ z\ sin^{-1}(z), \end{cases} \tag{2.2}$$

where a, b, c, d are positive parameters and $\mathrm{erf}(z) = \dfrac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$. The above systems have unique equilibrium at the origin. By performing direction calculations, the divergence of systems (2.1) and (2.2) can be expressed as follows:

$$\begin{aligned} \nabla V|_{(2.1)} &= -c + d\ \mathrm{erf}(z) + \frac{2dze^{-z^2}}{\sqrt{\pi}}, \\ \nabla V|_{(2.2)} &= -c + d\ sin^{-1}(z) + \frac{d\ z}{\sqrt{1-z^2}}, \end{aligned} \tag{2.3}$$

respectively. We note that if ∇V is a constant, then the volume's time evolution in phase space is given by $V(t) = V_0 e^{\nabla V t}$ where $V_0 = V(t = 0)$. When ∇V is negative, the phase space volume decreases exponentially, indicating a dissipative dynamical system that may exhibit stable attractors. If ∇V equals zero, the phase space volume remains constant, signifying a conservative dynamical system. Conversely, when ∇V is positive, the phase space volume increases, leading to only unstable fixed points, limit cycles, or potentially chaotic repellors, meaning the dynamics will diverge as t approaches infinity unless the initial conditions are precisely on one of the fixed points or stationary states.

The Jacobian matrix of the proposed jerk systems at the origin is

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a & -b & -c \end{pmatrix},$$

which has characteristic polynomial

$$\varphi(\lambda) = \lambda^3 + c\lambda^2 + b\lambda + a. \tag{2.4}$$

Therefore, based on the assumption that a, b, and c are greater than zero and according to the Routh-Hurwitz criterion (see, for instance, [[30], Theorem 6, pages 58-59]), the origin of both systems is asymptotically stable when $bc > a$. The discriminant $\varphi(\lambda)$ with respect to λ is

$$\Delta = -4ac^3 + b^2c^2 + 18abc - 4b^3 - 27a^2.$$

(1) If $\Delta > 0$, then the characteristic polynomial (2.4) has three distinct real roots and all of them are negative. Therefore, the equilibrium point is a stable node.
(2) If $\Delta < 0$, the characteristic polynomial (2.4) possesses three distinct roots: one real root and a pair of complex conjugate roots. In the case of $bc > a$, the real parts of these roots should be negative, resulting in a stable node-focus at the origin. However, due to $cb < a$, the origin becomes unstable. As a consequence, the remaining complex conjugate eigenvalues must have positive real parts. Thus, the origin is classified as a saddle-focus.

3. Hopf Bifurcations Analysis

Hopf bifurcation is a well-established phenomenon that is typically associated with the occurrence or non-occurrence of limit cycles. However, it is noteworthy that Hopf bifurcation can also manifest in the proposed chaotic jerk system , which includes both subcritical and supercritical versions. To investigate Hopf bifurcation in this system, various techniques such as bifurcation formulas [14], Lyapunov quantities [37, 38], and focus quantities [39, 40] can be employed.

To explore the appearance of Hopf bifurcation, we use the focus quantity method to derive the findings.

Theorem 3.1. *For system (2.1) with $b > 0$, a Hopf bifurcation occurs at the origin when $a = bc$. Moreover, the Hopf bifurcation is supercritical if*

$$32b^3cd^2 + 2\sqrt{\pi}bc^2d - 28\sqrt{\pi}b^2d - \pi c < 0 \tag{3.1}$$

and is subcritical if

$$32b^3cd^2 + 2\sqrt{\pi}bc^2d - 28\sqrt{\pi}b^2d - \pi c > 0. \tag{3.2}$$

Proof. The Jacobian matrix of system (2.1) at the origin is

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a & -b & -c \end{pmatrix}$$

and its characteristic polynomial at the origin is

$$\varphi(\lambda) = \lambda^3 + c\lambda^2 + b\lambda + a. \tag{3.3}$$

When $a = bc$, $\varphi(\lambda)$ has a simple pair of purely imaginary roots: $\lambda_{1,2}(a) = \pm i\sqrt{b}$ and a negative root $\lambda_3(a) = -c$. By setting (3.3) to zero, an application of the implicit function theorem yields the transversality condition[9, 21, 25]:

$$\begin{aligned} \frac{dRe\lambda_{1,2}}{da}|_{a=bc} &= \frac{-1}{2c\lambda + 3\lambda^2 + b}|_{\lambda=\pm\sqrt{b}i} \\ &= \frac{1}{2(c^2+b)} > 0. \end{aligned} \tag{3.4}$$

According to [11, 25], a Hopf bifurcation occurs at $a = bc$. By introducing the transformation

$$\begin{cases} x = -\frac{1}{b}y_1 - \frac{1}{b}y_2 + \frac{1}{c^2}y_3, \\ y = -\frac{i}{\sqrt{b}}y_1 + \frac{i}{\sqrt{b}}y_2 - \frac{1}{c}y_3, \\ z = y_1 + y_2 + y_3, \end{cases} \tag{3.5}$$

System (2.1) becomes

$$\begin{cases} \frac{dy_1}{dt} = i\sqrt{b}\, y_1 + (s_1 + is_2)\, y_1y_2 + (s_3 + is_4)\, y_1^2 + (s_5 + is_6)\, y_2^2 + (s_7 + is_8)\, y_3^2 + (s_9 + is_{10})\, y_1y_3 \\ \quad + (s_{11} + is_{12})\, y_2y_3 + O(3), \\ \frac{dy_2}{dt} = -i\sqrt{b}\, y_2 + (s_1 - is_2)\, y_1y_2 + (s_5 - is_6)\, y_1^2 + (s_3 - is_4)\, y_2^2 + (s_7 - is_8)\, y_3^2 + (s_{11} - is_{12})\, y_1y_3 \\ \quad + (s_9 - is_{10})\, y_2y_3 + O(3), \\ \frac{dy_3}{dt} = -cy_3 + t_1 y_1y_2 + (t_2 - it_3)\, y_1^2 + (t_2 + it_3)\, y_2^2 + t_4 y_3^2 + (t_5 + it_6)\, y_1y_3 + (t_5 - it_6)\, y_2y_3 + O(3), \end{cases} \tag{3.6}$$

where

$$s_1 = \frac{2db}{(c^2+b)\sqrt{\pi}}, \quad s_2 = \frac{2d\sqrt{b}\,c}{(c^2+b)\sqrt{\pi}}, \quad s_3 = \frac{2d\,b^2 + c\sqrt{\pi}}{2\,(c^2+b)\sqrt{\pi}\,b}, \quad s_4 = \frac{2dbc - \sqrt{\pi}}{\sqrt{\pi}\sqrt{b}\,(2c^2+2b)}$$

$$s_5 = \frac{2d\,b^2 - c\sqrt{\pi}}{2\,(c^2+b)\sqrt{\pi}\,b}, \quad s_6 = \frac{2dbc + \sqrt{\pi}}{\sqrt{\pi}\sqrt{b}\,(2c^2+2b)}, \quad s_7 = \frac{b\left(2c^3d + \sqrt{\pi}\right)}{2\,(c^2+b)\sqrt{\pi}\,c^3},$$

$$s_8 = \frac{\sqrt{b}\,\left(2c^3d + \sqrt{\pi}\right)}{2\,(c^2+b)\sqrt{\pi}\,c^2}, \quad s_9 = \frac{2dbc - \sqrt{\pi}}{(c^2+b)\,c\sqrt{\pi}}, \quad s_{10} = \frac{\left(-c^2+b\right)\sqrt{\pi} + 4db\,c^3}{2\sqrt{b}\sqrt{\pi}\,(c^2+b)\,c^2}$$

$$s_{11} = \frac{2db}{(c^2+b)\sqrt{\pi}}, \quad s_{12} = \frac{\left(-c^2-b\right)\sqrt{\pi} + 4db\,c^3}{2\sqrt{b}\sqrt{\pi}\,(c^2+b)\,c^2}$$

and

$$t_1 = \frac{4c^2d}{(c^2+b)\sqrt{\pi}}, \quad t_2 = \frac{2c^2d}{(c^2+b)\sqrt{\pi}}, \quad t_3 = \frac{c^2}{(c^2+b)\,b^{\frac{3}{2}}}, \quad t_4 = \frac{2c^3d + \sqrt{\pi}}{c\,(c^2+b)\sqrt{\pi}}$$

$$t_5 = \frac{c\,(4dbc - \sqrt{\pi})}{(c^2+b)\sqrt{\pi}\,b}, \quad t_6 = \frac{1}{(c^2+b)\sqrt{b}}.$$

By applying the recursive formula of [40] to system (3.6), we obtain a focus quantity of system (2.1) at the origin:

$$W = \frac{32b^3cd^2 + 2\sqrt{\pi}bc^2d - 28\sqrt{\pi}b^2d - \pi c}{b^2(c^2+4b)\pi(c^2+b)}. \tag{3.7}$$

Since $b > 0$, the criticality of the Hopf bifurcation is determined by the numerator of the above quantity and thus the conclusions of this theorem hold. □

Theorem 3.2. *Consider system (2.2) with $b > 0$, a Hopf bifurcation appears at the origin when $a = bc$. Furthermore, the Hopf bifurcation is supercritical if*

$$8b^3c\,d^2 + b\,c^2d - 14b^2d - c < 0, \tag{3.8}$$

and is subcritical if

$$8b^3c\,d^2 + b\,c^2d - 14b^2d - c > 0. \tag{3.9}$$

This theorem can be proven using the same approach as was used to prove Theorem (3.1).

4. Route to Self-Excited and Hidden Chaotic Attractors as the Parameter a Varies

To comprehend the chaotic mechanisms of these systems, the subsequent two subsections primarily focus on considering the following two special cases:

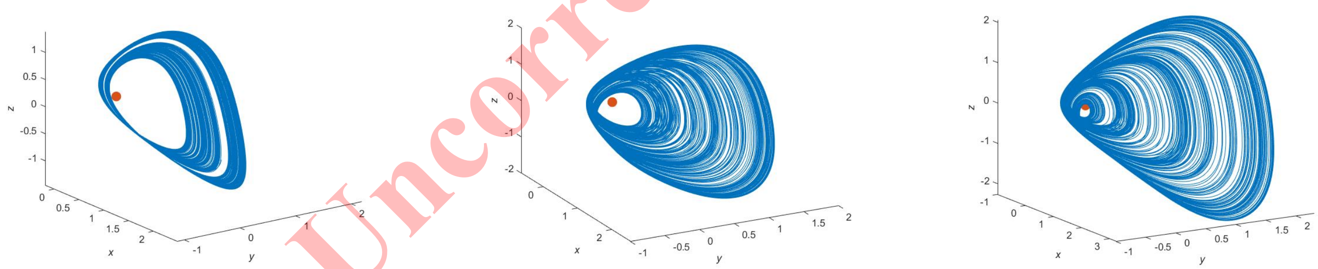
$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -ax - 1.2y - 1.09z - xy + 2.7\ z\ \mathrm{erf}(z). \end{cases} \tag{4.1}$$

$$\begin{cases} \dot{x} = y, \\ \dot{y} = z, \\ \dot{z} = -ax - y - 1.0009z - xy + 2.005\ z\ sin^{-1}(z). \end{cases} \tag{4.2}$$

To explore the dynamical behavior of systems, some numerical methods are applied to investigate the chaotic mechanisms such as phase portrait, Lyapunov exponent spectrum, cross-sections and bifurcation diagram.

4.1. **Hidden Attractors of System (4.1).** Consider system (4.1) with $a \in [1, 1.308]$. The system exhibits a unique equilibrium point at the origin, which remains stable throughout. When $a = 1.308$, the eigenvalues of the jacobian matrix at the origin are $\lambda_{1,2} = \pm 1.095445115i$ and $\lambda_3 = -1.090$. For $a = 1.12, 1.15$ and 1.21, the system displays a hidden chaotic attractor with a point attractor. This is illustrated in Figure 1 using fixed initial conditions $x(0) = y(0) = 1$ and $z(0) = -1$. Various projections of the cross sections, specifically the two-sided Poincare sections, for this system are depicted in Figure 2 at $a = 1.15$.

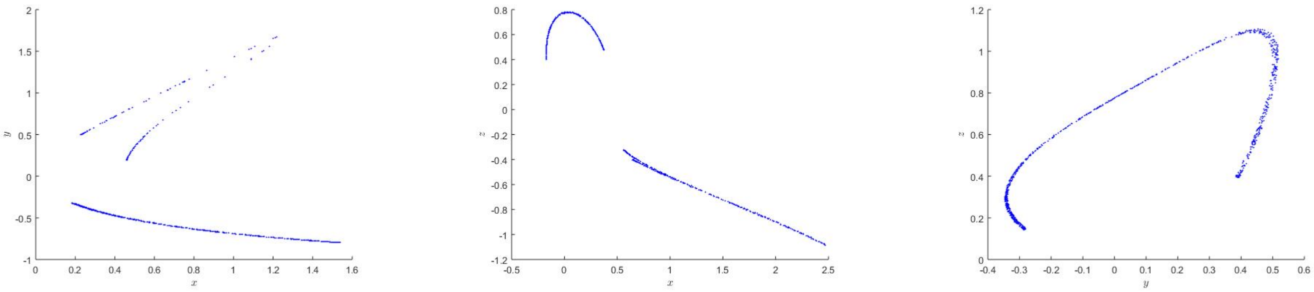
The Lyapunov exponent (LE) serves as the primary criterion for chaos, representing the rate of increase or decrease of small perturbations along the main axes of the phase space system. To obtain the Lyapunov exponents, the Wolf's algorithm [47] has been utilized with a time period of $t \in [0, 20000]$. The attractor exhibits Lyapunov exponents of $L_1 = 0.0929$, $L_2 = 0$ and $L_3 = -1.1416$, along with a Kaplan-Yorke dimension of 2.0814. Moreover, the jerk system possesses a positive maximal Lyapunov exponent (MLE), indicating its chaotic nature. Furthermore, the sum of all Lyapunov exponents is negative, indicating the dissipative nature of the system.



(A) $a = 1.12$ (B) $a = 1.15$ (C) $a = 1.21$

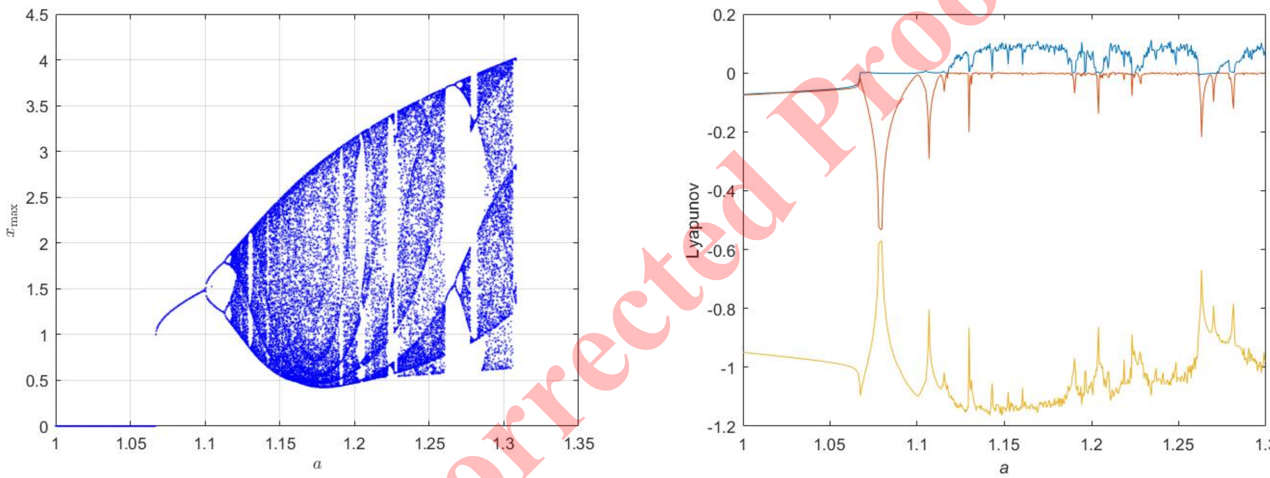
Figure 1. A hidden chaotic attractor of system (4.1) for $a = 1.12, 1.15, 1.21$ with initial condition $(1, 1, -1)$.

The bifurcation diagram shown in Figure 3(a) illustrates the local maxima of $x(t)$ in relation to the parameter a for system (4.1), using the initial condition $(x(0), y(0), z(0)) = (1, 1, -1)$. The figure reveals that the system can exhibit point attractors as the parameter a is varied, until reaching $a = 1.06$. At $a = 1.071$, a period 1-limit cycle emerges, followed by a hidden period 2-limit cycle at $a = 1.11$. Additionally, a hidden period 3-limit cycle is observed at $a = 1.13$. Moreover, the system demonstrates a hidden attractor for $a \in [1.14, 1.3079]$, with periodic windows occurring within the chaotic range on the bifurcation diagram. To complement the bifurcation diagrams, the corresponding Lyapunov exponents plotted against a are presented in Figure 3(b).



(A) $x-y$ (B) $x-z$ (C) $y-z$

FIGURE 2. Cross-sections of the hidden chaotic attractor of system (4.1) for $a = 1.15$ with initial condition $(1, 1, -1)$.

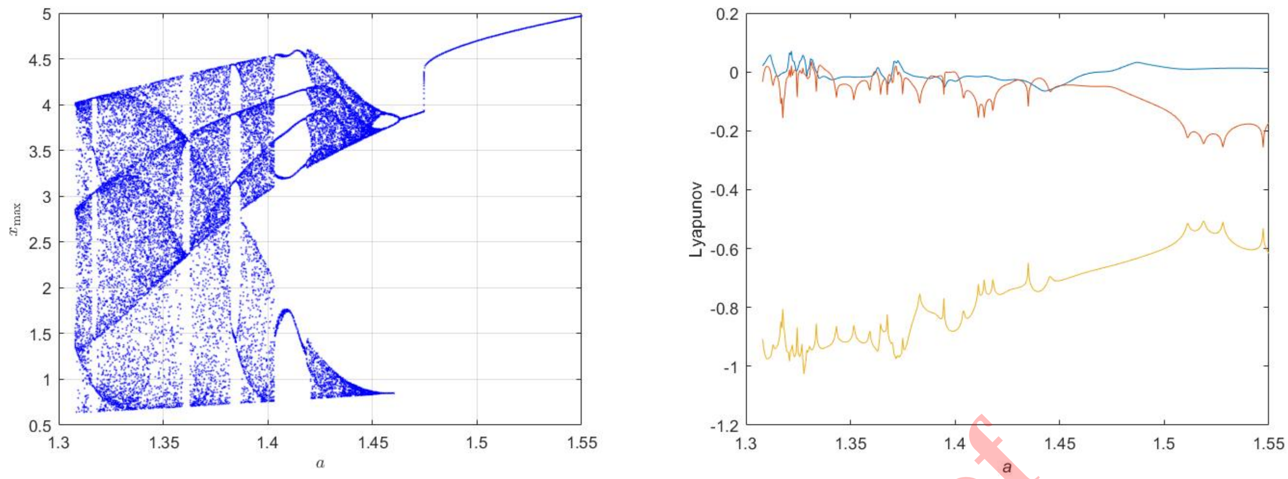


(A) (B)

FIGURE 3. Bifurcation diagrams of system (4.1), versus a parameter and corresponding Lyapunov exponents for initial conditions $(x(0), y(0), z(0)) = (1, 1, -1)$.

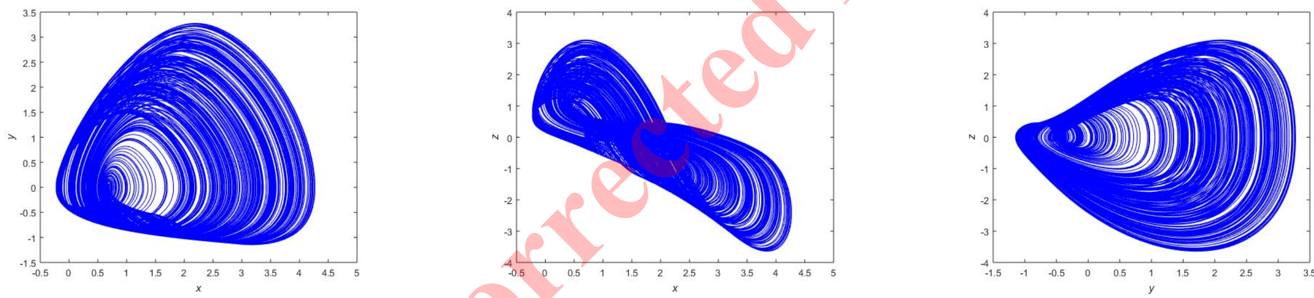
4.2. **Self-excited chaotic attractors of system (4.1) where $a \in [1.308, 1.55]$.** To investigate the occurrence of self-excited chaotic attractors in system (4.1) for values of a in the range $1.308 < a \leq 1.55$, it is observed that the equilibrium becomes unstable. When $a \in (1.308, 1.55]$ and fixed initial conditions of $(x(0), y(0), z(0)) = (1, 1, -1)$ are used, the bifurcation diagram of the system with respect to the parameter a is plotted in Figure 4(a). Within the range of 1.309 to 1.4, the system exhibits a self-excited chaotic attractor with an unstable equilibrium. Notably, there are period windows observed around $a = 1.37$ and $a = 1.38$, while a period 2-limit cycle is present at $a = 1.457$, and a period 1-limit cycle is observed at $a = 1.465$. As the parameter a varies from 1.5 to 1.55, the system does not display oscillatory behavior. Furthermore, the positive values of the Lyapunov exponents depicted in Figure 4(b) indicate that the underlying system is chaotic.

At $a = 1.35$, the jerk system (4.1) demonstrates the presence of a self-excited chaotic attractor, as depicted in Figure 5. Furthermore, at $a = 1.42$ and $a = 1.44$, typical chaotic attractors can be observed, as shown in Figure 6.



(A) (B)

FIGURE 4. Bifurcation diagrams of system (4.1) versus a parameter and corresponding Lyapunov exponents for initial conditions $(1, 1, -1)$.

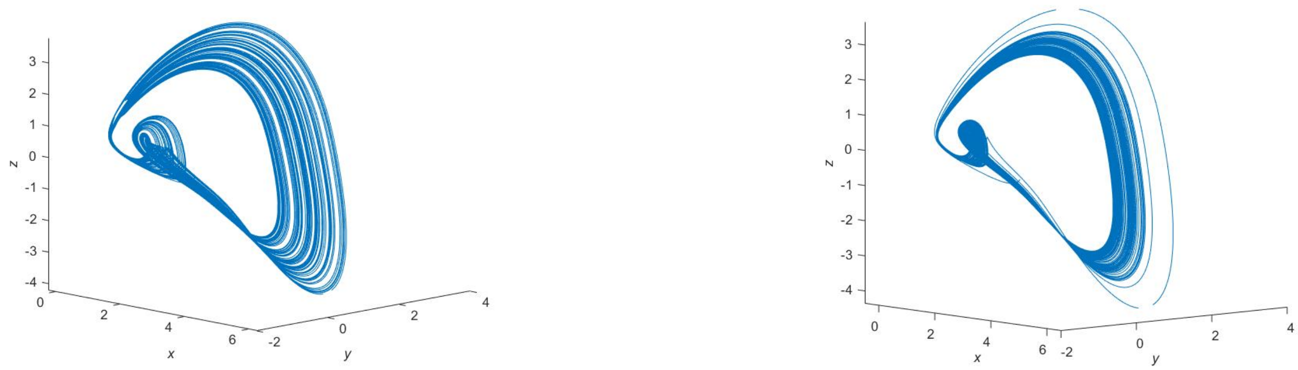


(A) (B) (C)

FIGURE 5. 2D views of the self-excited chaotic attractor of system (4.1) when $a = 1.35$ with initial condition $(1, 1, -1)$.

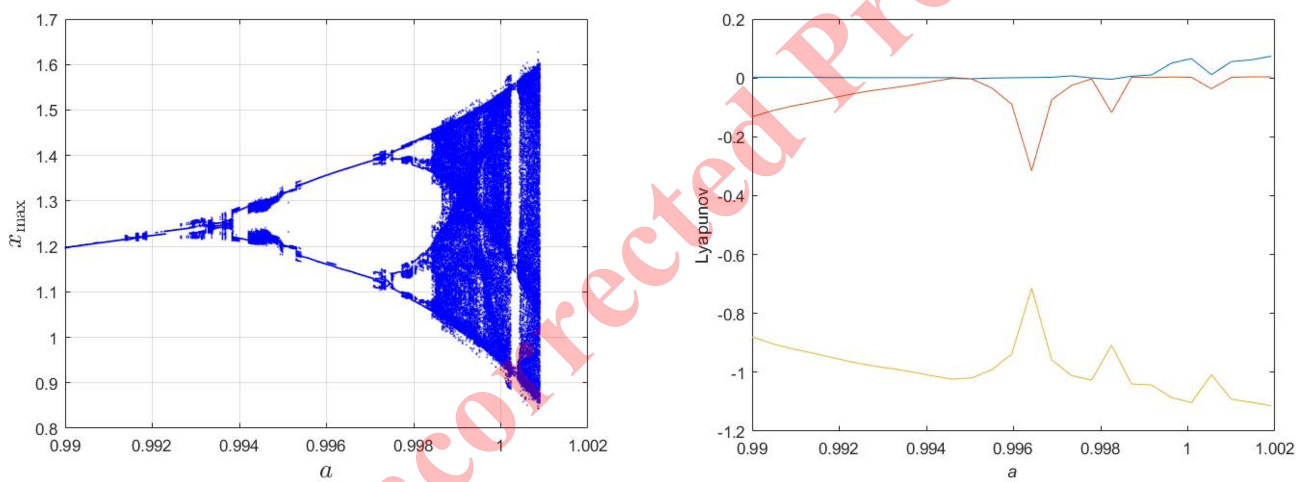
4.3. **Hidden chaotic attractors of system (4.2) when $a \in [0.99, 1.0009]$.** In this subsection, we thoroughly investigate how the dynamic behavior of system (4.2) changes with variations in the parameter a. We analyze this through the examination of bifurcation diagrams, Lyapunov exponents, phase portraits, and cross sections. The bifurcation diagram of system (4.2) is presented in Figure 7(a), illustrating the local maxima of $x(t)$ as a function of the parameter a, with initial conditions $x(0) = y(0) = 0$ and $z(0) = -0.5$. From the bifurcation diagram, we observe the occurrence of periodic behavior, a period-doubling route leading to hidden chaos, periodic windows, and hidden chaotic attractors. Furthermore, the positive values of the Lyapunov exponents displayed in Figure 7(b) confirm the chaotic nature of the underlying system.

The 2D views of the hidden chaotic attractor, along with a point attractor, are presented in Figure 8 for $a = 0.9988, 1, 1.0003$. Additionally, Figure 9 displays a cross-section in the $x(0) - y(0)$ plane at $z(0) = 0$, illustrating the basins of attraction of the two attractors. By employing Wolf's algorithm, the Lyapunov exponents are calculated for $a = 1$ as follows: $L_1 = 0.0537, L_2 = 0$, and $L_3 = -1.0858$. Consequently, the maximal Lyapunov exponent (MLE) of



(A) $a = 1.42$ (B) $a = 1.44$.

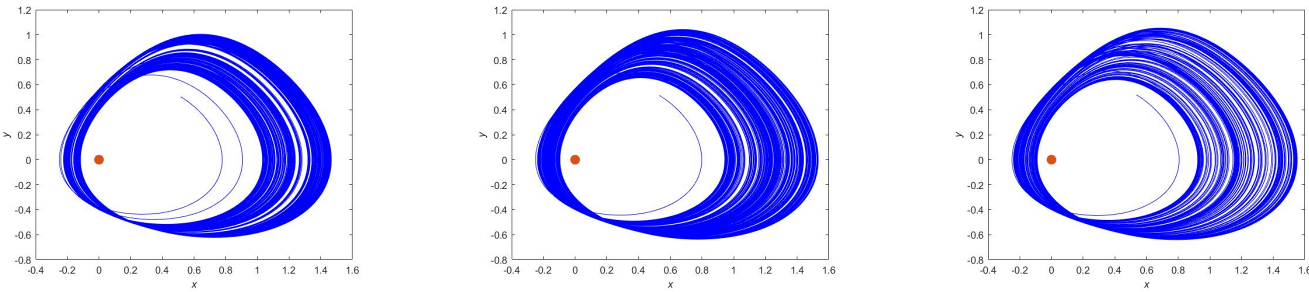
FIGURE 6. (a) 3D view of the chaotic attractors for a = 1.42, (b) 3D view of the chaotic attractors for a = 1.44.



(A) (B)

FIGURE 7. Bifurcation diagrams of system (4.2) versus a parameter and corresponding Lyapunov exponents for initial conditions $(0, 0, -0.5)$.

the jerk system is determined as $L_1 = 0.0537$, indicating the presence of chaotic behavior in system (4.2). Furthermore, the Kaplan-Yorke dimension of the proposed system is computed as $D_{KY} = 2 + \frac{L_1+L_2}{|L_3|} = 2.0495$.



(A) a=0.9988 (B) a=1 (C) a=1.0003

FIGURE 8. The hidden chaotic attractor onto the $x-y$ plane for system (4.2) with $a = 0.9988, 1$ and 1.0003 with initial condition $(0, 0, -0.5)$. Red point is stable equilibrium point at the origin.

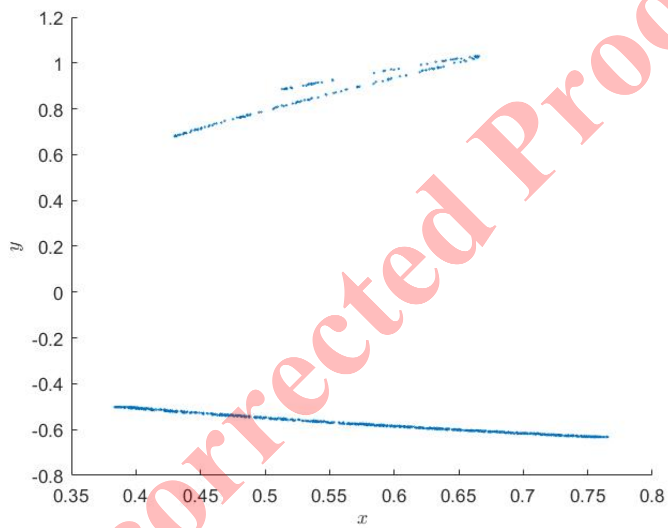


FIGURE 9. Cross-section of the basins of attraction of two attractors in the $x(0) - y(0)$ plane at $z(0) = 0$ for system (4.2) for $a = 1$. with initial conditions $(0, 0, -0.5)$.

5. CONCLUSIONS

The paper introduces and analyzes two specific cases of chaotic jerk systems with two nonlinearity terms. Our main focus is to investigate the connections between self-excited chaotic attractors and hidden chaotic attractors. To achieve this objective, we conducted stability and Hopf bifurcation analyses for the two proposed jerk systems. Additionally, we presented the dynamical properties of these two types of chaotic jerk systems through phase portraits, bifurcation diagrams, Lyapunov exponents and cross-sections.

For the system (4.1), it is determined that when a is in the range $[1, 1.308]$, the system can exhibit a hidden chaotic attractor with stable equilibrium points. The underlying mechanisms driving the chaotic dynamics are examined through the utilization of bifurcation diagrams. When a is in the interval $(1.308, 1.47]$ with $\mu = 0$, a self-excited chaotic attractor with unstable equilibrium points is observed in the system (4.1). Furthermore, for the proposed jerk system, hidden chaotic attractors with stable equilibrium points can be exhibited as the parameter a varies from 1.15 to 1.471. Finally, the system (4.2) displays a hidden attractor when $a \leq 1.0009$.

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