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The study of maximal surfaces by Lie symmetry

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Abstract

Maximal surfaces, a fascinating class of surfaces in differential geometry, are identified by having a mean curvature equal to zero. This distinctive feature gives rise to a nonlinear second-order partial differential equation. In this current article, we delve into the symmetries that underlie the maximal surface equation. Next, we identify one-dimensional optimal system of subalgebras that span these symmetries. It provides a powerful tool to analyze and manipulate the equation, making it easier to study. Finally, since we aim to not only explore the underlying symmetries of the maximal surface equation, we demonstrate how these symmetries can be harnessed to uncover and classify a wide range of maximal surfaces by using reduction methods.

Keywords. Catenoid, Helicoid, Maximal surface, Mean curvature, Symmetry group.
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1. Introduction

Lie symmetries play powerful role in the field of differential geometry, allowing us to solve a variety of geometric problems effectively and elegantly. These symmetries are closely related to the idea of invariance and offer a mathematical framework that help us comprehend and resolve complicated problems, such as determining geodesics, minimal or maximal surfaces in various geometric spaces.

Numerous researchers have delved into the exploration of symmetries within the context of geodesic equations and have widely reported classification results [1, 3]. This has led to the revelation that Noether symmetries form a subalgebra within the realm of homothetic algebra. Given the profound link between symmetries and conservation laws established by Noether symmetries, their significance in the field of science cannot be understated.

In essence, the symmetries extend their influence over all matters related to the action integral, even when these matters aren't directly connected to geodesic equations. This is because the symmetries inherently pertain to the symmetries of the action integral itself. Inspired by this intriguing perspective, Aslam and Qadir [2] investigated specific spaces, uncovering a compelling relationship between the Noether symmetries of minimal Lagrangians and the isometries inherent in this spaces. Furthermore, these authors [6] identified the symmetries within the framework of Lagrangians for minimal surfaces with fixed volumes in diverse spaces, encompassing Euclidean spaces, spaces characterized by constant curvature, and Schwarzschild spacetimes.

Recently, the first author et al. in [4] determined the Noether symmetries of hypersurfaces minimizing area with fixed volumes within select vacuum classes of pp-waves. Additionally, they uncovered the associated conservation laws. These spacetimes have undergone thorough examination in the literature related to Einstein's equations, particularly those accommodating a covariantly constant null bisector field.
Another important problem concerning about Lie symmetries is identifying the maximal surface by solving the maximal PDE equation. Maximal surfaces have taken the attention of researchers for years, both in the fields of physics and mathematics, and present a complex and intricate differential problem. These surfaces provide a rich field for

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investigation because of their singular viewpoint on the interaction between geometry and physics, which is typified by a mean curvature of zero. The application of Lie symmetries to reveal the properties of maximal surfaces demonstrates the power of mathematical symmetry in solving intricate, problems in differential geometry.

Furthermore, Lie symmetries, a potent mathematical tool that uncovers hidden structures and invariance within differential equations, prove instrumental in the study of maximal surfaces. By applying Lie symmetries to the differential equations obtaining these surfaces, scholars can find a multitude of details regarding their characteristics, symmetries, and classification. This method not only makes maximal surfaces easier to understand, but it also makes it easier to use them to solve practical problems, such as modeling the behavior of soap bubbles or optimizing membrane structures in engineering.

Here, as a new approach to Lie symmetry groups of maximal surfaces, we determine Lie symmetry groups of maximal surfaces in L^3 by a partial differential equation describing maximal immersions. We employ Lie symmetries to diminish the order of the equation. Some solutions of this equation are then obtained.

2. Preliminaries

Let we denote by L^3 the space $\mathbb{R}^3$ equipped metric $g = dx^2 + dy^2 - dz^2$. A spacelike surface M in L^3 is considered maximal if the mean curvature H at every point on the surface is equal to zero. Suppose M be represented as the graphical representation of the function $z = z(x, y)$ over a specific area of $\mathcal{D}$ into $\mathbb{R}^2$. We can demonstrate that the mean curvature of this surface can be defined by the equation

$$H = \frac{z_{xx}(1 - z_y^2) + z_{yy}(1 - z_x^2) + 2z_x z_y z_{xy}}{2(u_x^2 + z_y^2 - 1)^{\frac{3}{2}}}.$$

When the mean curvature H equals zero, it implies that the following equation holds

$$z_{xx}(1 - z_y^2) + z_{yy}(1 - z_x^2) + 2z_x z_y z_{xy} = 0. \tag{2.1}$$

In this case, M is called maximal surface, and the Equation (2.1) is said to be maximal equation of M.

We extend z to the second order function $z^{(2)} : \mathcal{D} \to Z^{(2)}$. Here, $Z^{(2)} = Z \times Z_1 \times Z_2 \cong \mathbb{R}^6$ corresponds to the Cartesian product space where its coordinates denoting the derivatives of z of orders ranging from 0 to 2, denoted as

$$z^{(2)} = (z; z_x, z_y; z_{xx}, z_{yy}).$$

The total space, denoted as $\mathcal{D} \times Z^{(2)}$, encompasses the input variables, the output variable, and the second-order derivatives of the output variable. This second-order space is known as the jet space of the domain $\mathcal{D} \times Z$.

Now, we contemplate the equation:

$$\mathcal{H}(x, y, z^{(2)}) = z_{xx}(1 - z_y^2) + z_{yy}(1 - z_x^2) + 2z_x z_y z_{xy}.$$

This equation can be associated with the linear subvariety $W = \{(x, y, z^{(2)}) \in \mathcal{D} \times Z^{(2)} | \mathcal{H}(x, y, z^{(2)}) = 0\}$, within $\mathcal{D} \times Z$, which is determined by the function $\mathcal{H}$ vanishing W. $\mathcal{H}$ with maximmal rank when happens that the Jacobian matrix

$$J_{\mathcal{H}}(x, y, z^{(2)}) = (\mathcal{H}_x, \mathcal{H}_y; \mathcal{H}_z; \mathcal{H}_{z_x}, \mathcal{H}_{z_y}; \mathcal{H}_{z_{xx}}, \mathcal{H}_{z_{xy}}, \mathcal{H}_{z_{yy}}),$$

satisfies the condition that the rank of $J_{\mathcal{H}}$ is equal to 1 whenever

$$\mathcal{H}(x, y, z^{(2)}) = 0. \tag{2.2}$$

So maximal equation yields

$$\begin{aligned} &\mathcal{H}_x = 0, \quad \mathcal{H}_y = 0, \quad \mathcal{H}_z = 0, \quad \mathcal{H}_{z_x} = 2z_y z_{xy} - 2z_x z_{yy}, \\ &\mathcal{H}_{z_y} = 2z_x z_{xy} - 2z_y z_{xx}, \quad \mathcal{H}_{z_{xx}} = 1 - z_y^2, \quad \mathcal{H}_{z_{xy}} = 2z_x z_y, \quad \mathcal{H}_{z_{yy}} = 1 - z_x^2. \end{aligned}$$

Therefore the following Jacobian satisfies the Equation (2.2)

$$J_{\mathcal{H}}(x, y, z^{(2)}) = (0, 0, 0; 2z_y z_{xy} - 2z_x z_{yy}, 2z_x z_{xy} - 2z_y z_{xx}; 1 - z_y^2, 2z_x z_y, 1 - z_x^2).$$

Alternatively, a symmetry group of a partial differential equation refers to a set of transformations denoted as G, which operates on an open set of the surface M encompassing both input and output variables in the space. This group exhibits the characteristic that if $z = z(x, y)$ represents a solution of the $\mathcal{H} = 0$, then $z_g = g.z(x, y)$ also represents a solution of the $\mathcal{H} = 0$ for every transformation g. The following Theorem represent an invariance algorithmic method employed to determine G.

Theorem 2.1. *Suppose $\mathcal{H}(x, y, z^{(2)}) = 0$ with maximal rank specify within an open set $M \subset \mathcal{D} \times Z$, if there exists a local set of transformations, denoted as G, that acts on M, and if the condition $Y[\mathcal{H}(x, y, z^{(2)})] = 0$ is satisfied whenever the equation $\mathcal{H}(x, y, z^{(2)}) = 0$ holds, for any infinitesimal vector fieldY belonging to the group G then it follows that the group G is a symmetry group of $\mathcal{H}(x, y, z^{(2)}) = 0$.*

Take into account the vector field $Y = \xi_1(x, y, z)\frac{\partial}{\partial x} + \xi_2(x, y, z)\frac{\partial}{\partial y} + \eta(x, y, z)\frac{\partial}{\partial z}$, on $M \in D \times Z$, the vector field $Y^{(1)} = \xi_1(x, y, z)\frac{\partial}{\partial x} + \xi_2(x, y, z)\frac{\partial}{\partial y} + \eta(x, y, z)\frac{\partial}{\partial z}$ is referred to as its first prolongation of Y, where

$$\eta^x = \eta_x + (\eta_z - \xi_{1_x})z_x - \xi_{2_x}z_y - \xi_{1_z}z_x^2 - \xi_{2_z}z_xz_y$$

and

$$\eta^y = \eta_y - \xi_{1_y}z_x + (\eta_z - \xi_{2_y})z_y - \xi_{1_z}z_xz_y - \xi_{2_z}z_y^2.$$

The vector field $Y^{(2)} = Y^{(1)} + \eta^{xx}\frac{\partial}{\partial z_{xx}} + \eta^{xy}\frac{\partial}{\partial z_{xy}} + \eta^{yy}\frac{\partial}{\partial z_{yy}}$ is referred to as its second prolongation of Y, where

$$\begin{aligned}
\eta^{xx} &= \eta_{xx} + (2\eta_{xz} - \xi_{1_{xx}})z_x - \xi_{2_{xx}}z_y + (\eta_{zz} - 2\xi_{1_{xz}})z_x^2 \\
&\quad - 2\xi_{2_{xz}}z_xz_y - \xi_{1_{zz}}z_x^3 - \xi_{2_{zz}}z_x^2z_y + (\eta_z - 2\xi_{1_x})z_{xx} - 2\xi_{2_x}z_{xy} \\
&\quad - 3\xi_{1_z}z_xz_{xx} - \xi_{2_z}z_yz_{xx} - 2\xi_{2_z}z_xz_{xy}, \\
\eta^{xy} &= \eta_{xy} + (\eta_{zy} - \xi_{1_{xy}})z_x + (\eta_{zx} - \xi_{2_{xy}})z_y - \xi_{1_{zy}}z_x^2 + (\eta_{zz} - \xi_{1_{zx}} \\
&\quad - \xi_{2_{zy}})z_xz_y - \xi_{2_{zx}}z_y^2 - \xi_{1_y}z_{xx} + (\eta_z - \xi_{1_x} - \xi_{2_y})z_{xy} - \xi_{2_x}z_{yy} \\
&\quad - \xi_{1_z}z_yz_{xx} - 2\xi_{2_z}z_yz_{xy} - 2\xi_{2_z}z_xz_{xy} - \xi_{2_z}z_xz_{yy} - \xi_{1_{zz}}z_x^2z_y - \xi_{2_{zz}}z_xz_y^2, \\
\eta^{yy} &= \eta_{yy} + (2\eta_{zy} - \xi_{2_{yy}})z_y - \xi_{1_{yy}}z_x + (\eta_{zz} - 2\xi_{2_{zy}})z_y^2 \\
&\quad - 2\xi_{1_{zy}}z_xz_y - \xi_{2_{zz}}z_y^3\xi_{1_{zz}}z_xz_y^2 + (\eta_z - 2\xi_{2_y})z_{yy} - 2\xi_{1_y}z_{xy} \\
&\quad - 3\xi_{2_z}z_yz_{yy} - \xi_{1_z}z_xz_{yy} - 2\xi_{1_z}z_yz_{xy}.
\end{aligned}$$

For Equation (2.1) the condition of Theorem 2.1 becomes

$$\eta^x(2z_yz_{xy} - 2z_xz_{yy}) + \eta^y(2z_xz_{xy} - 2z_yz_{xx}) + \eta^{xx}(1 - z_y^2) + \eta^{yy}(1 - z_x^2) + \eta^{xy}(2z_xz_y) = 0. \tag{2.3}$$

Substituting the functions η^x, η^y, η^{xx}, η^{xy} and η^{yy} defined by the first and second prolongation of $Y^{(1)}$, $Y^{(2)}$ and by removing any interdependencies among the derivatives of z's stemming from Equation (2.1), we discover:

$$\begin{aligned}
&\eta_{xx} + \eta_{yy} + (2\eta_{xz} - \xi_{1_{xx}} - \xi_{1_{yy}})z_x + (2\eta_{zy} - \xi_{2_{yy}} - \xi_{_{xx}})z_y + (\eta_{zz} \\
&-\eta_{yy} - 2\xi_{1_{xz}})z_x^2 + (\eta_{zz} - \eta_{xx} = 2\xi_{2_{yz}})z_y^2 - 2(\eta_{xy} - \xi_{2_{xz}} - \xi_{1_{yz}})z_xz_y \\
&-(\xi_{1_{zz}} - \xi_{1_{yy}})z_x^3 - (\xi_{2_{yy}} - \xi_{2_{xx}})z_y^3 - (2\xi_{2_{xy}} - \xi_{1_{xx}} + \xi_{1_{zz}})z_xz_y^2 + (2\xi_{1_{xy}} \\
&-\xi_{2_{yy}} + \xi_{2_{zz}})z_x^2z_y + 2(\xi_{2_y} - \eta_z)z_{xx} + 2(\xi_{1_x} - \eta_z)z_{yy} - 2(\xi_{1_y} - \xi_{2_x})z_{xy} \\
&+2(\eta_y - \xi_{2_z})z_yz_{xx} + 2(\xi_{1_z} - \eta_x)z_xz_{yy} - 2(\eta_y - \xi_{2_z})z_xz_{xy} - 2(\xi_{1_z} + \eta_x)z_yz_{xy} = 0.
\end{aligned}$$

We can now set the coefficients of the leftover independent partial derivatives of z to vanish. This action leads to the generation of a set of numerous PDEs for the coefficients ξ_1, ξ_2, and η within the infinitesimal operator. These PDEs are often referred to as the defining equations for G associated the equation $\mathcal{H}=0$:

$$\begin{aligned}
&\eta_{xx}+\eta_{yy}=0, \qquad \xi_{1_{xx}}+\xi_{1_{yy}}=2\eta_{xz}.\\
&\xi_{2_{xx}}+\xi_{2_{yy}}=2\eta_{yz}, \qquad \eta_{zz}-\eta_{yy}=2\xi_{1_{yz}},\\
&\eta_{zz}-\phi_{xx}=2\xi_{2_{yz}}, \qquad \eta_{xy}-\xi_{2_{xz}}-\xi_{1_{yz}}=0,\\
&\xi_{1_{yy}}-\xi_{1_{zz}}=0, \qquad \xi_{2_{xx}}-\xi_{2_{yy}}=0,\\
&\xi_{1_{xx}}-\xi_{1_{zz}}=2\xi_{2_{xy}}, \qquad \xi_{2_{yy}}-\xi_{2_{zz}}=2\xi_{1_{xy}},\\
&\xi_{2_y}=\eta_z, \qquad \eta_u=\xi_{1_x}\\
&\xi_{1_y}=-\xi_{2_x}, \qquad \eta_y=\xi_{2_z},\\
&\eta_x=\xi_{1_z}.
\end{aligned}$$

By integration, we find the following solutions

$$\begin{aligned}
\xi_1(x,y,z)&=a_7x-a_4y+a_6z+a_1,\\
\xi_2(x,y,z)&=a_4x+a_7y+a_5z+a_2,\\
\eta(x,y,z)&=a_6x+a_5y+a_7z+a_3
\end{aligned}$$

with $a_1,\cdots,a_7\in\mathbb{R}$, and the operator Y is expressed by:

$$Y=a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}+a_4(-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y})+a_5(z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z})+c_6(x\frac{\partial}{\partial x}+z\frac{\partial}{\partial x})+c_7(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}). \tag{2.4}$$

Proposition 2.2. *A partial differential equation, defined over a domain M that lies within $M\subset\mathcal{D}\times Z$, has a collection of infinitesimal symmetries forming a Lie algebra of operators in the space M denoted by $\mathfrak{g}$. Also, if $\mathfrak{g}$ is finite algebra, the symmetry equation constitutes a local Lie group of transformations that act on the domain M.*

The results are derived from $\mathfrak{g}$ is as follows :

Theorem 2.3. *The Lie algebra $\mathfrak{g}$ for the maximal equation consist of a set of seven vector fields that span it*

$$\begin{aligned}
&Y_1=\frac{\partial}{\partial x}, \qquad Y_2=\frac{\partial}{\partial y}, \qquad Y_3=\frac{\partial}{\partial z},\\
&Y_4=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}, \qquad Y_5=z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}, \qquad Y_6=z\frac{\partial}{\partial x}+x\frac{\partial}{\partial z},\\
&Y_7=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}.
\end{aligned}$$

As each one-parameter subgroup G_i generated by Y_i represents a symmetry group, every solution $z=z(x,y)$ undergoes the transformation to the following solutions

$$\begin{aligned}
&z^{(1)}=z(x-\epsilon,y),\\
&z^{(2)}=z(x,y-\epsilon),\\
&z^{(3)}=z(x,y)+\epsilon,\\
&z^{(4)}=z(x\cos\epsilon-y\sin\epsilon,x\sin\epsilon+y\cos\epsilon),\\
&z^{(5)}\cosh\epsilon-y\sinh\epsilon=z(x,y\cosh\epsilon-z^{(5)}\sinh\epsilon),\\
&z^{(6)}\cosh\epsilon-x\sinh\epsilon=z(x\cosh\epsilon-z^{(6)}\sinh\epsilon,y),\\
&z^{(7)}=e^{\epsilon}z(e^{-\epsilon}x,e^{-\epsilon}y),
\end{aligned}$$

where ϵ is an arbitrary real number. For every s-parameter subgroup K within the complete symmetry group G of the equation, there exists a set of group-invariant solutions. Therefore, a way to classify these solutions is to employ an optimal system of group-invariant solutions, from which all other solutions can be extracte

Table 1. Adjoint representation.

Ad	Y_1	Y_2	Y_3	Y_4
Y_1	Y_1	Y_2	Y_3	$Y_4 - aY_2$
Y_2	Y_1	Y_2	Y_3	$Y_4 + aY_1$
Y_3	Y_1	Y_2	Y_3	Y_4
Y_4	$Y_1 \cos a + Y_2 \sin a$	$Y_2 \cos a - Y_1 \sin a$	Y_3	Y_4
Y_5	Y_1	$Y_2 \cosh a + Y_3 \sinh a$	$Y_3 \cosh a + Y_2 \sinh a$	$Y_4 \cosh a + Y_6 \sinh a$
Y_6	$Y_1 \cosh a + Y_3 \sinh a$	Y_2	$Y_3 \cosh a + Y_1 \sinh a$	$Y_4 \cosh a - Y_5 \sinh a$
Y_7	$e^a Y_1$	$e^a Y_2$	$e^a Y_3$	Y_4

Table 2. Adjoint representation.

Ad	Y_5	Y_6	Y_7
Y_1	Y_5	$Y_6 + aY_3$	$Y_7 - aY_1$
Y_2	$Y_5 - aY_3$	Y_6	$Y_7 - aY_2$
Y_3	$Y_5 - aY_2$	$Y_6 - aY_1$	$Y_7 - aY_3$
Y_4	$Y_5 \cos a - Y_6 \sin a$	$Y_6 \cos a + Y_5 \sin a$	Y_7
Y_5	Y_5	$Y_6 \cosh a + Y_4 \sinh a$	Y_7
Y_6	$Y_5 \cosh a - Y_4 \sinh a$	Y_6	Y_7
Y_7	Y_5	Y_6	Y_7

Proposition 2.4. *If $z = z(x, y)$ represents a solution that is invariant under the subgroup K of the equation, and if $g \in G$ is any other group element, then the transformed function $z_g = \tilde{z}(x, y) = g \cdot z(x, y)$ becomes an invariant solution under the subgroup $\tilde{K} = gKg^{-1}$, which is the conjugate subgroup of K under the action of g.*

The challenge of categorizing group-invariant solutions can be simplified by focusing on the classification of subgroups within the full symmetry group G, considering them under conjugation. This process is essentially synonymous with the task of classifying subalgebras of the Lie algebra $\mathfrak{g}$ associated with the group G.

To achieve this, we undertake the computation of the adjoint representation AdG of the underlying Lie group G by employing the Lie series methodology.

$$Ad(\exp(aY_i)Y_j) = \sum_{m=0}^{\infty} \frac{a^m}{m!}(adY_i)^m(Y_j) = Y_j - a[Y_i, Y_j] + \frac{a^2}{2}[Y_i, [Y_i, Y_j]] - \cdots$$

the entry in the (i, j)-th position represents the action of the adjoint representation $Ad(\exp(\epsilon Y_i))Y_j$ on the elements Y_i and Y_j.

In the context of one-dimensional subalgebras, the classification quandary is fundamentally analogous to the task of categorizing the orbits within the adjoint representation. This is owing to the fact that each one-dimensional subalgebra is uniquely identified by a nontrivial vector in the Lie algebra g. The approach involves considering a general element denoted as Y and subjecting it to diverse adjoint transformations with the objective of rendering it as "simple" as feasibly achievable. Thus we will construct a one-dimensional optimal system of Equation (2.1) by considering an arbitrary element $y = \sum_{i=1}^{7} s_i Y_i$ of Equation (2.1) lie algebra $\mathfrak{g}$. The map $G_i^{a_i} : \mathfrak{g} \to \mathfrak{g}$ given by $y \to Ad(exp(a_i y_i))y$ is a linear, $i = 1, \cdots, 7$. By using the above Table the matrix $M_i^{a_i}$ of $G_i^{a_i}$ with respect to basis $\{Y_1, \cdots, Y_7\}$ is given by

$$M_1^{a_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -a_1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 1 & 0 \\ -a_1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_1^{a_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & -a_2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_1^{a_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -a_3 & 0 & 0 & 1 & 0 & 0 \\ -a_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -a_3 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_1^{a_4} = \begin{pmatrix} \cos a_4 & \sin a_4 & 0 & 0 & 0 & 0 & 0 \\ -\sin a_4 & \cos a_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos a_4 & -sin a_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos a_4 & \sin a_4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_1^{a_5} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cosh a_5 & \sinh a_5 & 0 & 0 & 0 & 0 \\ 0 & \sinh a_5 & \cosh a_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cosh a_5 & \sinh a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sinh a_5 & 0 & \cosh a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_1^{a_6} = \begin{pmatrix} \cosh a_6 & 0 & \sinh a_6 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ \sinh a_6 & 0 & \cosh a_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cosh a_6 & -\sinh a_6 & 0 & 0 \\ 0 & 0 & 0 & -\sinh a_6 & \cosh a_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, M_1^{a_7} = \begin{pmatrix} e^{a_7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{a_7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{a_7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It can be observed that

$$\begin{aligned} &G_7^{a_7} \circ G_6^{a_6} \circ \cdots \circ G_1^{a_1} : y \mapsto (-a_1 s_7 + a_2 s_4 - a_3 s_6 + [\cos a_4 + \cosh a_6 + e^{a^7} + 1]s_1 - \sin a_4 s_2 + \sin a_6 s_3)Y_1 \\ &+ (\sin a_4 s_1 + [\cos a_4 + \cos a_5 + e^{a_7} + 1]s_2 + \sinh a_5 s_3 - a_3 s_5)Y_2 \\ &+ (a_1 s_6 - a_2 s_5 + [1 + \cos h a_5 + \cosh a_6 + e^{a_7}]s_3 - a_3 s_7 + \sinh a_5 s_2)Y_3 \\ &+ ([1 + \cosh a_5 + \cosh a_6]s_4 - \sinh a_6 + \sinh a_5 s_6)Y_4 + ([\sinh a_5 - \sinh a_6]s_4 + [\cosh a_6 + \cos a_4 + 1]s_5)Y_5 \\ &+ ([\cosh a_5 + 1 + \cos a_4]s_6 - \sin a_4 s_5)Y_6 + (s_7 + \sin a_4 s_6)Y_7. \end{aligned}$$

By appropriately choosing values for a_i , we can readily eliminate the coefficient of Y_j in various scenarios. This allows for the reduction of y, and a one-dimensional optimal system is consequently established

$$Y_1, Y_2, Y_3, \tag{2.5}$$

$$Y_4, Y_5, Y_6, \tag{2.6}$$

$$Y_3 + Y_4, Y_1 + Y_5, Y_2 + Y_6, \tag{2.7}$$

$$Y_4 + Y_7, Y_5 + Y_7, Y_6 + Y_7. \tag{2.8}$$

For every one-parameter subgroup, there exists a corresponding set of group-invariant solutions, which are determined by a reduced ordinary differential equation (ODE) whose form depends on the specific subgroup.

In Equation (2.5), if we take $Y_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$, the global solutions are $A_1 = \dfrac{y}{x}$ and $A_2 = \dfrac{z}{x}$. Consequently, we obtain $z = xk(\frac{y}{x})$.

Upon substitution into the Equation (2.1), we obtain

$$z = c_1 x + c_2 y, \ \ c_1, c_2 \in \mathbb{R}. \tag{2.9}$$

Similarly, planes are identified for both Y_1 and Y_2.

In Equation (2.6), when we examine $Y_4 = -y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}$, the global invariants are given by $A_1 = \sqrt{x^2 + y^2}$ and $A_2 = z$. Consequently, a solution invariant under the group takes the form $z = k(\sqrt{x^2 + y^2})$.

Take

$$r=\sqrt{x^2+y^2},\ \theta=arctg(\frac{y}{x}),$$

and through solutions within Equation (2.1), we obtain

$$k''=\frac{1}{r}(k'^3-k').$$

- When $k'=0$, it implies $z=n$, so the maximal surface is a flat.
- In the case where $k'\neq 0$, the Bernoulli equation $h'=\frac{1}{r}(h^3-h)$, $h=k'$ is obtained, with the solution $h=k'$ for $h=\frac{c_1}{\sqrt{r^2+c_1^2}}$. The corresponding value of k is found to be $c_1>0$. Therefore, the maximal surface is a catenoid (see [5]).

For Y_5, we detect also the catenoid rotating around the y-axis:

$$z=\sqrt{y^2-c_1^2\cosh^2(\frac{x}{c_1}+c_2)},\ c_1,c_2\in\mathbb{R}^*. \tag{2.10}$$

For Y_6, we find the catenoid rotating around the x-axis. (2.7) Consider

$$Y=Y_3+Y_4=-y\frac{\partial}{\partial x}+x\frac{\partial}{\partial y}+\frac{\partial}{\partial z}.$$

The global invariants of this group are $A_1=\sqrt{x^2+y^2}$, $A_2=z-arctg\frac{y}{x}$. Thus $z=\theta+k(r)$. By substituting in the Equation (2.1) we get

$$k''(1-\frac{1}{r^2})+\frac{1}{r}k'(1-k'^2-\frac{2}{r^2})=0.$$

- If $k'=0$ as a result
  $z=arctg\frac{y}{x}+c_1,\ c_1\in\mathbb{R},$
  subsequently, the helicoid is discovered.
- Persume $k'\neq 0$, and indicate $k'=\frac{h}{r}$. We obtain the differential equation $h'(r^2-1)-\frac{h}{r}(h^2+1)=0$, with the general solution $h=\sqrt{\frac{r^2-1}{(c_1^2-1)r^2+1}}$, $c_1>1$, and

$$k(r)=c_2\ln(\sqrt{r^2+1}\pm\sqrt{r^2-c_2^2})+arctg(\pm\frac{1}{c_2}\sqrt{\frac{r^2-c_2^2}{r^2+1}})-c_2\ln\sqrt{r^2+c_2^2}+c,$$

where $c_2=\frac{1}{\sqrt{c_1^2-1}}$, $c\in\mathbb{R}$. In this instance, we derive Scherk's second surface.

$$z=arctg\theta+k(r),$$

which is helicoidal surface (2.8). Consider

$$Y=Y_4+Y_7=(x-y)\frac{\partial}{\partial x}+(x+y)\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}.$$

We determine the global invariants as $A_1=\theta-\ln r$ and $A_2=\frac{z}{r}$. Consequently, we can express z as $rk\sqrt{\theta-\ln r}$. By substituting this into Equation (2.1), the resulting differential equation is given by:

$$k''(2+k^2)+k-2k'-k'^3+(k-k')^3=0.$$

In this specific scenario, finding a solution presents a more intricate challenge and calls for a thorough investigation in future studies. Moreover, we plan to delve into the classification of the s-subalgebra for $s>1$. By employing this theoretical framework, our objective is to deduce all solutions of Equation (2.1).

3. Conclusion

In summary, this study has delved into the profound world of maximal surfaces and their underlying symmetries, demonstrating the remarkable power of mathematical symmetry in solving intricate problems in the realm of differential geometry. By applying Lie symmetries to the differential equations governing these surfaces, we have uncovered a wealth of information about their characteristics, symmetries, and classification. The pursuit of Lie symmetry groups for maximal surfaces in L^3 through the partial differential equation describing maximal immersions has yielded valuable reductions in equation order and, consequently, solutions to this fascinating problem. This work highlights the enduring importance of symmetry in uncovering the hidden structures and invariance within differential equations, making it a powerful tool for exploring the intricate interplay between geometry and physics in the world of maximal surfaces.

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