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Dynamic Insights into Gaseous Diffusion: Analytical Soliton and Wave Solutions via Chaffee–Infante Equation in Homogeneous Media

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Abstract

Gaseous diffusion (GD) has been used in various fields, including electromagnetic wave fields, high-energy physics, fluid dynamics, coastal engineering, ion-acoustic waves in plasma physics, and optical fibers. GD involves random molecular movement from areas of high partial pressure to areas of low partial pressure. Researchers have developed models to describe this phenomenon, among these models is the $(2+1)$-dimensional Chaffee–Infante (CI)-equation. This research explores analytical soliton and wave solutions of Gaseous diffusion through a homogeneous medium considering two analytical methods, the Riccati equation and F-expansion methods. Thirty-seven different solutions have been identified and some of these solutions have been illustrated graphically. The figures show a range of bright, dark, singular, singular-periodic, and kink-type soliton wave solutions.

Keywords. $(2+1)$-dimensional Chaffee–Infante, Gaseous Diffusion (GD), Riccati equation method, F-expansion method.
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1. INTRODUCTION

One of the most significant natural phenomena, widely used in numerous fields such as physics, biology and chemistry is gaseous diffusion. Gaseous diffusion is the movement of molecules under a concentration gradient. GD has produced most of the enriched uranium in the world [1]. GD in materials science is widespread, for example, in processes such as sintering, corrosion, steel hardening, and semiconductor manufacturing. The diffusion of gas in a homogeneous medium involves studying a particle with a homogeneous temperature under the influence of an external force. Scientists have developed models to describe this phenomenon, among which is the $(2+1)$-dimensional Chaffee-Infante (CI) equation [2]. CI equation can depict the physical phenomena of particle diffusion ,which has been extensively used in electromagnetic wave fields, fluid dynamics, high-energy physics, coastal engineering, fluid mechanics, and ion-acoustic waves in plasma physics, optical fibers, and other fields [1]. CI equation can depict the physical phenomena of particle diffusion ,which has been extensively used in electromagnetic wave fields, fluid dynamics, high-energy physics, coastal engineering, fluid mechanics, and ion-acoustic waves in plasma physics, optical fibers, and other fields [1]. There are many techniques can be used to solve a wide of higher-dimensional nonlinear equations in the applied sciences and mathematical physics such as Lie and symmetry analysis [2–14], the inverse scattering transformation method [15], the Darboux transformation method [16, 17], generalized exponential rational function (GERF) technique [1, 18, 19], the Riccati equation method [20, 21], the (G/G) expansion [23–25], the tanh-coth (TC) method [26, 27], F-expansion method [28, 29], the Backlund transformation method [30, 31] ,and extra methods have been developed to get the exact solutions of nonlinear equations. Recently CI equation has been widely studied by many researchers. Khater and Ghanbari utilized five methods to obtain the solitary wave solutions for the CI equation: the $\exp(-\phi)$-expansion method, the extended (G'/G)-expansion method, the extended simplest equation method, the extended tanh expansion method, and the modified Khater method [32]. Moreover, differential quadrature method (DQM) is also an effective

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technique [33]. Also, many of these techniques are very effective in case of fractional calculus. Other techniques can be found here [34–36] .In [1], the GERF method is used to produce a great number of analytical soliton solutions. According to Akbar et al.[37] , The first integral method has been employed to find the analytic solutions of the CI equation. In [38], authors used the modified Khatter method to seek the CI equation with numerous new closed-form solutions. The solitary wave solution is founded by the extended sinh-Gordon expansion technique [39]. There are various other techniques for the construction of different kinds of solutions. In this article, we will apply two distinct methods namely the Riccati equation and F-expansion method to find the analytic solutions of the CI- equation which has the form:

$$\psi_{xt} + (-\psi_{xx} + \alpha\psi^3 - \alpha\psi)_x + \sigma\psi_{yy} = 0, \tag{1.1}$$

Where, α is the diffusion coefficient and σ is degradation coefficient.
The paper is arranged as follows. Section 2 introduces the Riccati equation method with application to the (2 + 1)-dimensional Chaffee–Infante equation. In section 3, the soliton wave solutions of the CI- equation are argued via F-expansion method. The paper ends with conclusions in section 4.

2. THE RICCATI EQUATION METHOD FOR THE (2 + 1)- DIMENSIONAL CI- EQUATION

In this section the Riccati equation method is discussed and applied to find the soliton solutions of the CI- equation. The method can be summarized in the following steps:
Step 1: Reduce the order of the partial differential equation via wave transformation. Consider

$$\psi(x, y, t) = \psi(\tau), \qquad \tau = \mu x + \lambda y - \gamma t. \tag{2.1}$$

Substituting Eq. (2.1) into (1.1) reduces it to:

$$-\mu^3\psi''' - \mu\alpha\psi' + 3\mu\alpha\psi^2\psi' - \mu\gamma\psi'' + \sigma\lambda^2\psi'' = (-\mu^3\psi'' - \mu\alpha\psi + \mu\alpha\psi^3)' + \left(\sigma\lambda^2 - \mu\gamma\right)\psi'' = 0. \tag{2.2}$$

Which can be simplified after integration to:

$$-\mu^3\psi'' + \left(\sigma\lambda^2 - \mu\gamma\right)\psi' - \mu\alpha\psi + \mu\alpha\psi^3 = 0. \tag{2.3}$$

Step 2: Assume that the solution of the reduced equation in the a series form:

$$\psi(\tau) = \sum_{i=0}^{N} A_i\phi^i(\tau) \tag{2.4}$$

where A_i are real constants will be determined, N is positive integer which result via the balancing principle with higher order non-linear and linear terms in Eq. (2.3), by balancing ψ'' and ψ^3 gives N=1, and Equation (2.4) is written as:

$$\psi(\tau) = A_0 + A_1\phi. \tag{2.5}$$

as $\phi(\tau)$ satisfies the following Riccati equation:

$$\phi'(\tau) = a\phi^2(\tau) + b\phi + c \tag{2.6}$$

where $a \neq 0$, b and c will be determined later. The solutions of Riccati equation can be written as follows: **Case (1)**: $\Omega > 0$,

$$\phi(\xi) = -\frac{b}{2a} - \frac{\sqrt{\Omega}}{2a}\tanh\left(\frac{\sqrt{\Omega}}{2}\xi + \xi_0\right), \tag{2.7}$$

$$\phi(\xi) = -\frac{b}{2a} - \frac{\sqrt{\Omega}}{2a}\coth\left(\frac{\sqrt{\Omega}}{2}\xi + \xi_0\right). \tag{2.8}$$

Case (2): $\Omega < 0$,

$$\phi(\xi) = -\frac{b}{2a} - \frac{\sqrt{-\Omega}}{2a}\tan\left(\frac{\sqrt{-\Omega}}{2}\xi + \xi_0\right). \tag{2.9}$$

$$\phi(\xi) = -\frac{b}{2a} - \frac{\sqrt{-\Omega}}{2a}\cot\left(\frac{\sqrt{-\Omega}}{2}\xi + \xi_0\right). \tag{2.10}$$

Case (3): $\Omega = 0$,

$$\phi(\xi) = -\frac{b}{2a} - \frac{1}{a\xi + \xi_0}, \tag{2.11}$$

where $\Omega = b^2 - 4ac$,and ξ_0 is the integration constant. **Step 3**: Inserting Eq. (2.5) together with Eq. (2.6) into Eq. (2.3), collecting all coefficients of each power of f^i,$0 \le i \le N$ in the resulting equation where these coefficients must vanish. This gives a system of algebraic equations involving the parameters $A_i, (i = 1, 2, 3), a, b, \mu, \lambda, \gamma$ and c

$$\begin{cases} \psi = A_0 + A_1\varphi, \\ \psi^{'} = A_1\varphi^{'} = aA_1\varphi^2 + bA_1\varphi + cA_1, \\ \psi^{''} = A_1\varphi^{''} = 2a^2A_1\varphi^3 + 3abA_1\varphi^2 + \left(b^2A_1 + 2acA_1\right)\varphi + cbA, \\ \psi^3 = A_1^3\varphi^3 + 3A_0A_1^2\varphi^2 + 3A_0^2A_1\varphi + A_0^3, \end{cases} \tag{2.12}$$

Substituting (2.12) into (2.3), gathering coefficients of each φ^i power, and setting the sum to zero yields the subsequent algebraic equations:

$$\text{Coefficient of } \varphi^3 = -2\mu^3a^2A_1 + \mu\alpha\ A_1^3, \tag{2.13}$$

$$\text{Coefficient of } \varphi^2 = -3\mu^3abA_1 + aA_1\sigma\lambda^2 - kmaA_1 + 3\mu\alpha A_0A_1^2, \tag{2.14}$$

$$\text{Coefficient of } \varphi = -k^3b^2A_1 - 2\mu^3acA_1 + \ bA_1\sigma\lambda^2 - bA_1\mu\gamma + 3\mu\alpha A_0^2A_1 - \mu\alpha A_1, \tag{2.15}$$

$$\text{Coefficient of } \varphi^0 = -\mu^3cbA + cA_1\sigma\lambda^2 - cA_1\mu\gamma + \ \mu\alpha A_0^3 - \mu\alpha A_0, \tag{2.16}$$

Solving the algebraic system (2.13)-(2.16) using Maple package confers six different groups of solutions of the system which result in twelve different solutions of the CI Equation (1.1) as:

Group1:

$$A_0 = \frac{b\mu}{\sqrt{2\alpha}}, \qquad A_1 = \sqrt{\frac{2}{\alpha}}\mu a, \quad c = \frac{b^2\mu^2 - 2\alpha}{4a\mu^2}, \qquad \gamma = \frac{\sigma\lambda^2}{\mu}, \quad \text{and} \quad \Omega = \frac{2\alpha}{\mu^2}\ . \tag{2.17}$$

Case1: $\Omega > 0$

$$\psi_1(\tau) = -\tanh\left(\frac{\sqrt{2\alpha}}{2\mu}(\mu x + \lambda y - \gamma t)\right), \tag{2.18}$$

$$\psi_2(\tau) = -\coth\left(\frac{\sqrt{2\alpha}}{2\mu}(\mu x + \lambda y - \gamma t)\right). \tag{2.19}$$

The kink wave ψ_1 is presented for $a = 1,\ \ b = 3,\ c = 1.25,\ \alpha = 2, t = 1$ *and* $\gamma = \lambda = \mu = 1$ at Figure 1, while the singular-type soliton solution (2.19) *at* $a = 1,\ \ b = 3,\ c = 1.25,\ \alpha = 2, t = 1$ *and* $\gamma = \lambda = \mu = 1$ at Figure 2.

Case (2): $\Omega < 0$,

$$\psi_3(\tau) = i\tan\left(\frac{\sqrt{-2\alpha}}{2\mu}(\mu x + \lambda y - \gamma t)\right), \tag{2.20}$$

$$\psi_4(\tau) = -i\cot\left(\frac{\sqrt{-2\alpha}}{2\mu}(\mu x + \lambda y - \gamma t)\right). \tag{2.21}$$

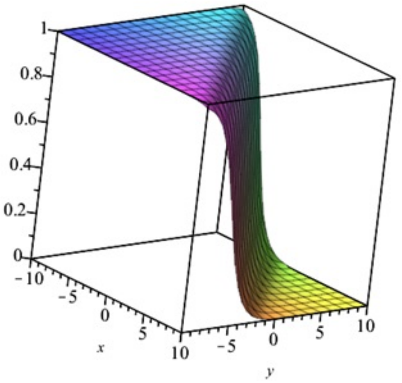

FIGURE 1. Kink wave ψ_1 *at* $a = 1,\ \ b = 3,\ c = 1.25,\ \alpha = 2, t = 1$ *and* $\gamma = \lambda = \mu = 1$.

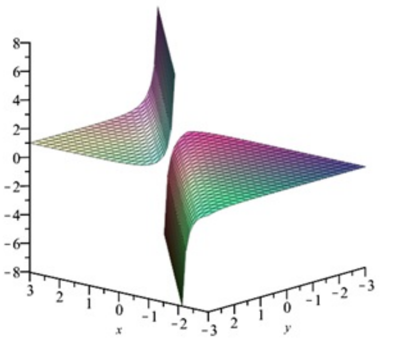

FIGURE 2. singular soliton solution ψ_2 *at* $a = 1,\ \ b = 3,\ c = 1.25,\ \alpha = 2, t = 1$ *and* $\gamma = \lambda = \mu = 1$.

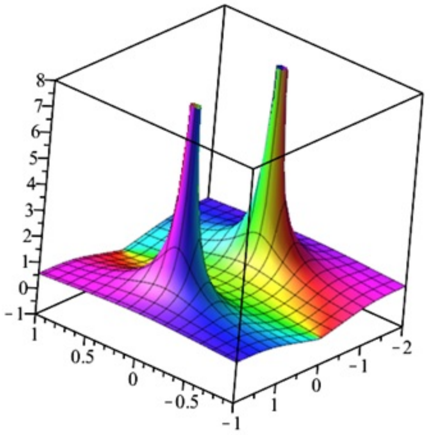

FIGURE 3. The double soliton solution ψ_3 *at* $a = 1,\ \ b = 3,\ c = 1.25,\ \alpha = 2, t = 1$ *and* $\gamma = \lambda = \mu = 1$.

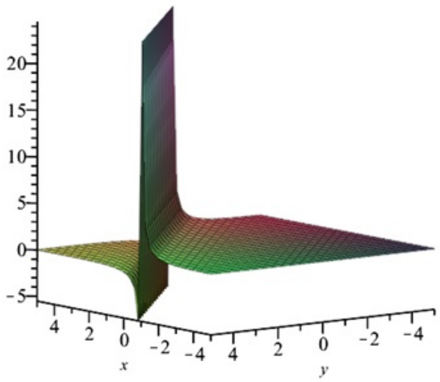

FIGURE 4. solitary wave solution ψ_6 *at* $a = 1,\ \ b = 3,\ c = 2,\ \alpha = 2, t = 1$ *and* $\gamma = 4,\ \ \lambda = \mu = 1$.

Figure 3 indicates the double soliton solution (2.20) *at* $a = 1,\ \ b = 3,\ c = 1.25,\ \alpha = -10$ *and* $-\gamma = \lambda = \mu = 5$.

Group2:

$$A_0=\frac{\sqrt{2}b\mu+\sqrt{\alpha}}{2\sqrt{\alpha}}\quad,\quad A_1=\sqrt{\frac{2}{\alpha}}\mu a\quad,c=\frac{2b^2\mu^2-\alpha}{8a\mu^2}\quad,\gamma=\frac{\sigma\lambda^2+\frac{3}{\sqrt{2}}\sqrt{\alpha}\,\mu^2}{\mu},and\Omega=\frac{\alpha}{2\mu^2}\ . \tag{2.22}$$

Case1: $\Omega > 0$

$$\psi_5(\tau)=\frac{1}{2}-\frac{1}{2}\tanh\left(\frac{\sqrt{\frac{\alpha}{2}}}{2\mu}(\mu x+\lambda y-\gamma t)\right), \tag{2.23}$$

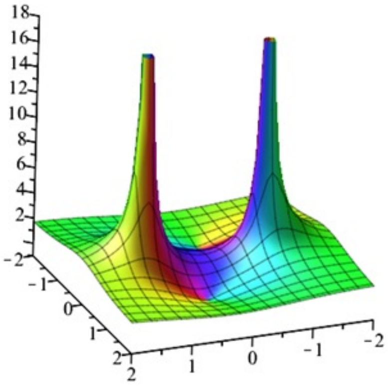

FIGURE 5. ψ_8 *at* $a = 1, \quad b = 3, \; c = 2, \; \alpha = 2, t = 1$ *and* $\gamma = 4, \quad \lambda = \mu = 1.$

$$\psi_6(\tau) = \frac{1}{2} - \frac{1}{2}\coth\left(\frac{\sqrt{\frac{\alpha}{2}}}{2\mu}(\mu x + \lambda y - \gamma t)\right). \tag{2.24}$$

The solitary wave solution ψ_6 *at* $a = 1, \quad b = 3, \; c = 2, \; \alpha = 2, t = 1$ *and* $\gamma = 4, \quad \lambda = \mu = 1.$ is shown in Figure 4.

Case (2): $\Omega < 0$,

$$\psi_7(\tau) = \frac{1}{2} + \frac{1}{2}i\tan\left(\frac{\sqrt{-\frac{\alpha}{2}}}{2\mu}(\mu x + \lambda y - \gamma t)\right), \tag{2.25}$$

$$\psi_8(\tau) = \frac{1}{2} - \frac{1}{2}i\cot\left(\frac{\sqrt{-\frac{\alpha}{2}}}{2\mu}(\mu x + \lambda y - \gamma t)\right). \tag{2.26}$$

The solution, ψ_8 *,at* $a = 1, \quad b = 3, \; c = 2, \; \alpha = 2, t = 1$ *and* $\gamma = 4, \quad \lambda = \mu = 1$ is depicted in Figure 5.

Group3:

$$A_0 = \frac{-b\mu}{\sqrt{2\alpha}} \quad , \quad A_1 = -\sqrt{\frac{2}{\alpha}}\mu a \quad , c = \frac{b^2\mu^2 - 2\alpha}{4a\mu^2} \quad , \gamma = \frac{\sigma\lambda^2}{\mu}, \textit{and } \Omega = \frac{2\alpha}{\mu^2}\ . \tag{2.27}$$

Case1: $\Omega > 0$

$$\psi_9(\tau) = \tanh\left(\frac{\sqrt{2\alpha}}{2\mu}(\mu x + \lambda y - \gamma t)\right), \tag{2.28}$$

$$\psi_{10}(\tau) = \coth\left(\frac{\sqrt{2\alpha}}{2\mu}(\mu x + \lambda y - \gamma t)\right). \tag{2.29}$$

Case (2): $\Omega < 0$,

$$\psi_{11}(\tau) = -i\tan\left(\frac{\sqrt{-2\alpha}}{2\mu}(\mu x + \lambda y - \gamma t)\right), \tag{2.30}$$

$$\psi_{12}(\tau) = i\cot\left(\frac{\sqrt{-2\alpha}}{2\mu}(\mu x + \lambda y - \gamma t)\right). \tag{2.31}$$

Group4:

$$A_0 = \frac{\sqrt{2}b\mu - \sqrt{\alpha}}{2\sqrt{\alpha}} \quad , \quad A_1 = \sqrt{\frac{2}{\alpha}}\mu a \quad ,c = \frac{2b^2\mu^2 - \alpha}{8a\mu^2} \quad ,\gamma = \frac{\sigma\lambda^2 - \frac{3}{\sqrt{2}}\sqrt{\alpha}\ \mu^2}{\mu}, \ \Omega = \frac{\alpha}{2\mu^2} \tag{2.32}$$

Case1: $\Omega > 0$

$$\psi_{13}(\tau) = -\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{\sqrt{\frac{\alpha}{2}}}{2\mu}(\mu x + \lambda y - \gamma t)\right), \tag{2.33}$$

$$\psi_{14}(\tau) = -\frac{1}{2} - \frac{1}{2}\coth\left(\frac{\sqrt{\frac{\alpha}{2}}}{2\mu}(\mu x + \lambda y - \gamma t)\right) \tag{2.34}$$

Case (2): $\Omega < 0$,

$$\psi_{15}(\tau) = -\frac{1}{2} + \frac{1}{2}i\tan\left(\frac{\sqrt{-\frac{\alpha}{2}}}{2\mu}(\mu x + \lambda y - \gamma t)\right), \tag{2.35}$$

$$\psi_{16}(\tau) = -\frac{1}{2} - \frac{1}{2}i\cot\left(\frac{\sqrt{-\frac{\alpha}{2}}}{2\mu}(\mu x + \lambda y - \gamma t)\right). \tag{2.36}$$

Group 5:

$$A_0 = \frac{-\sqrt{2}b\mu - \sqrt{\alpha}}{2\sqrt{\alpha}} \quad , \quad A_1 = -\sqrt{\frac{2}{\alpha}}\mu a \quad ,c = \frac{2b^2\mu^2 - \alpha}{8a\mu^2} \quad ,\gamma = \frac{\sigma\lambda^2 + \frac{3}{\sqrt{2}}\sqrt{\alpha}\ \mu^2}{\mu}, \ and\ \Omega = \frac{\alpha}{2\mu^2} \tag{2.37}$$

Case1: $\Omega > 0$

$$\psi_{17}(\tau) = -\frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\sqrt{\frac{\alpha}{2}}}{2\mu}(\mu x + \lambda y - \gamma t)\right), \tag{2.38}$$

$$\psi_{18}(\tau) = -\frac{1}{2} + \frac{1}{2}\coth\left(\frac{\sqrt{\frac{\alpha}{2}}}{2\mu}(\mu x + \lambda y - \gamma t)\right). \tag{2.39}$$

Case (2): $\Omega < 0$,

$$\psi_{19}(\tau) = -\frac{1}{2} - \frac{1}{2}i\tan\left(\frac{\sqrt{-\frac{\alpha}{2}}}{2\mu}(\mu x + \lambda y - \gamma t)\right), \tag{2.40}$$

$$\psi_{20}(\tau) = -\frac{1}{2} + \frac{1}{2}i\cot\left(\frac{\sqrt{-\frac{\alpha}{2}}}{2\mu}(\mu x + \lambda y - \gamma t)\right). \tag{2.41}$$

Group 6:

$$A_0 = \frac{-\sqrt{2}b\mu + \sqrt{\alpha}}{2\sqrt{\alpha}} \quad , \quad A_1 = -\sqrt{\frac{2}{\alpha}}\mu a \quad ,c = \frac{2b^2\mu^2 - \alpha}{8a\mu^2} \quad ,\gamma = \frac{\sigma\lambda^2 - \frac{3}{\sqrt{2}}\sqrt{\alpha}\ \mu^2}{\mu}, \ \Omega = \frac{\alpha}{2\mu^2} \tag{2.42}$$

Case1: $\Omega > 0$

$$\psi_{21}(\tau) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\sqrt{\frac{\alpha}{2}}}{2\mu}(\mu x + \lambda y - \gamma t)\right), \tag{2.43}$$

$$\psi_{22}(\tau) = \frac{1}{2} + \frac{1}{2}\coth\left(\frac{\sqrt{\frac{\alpha}{2}}}{2\mu}(\mu x + \lambda y - \gamma t)\right). \tag{2.44}$$

Case (2): $\Omega < 0$,

$$\psi_{23}(\tau) = \frac{1}{2} - \frac{1}{2} i \tan\left(\frac{\sqrt{-\frac{\alpha}{2}}}{2\mu}(\mu x + \lambda y - \gamma t)\right), \, \psi_{24}(\tau) = \frac{1}{2} + \frac{1}{2} i \cot\left(\frac{\sqrt{-\frac{\alpha}{2}}}{2\mu}(\mu x + \lambda y - \gamma t)\right). \tag{2.45}$$

Case (3): $\Omega = 0$

$$\psi_{25}(\tau) = -\frac{1}{a(\mu x + \lambda y - \gamma t)}. \tag{2.46}$$

3. F-EXPANSION METHOD FOR THE (2 + 1)- DIMENSIONAL CI- EQUATION

F-expansion method [21, 40] mainly starts from the reduced Equation (2.3), where the solution of the ODE is assumed to be:

$$\psi(\tau) = \sum_{i=0}^{N} s_i F^i(\tau), \tag{3.1}$$

where, $N = 1$. This value is obtained from the balancing between the higher order nonlinear and linear terms in Eq. (2.3) and leads to:

$$\begin{cases} \psi = s_0 + s_1 F, \\ \psi' = s_1 F' = s_1 (pF^4 + QF^2 + R)^{0.5}, \\ \psi'' = s_1 F'' = 2 s_1 p F^3 + s_1 Q F, \\ \psi^3 = {s_0}^3 + 3{s_0}^2 s_1 F + 3 s_0 {s_1}^2 F^2 + {s_1}^3 F^3. \end{cases} \tag{3.2}$$

s_i are real constants and $F(\tau)$ satisfies the following first order ODE:

$$F'(\xi) = (PF^4(\xi) + QF^2 + R)^{0.5} \tag{3.3}$$

as $P \neq 0$, Q and R are real constants. They will be determined later. The solutions of Eq. (3.3)are illustrated in terms of Jacobian elliptic functions, for more details see [41]. Inserting Eq. (3.1) together with Eq. (3.3) into Eq. (2.3), collecting all coefficients of each power of F^i, $i = 0, 1, \ldots, N$ to zero, results in the following system of algebraic equations:

$$Coefficient\ of\ F^6: \ 4p^2{s_1}^2\mu^6 - 4\alpha p{s_1}^4\mu^4 + \alpha^2\mu^2{s_1}^6 = 0, \tag{3.4}$$

$$Coefficient\ of\ F^5: -12\alpha p s_0 {s_1}^3\mu^4 + 6 s_0 \alpha^2\mu^2{s_1}^5 = 0, \tag{3.5}$$

$$\begin{aligned} &Coefficient\ of\ F^4: \ 4Qp{s_1}^2\mu^6 - 12\alpha p{s_0}^2{s_1}^2\mu^4 - 2Q\alpha{s_1}^4\mu^4 + 15\alpha^2{s_0}^2\mu^2{s_1}^4 + 4\alpha p{s_1}^2\mu^4 \\ &- 2\alpha^2\mu^2{s_1}^4 - p{s_1}^2\sigma^2\lambda^4 + 2p\mu\gamma\sigma{s_1}^2\lambda^2 - p{s_1}^2\mu^2\gamma^2 = 0, \end{aligned} \tag{3.6}$$

$$Coefficient\ of F^3: -4\alpha p s_1{s_0}^3\mu^4 - 6Q\alpha s_0{s_1}^3\mu^4 + 20\alpha^2\mu^2{s_0}^3{s_1}^3 + 4p s_0 s_1\alpha\mu^4 - 8s_0\alpha^2\mu^2{s_1}^3 = 0, \tag{3.7}$$

$$\begin{aligned} &Coefficient\ of F^3: -4\alpha p s_1{s_0}^3\mu^4 - 6Q\alpha s_0{s_1}^3\mu^4 + 20\alpha^2\mu^2{s_0}^3{s_1}^3 + 4p s_0 s_1\alpha\mu^4 - 8s_0\alpha^2\mu^2{s_1}^3 \\ &- Q{s_1}^2\sigma^2\lambda^4 + 2Q\mu\gamma\sigma{s_1}^2\lambda^2 - Q{s_1}^2\mu^2\gamma^2 = 0, \end{aligned} \tag{3.8}$$

$$Coefficient\ of F^1: -2Q\alpha s_1{s_0}^3\mu^4 + 6s_1\alpha^2\mu^2{s_0}^5 + 2\alpha Q s_0 s_1\mu^4 - 8s_1\alpha^2\mu^2{s_0}^3 + 2s_0 s_1\alpha^2\mu^2 = 0, \tag{3.9}$$

$$Coefficient\ of F^0: \alpha^2\mu^2{s_0}^6 - 2\alpha^2\mu^2{s_0}^4 + \alpha^2\mu^2{s_0}^2 - R{s_1}^2\sigma^2\lambda^4 + 2R\mu\gamma\sigma{s_1}^2\lambda^2 - R{s_1}^2\mu^2\gamma^2 = 0, \tag{3.10}$$

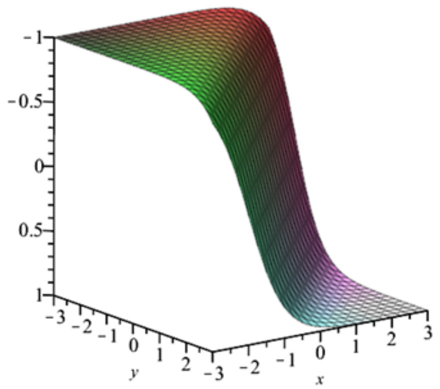

FIGURE 6. Kink wave ψ_{13} *at* $p = 1, \ \ Q = -2, \ R = 1, \ \alpha = 2, t = 1, \ \gamma = \lambda = \mu = 1$ *and* $s_1 = 1$.

Solving the algebraic system (3.4)-(3.10)using MAPLE package confers five different groups of solutions of the system which result in numerous different solutions of the CI Equation (1.1) as: **Group1:**

$$s_0 = 0 \ \ , \ s_1 \ = \ s_1 \ \ , p = \frac{{s_1}^2\alpha}{2\mu^2} \ \ , Q = \frac{-\alpha}{\mu^2}, \gamma = \frac{\sigma\lambda^2}{\mu}, \tag{3.11}$$

Case 1: If $P = m^2, \ Q = -1 - m^2, \ m = 1,$
A kink-type solution is obtained:

$$\psi_{26}(\tau) = \left(s_1 \tanh(\mu x + \lambda y - \gamma t)\right). \tag{3.12}$$

This solution is plotted in Figure 6 at:

$$p = 1, \ \ Q = -2, \ R = 1, \ \alpha = 2, t = 1, \ \gamma = \lambda = \mu = 1 \ \ and \ s_1 = 1. \tag{3.13}$$

Case 2: If $P = -m^2, Q = 2m^2 - 1, \ m = 1,$
Bright soliton solutions is obtained:

$$\psi_{27}(\tau) = s_1 \left(sech(\mu x + \lambda y - \gamma t)\right) \tag{3.14}$$

This solution is illustrated in Figure 7 *at* $p = -1, \ \ Q = 1, \ R = 1, \ \alpha = -1, t = 1, \ \gamma = \lambda = \mu = 1, \ and \ s_1 = \sqrt{2}$.
Case 3: If $P = 1, \ Q = -1 - m^2,$
The following solutions are obtained:
a- Singular solution with $m = 1$

$$\psi_{28}(\tau) = s_1 \left(\coth(\mu x + \lambda y - \gamma t)\right), \tag{3.15}$$

which is graphed *at* $p = 1, \ \ Q = -2, \ R = 1, \ \alpha = 2, t = 1, \ \gamma = \lambda = \mu = 1 \ \ and \ s_1 = 1$,in Figure 8.
b- Singular periodic solution with $m = 0$

$$\psi_{29}(\tau) = s_1 \left(\csc(\mu x + \lambda y - \gamma t)\right). \tag{3.16}$$

This solution is depicted in Figure 9 *at* $p = 1, \ \ Q = -1, \ R = 1, \ \alpha = 1, t = 1, \ \gamma = \lambda = \mu = 1 \ and \ s_1 = \sqrt{2}$.
Case 4: If $P = 1 - m^2, Q = 2m^2 - 1, m = 0$
A singular periodic solution is obtained as:

$$\psi_{30}(\tau) = s_1 \left(\sec(\mu x + \lambda y - \gamma t)\right). \tag{3.17}$$

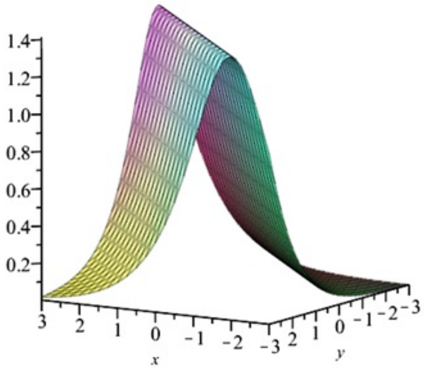

FIGURE 7. Bright kink soliton solution ψ_{14} *at* $p = -1,\ Q = 1,\ R = 1,\ \alpha = -1, t = 1,\ \gamma = \lambda = \mu = 1$ *and* $s_1 = \sqrt{2}$.

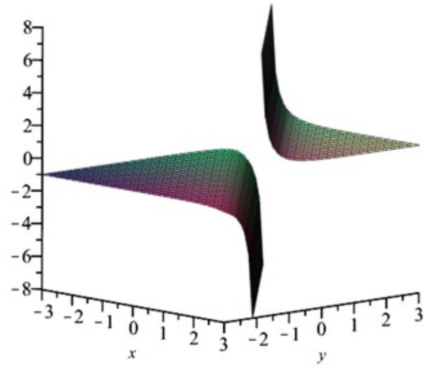

FIGURE 8.

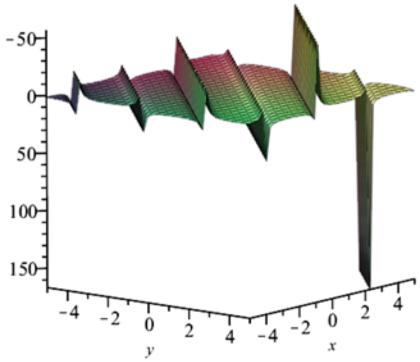

FIGURE 9. Singular periodic solution ψ_{16} *at* $p = 1,\ Q = -1,\ R = 1,\ \alpha = 1, t = 1,\ \gamma = \lambda = \mu = 1,$ *and* $s_1 = \sqrt{2}$.

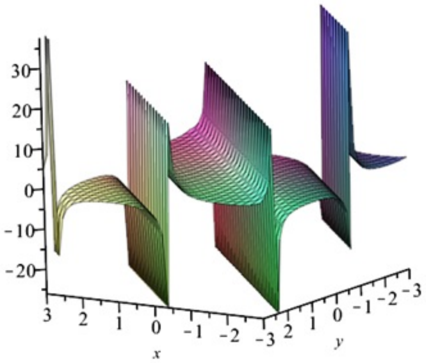

FIGURE 10. Singular periodic solution ψ_{17} *at* $p = 1,\ Q = -1,\ R = 1,\ \alpha = 1, t = 1,\ \gamma = \lambda = \mu = 1$ *and* $s_1 = \sqrt{2}$.

This solution is illustrated in Figure 10 for $p = 1,\ \ Q = -1,\ R = 1,\ \alpha = 1, t = 1,\ \gamma = \lambda = \mu = 1,\ and\ s_1 = \sqrt{2}$.

Group2:

$$s_0 = \frac{1}{2}\ \ ,\ s_1\ =\ s_1\ \ ,p = \frac{{s_1}^2\alpha}{2\mu^2}\ \ ,Q = \frac{-\alpha}{4\mu^2}, \gamma = \frac{\sigma\lambda^2 + \frac{3}{\sqrt{2}}\sqrt{\alpha}\ \mu^2}{\mu}, R = \frac{\alpha}{32\mu^2{s_1}^2} \tag{3.18}$$

Case 1: If $P = m^2,\ Q = -1 - m^2, m = 1$.
A kink-type soliton solution is obtained:

$$\psi_{31}(\tau) = \frac{1}{2} + (s_1 \tanh(\mu x + \lambda y - \gamma t))\,. \tag{3.19}$$

Case 2: If $P = -m^2, Q = 2m^2 - 1,\ m = 1$.

A Bright and dark soliton solution is obtained:

$$\psi_{32}(\tau) = \frac{1}{2} + s_1 \left(sech(\mu x + \lambda y - \gamma t) \right). \tag{3.20}$$

Case 3: If $P = 1,\ Q = -1 - m^2$.
The following solutions are obtained:
a-Singular solution with $m = 1$

$$\psi_{33}(\tau) = \frac{1}{2} + s_1 \left(\coth(\mu x + \lambda y - \gamma t) \right), \tag{3.21}$$

b-Singular periodic solution with $m = 0$.

$$\psi_{34}(\tau) = \frac{1}{2} + s_1 \left(\csc(\mu x + \lambda y - \gamma t) \right). \tag{3.22}$$

Case 4: If $P = 1 - m^2, Q = 2m^2 - 1,\ m = 0$.
A singular periodic solution is obtained:

$$\psi_{35}(\tau) = \frac{1}{2} + s_1 \left(\sec(\mu x + \lambda y - \gamma t) \right). \tag{3.23}$$

Group3:

$$s_0 = \frac{-1}{2} \quad , \ s_1 \ = \ s_1 \ \ , p = \frac{{s_1}^2 \alpha}{2\mu^2} \quad , Q = \frac{-\alpha}{4\mu^2}, \gamma = \frac{\sigma\lambda^2 + \frac{3}{\sqrt{2}}\sqrt{\alpha}\ \mu^2}{\mu}, R = \frac{\alpha}{32\mu^2 {s_1}^2}, \tag{3.24}$$

Case 1: If $P = m^2,\ Q = -1 - m^2, m = 1$.
A kink-type soliton solution is obtained:

$$\psi_{36}(\tau) = -\frac{1}{2} + \left(s_1 \tanh(\mu x + \lambda y - \gamma t) \right). \tag{3.25}$$

Case 2: If $P = -m^2, Q = 2m^2 - 1,\ m = 1$.
Bright and dark soliton solutions is obtained:

$$\psi_{37}(\tau) = -\frac{1}{2} + s_1 \left(sech(\mu x + \lambda y - \gamma t) \right). \tag{3.26}$$

Case 3: If $P = 1,\ Q = -1 - m^2$.
The following solutions are created:
a-Singular solution with $m = 1$

$$\psi_{38}(\tau) = -\frac{1}{2} + s_1 \left(\coth(\mu x + \lambda y - \gamma t) \right), \tag{3.27}$$

b-Singular periodic solution with $m = 0$

$$\psi_{39}(\tau) = -\frac{1}{2} + s_1 \left(\csc(\mu x + \lambda y - \gamma t) \right). \tag{3.28}$$

Case 4: If $P = 1 - m^2, Q = 2m^2 - 1, m = 0$.
A singular periodic solution is created:

$$\psi_{40}(\tau) = -\frac{1}{2} + s_1 \left(\sec(\mu x + \lambda y - \gamma t) \right). \tag{3.29}$$

Group4:

$$s_0=\frac{1}{2}\quad,\ s_1\ =\ s_1\quad,p=\frac{{s_1}^2\alpha}{2\mu^2}\quad,Q=\frac{-\alpha}{4\mu^2},\gamma=\frac{\sigma\lambda^2-\frac{3}{\sqrt{2}}\sqrt{\alpha}\ \mu^2}{\mu},R=\frac{\alpha}{32\mu^2{s_1}^2},\tag{3.30}$$

Case 1: If $P=m^2,\ Q=-1-m^2,m=1$.
A kink-type soliton solution is obtained:

$$\psi_{41}(\tau)=\frac{1}{2}+\left(s_1\tanh(\mu x+\lambda y-\gamma t)\right).\tag{3.31}$$

Case 2: If $P=-m^2,Q=2m^2-1,\ m=1$
Bright and dark soliton solution is given:

$$\psi_{42}(\tau)=\frac{1}{2}+s_1\left(sech(\mu x+\lambda y-\gamma t)\right).\tag{3.32}$$

Case 3: If $P=1,\ Q=-1-m^2$.
The following solutions are: *a-Singular solution with* $m=1$

$$\psi_{43}(\tau)=\frac{1}{2}+s_1\left(\coth(\mu x+\lambda y-\gamma t)\right),\tag{3.33}$$

b-Singular periodic solution with $m=0$.

$$\psi_{44}(\tau)=\frac{1}{2}+s_1\left(\csc(\mu x+\lambda y-\gamma t)\right).\tag{3.34}$$

Case 4: If $P=1-m^2,Q=2m^2-1,\ m=0$.
A singular periodic solution is given in the following form:

$$\psi_{45}(\tau)=\frac{1}{2}+s_1\left(\sec(\mu x+\lambda y-\gamma t)\right).\tag{3.35}$$

Group5:

$$s_0=\frac{-1}{2}\quad,\ s_1\ =\ s_1\quad,p=\frac{{s_1}^2\alpha}{2\mu^2}\quad,Q=\frac{-\alpha}{4\mu^2},\gamma=\frac{\sigma\lambda^2-\frac{3}{\sqrt{2}}\sqrt{\alpha}\ \mu^2}{\mu},R=\frac{\alpha}{32\mu^2{s_1}^2},\tag{3.36}$$

Case 1: If $P=m^2,\ Q=-1-m^2,m=1$.
A kink-type soliton solution is created:

$$\psi_{46}(\tau)=-\frac{1}{2}+\left(s_1\tanh(\mu x+\lambda y-\gamma t)\right).\tag{3.37}$$

Case 2: If $P=-m^2,Q=2m^2-1,\ m=1$
Bright and dark soliton solution is given:

$$\psi_{47}(\tau)=-\frac{1}{2}+s_1\left(sech(\mu x+\lambda y-\gamma t)\right).\tag{3.38}$$

Case 3: If $P=1,\ Q=-1-m^2$.

The following solutions are created:
a-Singular solution with $m = 1$.

$$\psi_{48}(\tau) = -\frac{1}{2} + s_1 \left(\coth(\mu x + \lambda y - \gamma t)\right), \tag{3.39}$$

b-Singular periodic solution with $m = 0$.

$$\psi_{49}(\tau) = -\frac{1}{2} + s_1 \left(\csc(\mu x + \lambda y - \gamma t)\right). \tag{3.40}$$

Case 4: If $P = 1 - m^2, Q = 2m^2 - 1, m = 0$.
A singular periodic solution is obtained:

$$\psi_{50}(\tau) = -\frac{1}{2} + s_1 \left(\sec(\mu x + \lambda y - \gamma t)\right). \tag{3.41}$$

4. CONCLUSIONS

In this article, the dynamical behavior of gas diffusion was examined by investigating the (2 + 1)-dimensional Chaffee–Infante equation. Firstly, the Riccati equation method is applied reveals six groups of soliton solutions. The F-expansion method is then applied, confers five different groups of solutions, which result in twenty-five different solutions. The obtained solutions are varying between kink-type soliton, singular soliton, singular periodic dark and bright soliton solutions. Many of these solutions are essential for understanding the behavior of high frequency waves. Such results are tremendously recommended in advanced research and innovation.

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