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Quintic B-Spline Method for Numerical Solution of Second-Order Singularly Perturbed Delay Differential Equations

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Abstract

This article presents a quintic B-spline method to find an approximate solution of the second-order singularly perturbed differential equation in which the convection term occurs with negative shift. The proposed method gives rise to a penta-diagonal linear system. Thomas algorithm is employed to solve the obtained system of equations. The method's convergence is examined through truncation error analysis, and the existence and uniqueness of the solution are also established. Maximum absolute error is tabulated for two numerical examples, proving the proposed method's efficiency. Graphs are drawn to show the behavior of the solution. A comparative study shows that the obtained solution is better than the previous solutions in the literature. The method is found to be fourth-order convergence. The effect of the delay parameter on the boundary region is also discussed in the example.

Keywords. Singularly perturbed delay differential equation, Boundary value problem, Spline methods, Quintic B-spline method, Existence and uniqueness, Uniform mesh, Convergence.
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1. Introduction

Singular perturbation is a phenomenon in which the equation or solution of the equation changes abruptly when the perturbation parameter ε is equated to zero. For example, suppose we have a n^{th}-order differential equation, and ε occurs with n^{th}-order derivative. Then, the order of the differential equation will reduce, and one of the boundary conditions becomes redundant as ε tends to zero. The given differential equation is singularly perturbed if ε occurs with the highest order derivative. Such an equation is called a singularly perturbed differential equation (SPDE). Perturbation theory plays a significant role in finding an approximate solution to regular and singular perturbation boundary value problems (BVPs) [3, 26].

Models involving time delays are seen in different fields like biology, population dynamics, economics, physics, engineering, etc. A review of applications of time delay in the area of engineering is conducted by Kyrychko and Hogan [20]. If delay parameter δ appears in an SPDE, it is called a singularly perturbed delay differential equation (SPDDE). The existence of delay term implies that the present value of the dependent variable or its derivatives depends on the past values of the independent variable. SPDDEs are called neutral if the delay term shows up with the highest order derivative; otherwise, it is called retarded. Following is an example of retarded delay differential equation (DDE) of order k.

$$\begin{aligned} y^{(k)}(w) &= g\Big(w, y(w), y'(w-\delta_1), y''(w-\delta_2)\dots, y^{(k-1)}(w-\delta_{k-1})\Big), \\ &\quad w \in [w_0, w_0+\theta], \quad \delta_i \geq 0, \quad \forall\, 1 \leq i \leq k-1, \\ y(w_0) &= \tau_0, \quad y^{(n)}(w) = \tau_n\,, n = 1, 2, \dots, k-1, \quad w \in [w_0 - \delta_i, w_0]\,. \end{aligned} \tag{1.1}$$

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On the other hand, neutral DDE is of the form:

$$\begin{aligned} y^{(k)}(w) &= g\left(w, y(w), y'(w-\eta_1), y''(w-\eta_2), \ldots, y^{(k-1)}(w-\eta_{k-1}), y^{(k)}(w-\eta_k)\right), \\ &\quad w \in [w_0, w_0+\theta], \quad \eta_i \geq 0, \quad \forall 1 \leq i \leq k, \\ y(w_0) &= \tau_0, \quad y^{(n)}(w) = \tau_n \quad n = 1, 2, \ldots, k, \quad w \in [w_0 - \eta_i, w_0]. \end{aligned} \tag{1.2}$$

SPDDEs can further be classified based on parameters like boundary conditions, large or fixed delay, linear and nonlinear, etc. A variety of these equations can be found in the literature. Senu *et al.* [33] have developed a hybrid numerical method based on Newton's interpolation and two-derivative type Runge-Kutta method to solve third-order SPDDE with constant delay. Mahendran and Subburayan [25] have presented a finite difference method (FDM) to find the approximate solution of third-order SPDDE on Shishkin mesh. They have checked the stability and estimated different bounds of the derivatives of the solution. Liu *et al.* [21] have addressed a system of semi-linear initial value SPDEs on an adaptive grid with the help of Euler's backward formula. They have also applied the adaptive grid technique to a nonlinear system. Fadugba *et al.* [11] have presented an exponential polynomial single-step method to solve first-order SPDDE. The stability of the method is discussed, and the solution is compared with the two other methods. Elango [8] has deployed FDM on a Shishkin mesh to solve second-order SPDDE with non-local integral boundary conditions, and it is found that the order of convergence is five. Ejere *et al.* [7] have discussed an exponentially fitted method for finding the numerical solution of SPDDE with large delay, and they have divided the given domain into six parts to obtain ε-uniform convergence results.

Difference methods and spline techniques are commonly used to solve SPDEs and SPDDEs. Gupta and Kaushik [13] have presented a hybrid difference scheme for fourth-order SPDE on a non-uniform mesh. They first transformed the SPDE into a system of two coupled differential equations and applied compact FDM to solve an obtained system of equations. Singh and Kumar [32] have also solved fourth-order SPDE using the quadratic B-spline method. They have converted the problem into a weekly coupled system of second-order BVPs, and the method is shown to be second-order convergence. Chakravarthy *et al.* [5] have studied the numerical solution of second-order SPDDE with a large delay using cubic B-spline in tension. The advantage of spline techniques is that one can find the solution at any point of the domain and not only at the nodal points. Due to this, spline methods are widely used to find the solution of different types of BVPs. Goh *et al.* [12] have developed a quartic B-spline method for solving second-order singular BVP. Roul and Thula [29] have introduced the quartic B-spline method to find the approximate solution of Bratu type and Lane-Emden problems. The quintic B-spline method is a powerful numerical interpolation method. It employs B-splines of fifth-degree, assuring higher order continuity and smoothness than lower order splines. It motivates the researchers to apply quintic B-spline to a variety of boundary value problems [1, 9, 19]. Bawa and Natesan [2] have approximated self-adjoint SPDE using the quintic B-spline method. Lodhi and Mishra [22–24] have used quintic B-spline methods to find the numerical solution of second and fourth-order SPDEs. Iqbal *et al.* [14] have solved Kuramoto–Sivashinsky equation. Authors have applied new approximations of quintic B-spline in special derivatives and the finite forward differences in temporal integration. An inverse Rosenau equation with Dirichlet's boundary conditions was handled by Saeedi and Pourgholi [30]. Thus, the quintic B-spline technique is well suited for various differential equations arising in different physical phenomena. One can refer to [4, 27] for a detailed discussion of spline techniques.

Nowadays, researchers are paying much attention to developing an efficient numerical method for solving second-order SPDDEs of retarded type with delay in convection term. Kadalbajoo and Sharma have presented papers on treating these types of BVPs. In [15], they used a combination of central and forward difference methods to obtain the numerical solution. A discrete invariant embedding algorithm is used to prove the stability and convergence of the method. An upward finite difference scheme [17] has been applied to find the solution of SPDDEs with smaller and bigger delay arguments. The fitted mesh method [16] has been utilized to solve SPDDEs with left and right boundary layers. Singh and Reddy [31] have considered the same problem with the boundary layer at one end. After applying Taylor's series expansion to delay term, they replaced the resulting SPDE with the first-order differential equation. The obtained equation was solved using a proper fitting factor and linear interpolation formulas. An exponentially fitted scheme [6, 10] has been utilized to discuss the solution of SPDDEs on a boundary layer. Kanth and Kumar [28] have developed a hybrid numerical scheme for solving linear and nonlinear SPDDEs based on tension spline in the boundary region and midpoint approximation in the outer region on a piecewise uniform mesh. Woldaregay *et al.* [34] have developed the uniform convergence numerical method by using exponentially fitted FDM to solve SPDDEs with delay on the first derivative term.

This work aims to get a better numerical solution of second-order SPDDEs through a quintic B-spline method (QBSM) with improved order of convergence. The remaining part of the manuscript is presented as follows. Statement of the problem is given in section 2, followed by the description of the quintic B-spline method in section 3. Existence and uniqueness are described in section 4. Convergence is discussed in section 5. Numerical results and discussion are described in section 6 and the conclusion towards the end of the paper in section 7.

2. Statement of Problem

Consider the following SPDDEs of retarded type:

$$\varepsilon y''(w) + a(w)\, y'(w-\delta) + b(w)\, y(w) = f(w), \quad w \in \Omega = (0,1), \tag{2.1}$$

with the boundary conditions

$$y(w) = \phi(w), \quad -\delta \le w \le 0, \qquad y(1) = \gamma. \tag{2.2}$$

where, ε is a small parameter with $0 < \varepsilon << 1$ and δ is a delay parameter $0 < \delta << 1$; to assure the existence of the solution, we assume that the functions $a(w), b(w), f(w)$ and $\phi(w)$ are sufficiently smooth and γ is constant. We also assume that $a(w) \ge M > 0$ for all $w \in (0,1)$. It implies that the boundary layer [18] lies in the neighbourhood of $w = 0$, i.e. the left-end point and if $a(w) \le M < 0$ then the boundary layer lies in the neighbourhood of $w = 1$, i.e. the right-end point. Solution of SPDDEs has a multiscale character, i.e. there are thin transition layers in the domain where the solution varies rapidly, known as inner region, and the layers in which the solution varies slowly is known as outer region. Using Taylor's series expansion, we get

$$y'(w-\delta) \approx y'(w) - \delta y''(w). \tag{2.3}$$

Using Equation (2.3) in Equations (2.1) and (2.2), we get

$$(\varepsilon - \delta a(w)) y''(w) + a(w) y'(w) + b(w) y(w) = f(w), \tag{2.4}$$

$$y(0) = \phi_0, \quad y(1) = \gamma. \tag{2.5}$$

3. Quintic B-Spline Method

This section describes a QBSM to find the numerical solution of Equations (2.4) and (2.5) on a uniform mesh. Let $\pi = \{0 = w_0 < w_1 < w_2 < \cdots < w_N = 1\}$ be a partition of Ω and $h = w_{i+1} - w_i$. Insert three nodes on each side of the partition π as $w_{-3} < w_{-2} < w_{-1} < w_0$ and $w_N < w_{N+1} < w_{N+2} < w_{N+3}$. Let $L_2(0,1)$ be the set of all square-integrable functions. Then $L_2(0,1)$ is a vector space, and X is a subspace of $L_2(0,1)$. QBSM basis functions are defined as follows.

$$\text{ß}_i(w) = \frac{1}{h^5}\begin{cases} (w-w_{i-3})^5, & \text{if} \quad w \in [w_{i-3}, w_{i-2}] \\ h^5 + 5h^4(w-w_{i-2}) + 10h^3(w-w_{i-2})^2 + 10h^2(w-w_{i-2})^3 & \\ +5h(w-w_{i-2})^4 - 5(w-w_{i-2})^5, & \text{if} \quad w \in [w_{i-2}, w_{i-1}] \\ 26h^5 + 50h^4(w-w_{i-1}) + 20h^3(w-w_{i-1})^2 - 20h^2(w-w_{i-1})^3 & \\ -20h(w-w_{i-1})^4 + 10(w-w_{i-1})^5, & \text{if} \quad w \in [w_{i-1}, w_i] \\ 26h^5 + 50h^4(w_{i+1}-w) + 20h^3(w_{i+1}-w)^2 - 20h^2(w_{i+1}-w)^3 & \\ -20h(w_{i+1}-w)^4 + 10(w_{i+1}-w)^5, & \text{if} \quad w \in [w_i, w_{i+1}] \\ h^5 + 5h^4(w_{i+2}-w) + 10h^3(w_{i+2}-w)^2 + 10h^2(w_{i+2}-w)^3 & \\ +5h(w_{i+2}-w)^4 - 5(w_{i+2}-w)^5, & \text{if} \quad w \in [w_{i+1}, w_{i+2}] \\ (w_{i+3}-w)^5, & \text{if} \quad w \in [w_{i+2}, w_{i+3}] \\ 0, & \text{otherwise for i} = 0, 1, 2, \ldots, \text{N}. \end{cases} \tag{3.1}$$

TABLE 1. Values of $ß_i(w), ß_i'(w), ß_i''(w)$ and $ß_i'''(w)$ at nodal points.

$ß\,(w)$	w_{i-3}	w_{i-2}	w_{i-1}	w_i	w_{i+1}	w_{i+2}	w_{i+3}
$ß_i\,(w)$	0	1	26	66	26	1	0
$ß_i'\,(w)$	0	5/h	50/h	0	-50/h	-5/h	0
$ß_i''\,(w)$	0	$20/h^2$	$40/h^2$	$-120/h^2$	$40/h^2$	$20/h^2$	0
$ß_i'''\,(w)$	0	$60/h^3$	$-120/h^3$	0	$120/h^3$	$-60/h^3$	0

The B-spline functions $ß_i\,(w)$ are four times continuously differentiable on the entire real line. The values of $ß_i(w), ß_i'(w)$, $ß_i''(w)$, and $ß_i'''(w)$ at the nodal points are given in Table 1.
The set $\Pi = \{ß_{-2}, ß_{-1}, ß_0, ß_1, \ldots\ldots, ß_{N-1}, ß_N, ß_{N+1}, ß_{N+2}\}$ is linearly independent on the interval $(0,1)$. Thus, the $\psi_5\,(\pi) = span \prod$ is $(N{+}5)$-dimensional, and also one can easily prove that $\psi_5\,(\pi) \subseteq_{subpace} X$. Define the spline function:

$$\Im(w) = \sum_{i=-2}^{N+2} C_i ß_i(w). \tag{3.2}$$

Assume that it is an approximate solution of Equations (2.4)-(2.5), where the constants $C_i{}'s$ need to be determined and $ß_i(w)'s$ are the fifth-degree spline functions defined in Equation (3.1). To solve equation (2.1) along with the boundary conditions (2.2), the $\Im\,(w)$ is evaluated at nodal points $w = w_i \ (i = 0, 1, 2, \ldots, N)$. Using the Table 1 values in Equation (3.1), we obtain

$$y\,(w_i) = \Im(w_i) = C_{i-2} + 26C_{i-1} + 66C_i + 26C_{i+1} + C_{i+2}. \tag{3.3}$$

$$\mathfrak{M}_i = \Im'(w_i) = \frac{1}{h}\left(-5C_{i-2} - 50C_{i-1} + 50C_{i+1} + 5C_{i+2}\right). \tag{3.4}$$

$$M_i = Im''(w_i) = \frac{1}{h^2}\left(20C_{i-2} + 40C_{i-1} - 120C_i + 40C_{i+1} + 20C_{i+2}\right). \tag{3.5}$$

$$T_i = \Im'''(w_i) = \frac{1}{h^3}\left(-60C_{i-2} + 120C_{i-1} - 120C_{i+1} + 60C_{i+2}\right). \tag{3.6}$$

The approximate values of $y'\,(w_i)\,,\ y''\,(w_i)\,,\ y'''\,(w_i)$ can be encountered using $\mathfrak{M}_i, M_i, T_i$. Substitute $\Im(x)$from Equation (3.1) into Equation (2.1) and the boundary conditions (2.2).

$$\left(\varepsilon - \delta a\,(w)\right)\Im''\,(w) + a\,(w)\,\Im'\,(w) + b\,(w)\,\Im\,(w) = f\,(w)\,. \tag{3.7}$$

$$\Im\,(0) = \phi_0, \qquad \Im\,(1) = \gamma. \tag{3.8}$$

At nodal points $w = w_i \ (i = 0, 1, 2, \ldots, N)$ Equation (3.7) becomes

$$\left(\varepsilon - \delta a\,(w_i)\right)\Im''\,(w_i) + a\,(w_i)\,\Im'\,(w_i) + b\,(w_i)\,\Im\,(w_i) = f\,(w_i)\,. \tag{3.9}$$

Using Equations (3.3)-(3.6) in Equation (3.9), we obtain

$$\begin{aligned} &\tfrac{A_i}{h^2}\left(20C_{i-2} + 40C_{i-1} - 120C_i + 40C_{i+1} + 20C_{i+2}\right) + \tfrac{a_i}{h}\left(-5C_{i-2} - 50C_{i-1} + 50C_{i+1} + 5C_{i+2}\right) \\ &+ b_i\left(C_{i-2} + 26C_{i-1} + 66C_i + 26C_{i+1} + C_{i+2}\right) = f_i, \end{aligned} \tag{3.10}$$

where $A_i = \varepsilon - \delta a\,(w_i)\,,\ a_i = a\,(w_i)\,,\ b_i = b\,(w_i)\,,\ f_i = f\,(w_i)$. After simplifications, we get

$$\mu_1\,(w_i)\,C_{i-2} + \mu_2\,(w_i)\,C_{i-1} + \mu_3\,(w_i)\,C_i + \mu_4\,(w_i)\,C_{i+1} + \mu_5\,(w_i)\,C_{i+2} = f_i\,h^4, \tag{3.11}$$

where

$$\begin{aligned} &\mu_1\,(w_i) = 20A_i + 5ha_i + h^2 b_i, \quad \mu_2\,(w_i) = 40A_i + 50ha_i + 26h^2 b_i, \\ &\mu_3\,(w_i) = -120A_i + 66h^2 b_i, \quad \mu_4\,(w_i) = 40A_i - 50ha_i + 26h^2 b_i, \\ &\mu_5\,(w_i) = 20A_i - 5ha_i + h^2 b_i, \quad \text{for } \ i = 0, 1, 2, \ldots, N. \end{aligned}$$

We obtain the following equations using the boundary conditions (3.8)

$$C_{-2}+26C_{-1}+66C_0+26C_1+C_2=\phi_0, \tag{3.12}$$

$$C_{N-2}+26C_{N-1}+66C_N+26C_{N+1}+C_{N+2}=\gamma. \tag{3.13}$$

Equations (3.11), along with (3.12) and (3.13), form a system of $(N+3)$ equations in $(N+5)$unknowns. To obtain two more equations, differentiate Equation (3.7), we get

$$A\left(w\right)\Im'''\left(w\right)+P\left(w\right)\Im''\left(w\right)+Q\left(w\right)\Im'\left(w\right)+R\left(w\right)\Im\left(w\right)=f'\left(w\right), \tag{3.14}$$

where

$$\begin{aligned}&P\left(w\right)=a\left(w\right)-\delta a'\left(w\right),\\&Q\left(w\right)=a'\left(w\right)+b\left(w\right),\\&R\left(w\right)=b'\left(w\right).\end{aligned}$$

Using nodal points $w=w_0=0,\ w=w_N=1$ in the above equation and simplifying, we obtain

$$\begin{aligned}&\left(-60A_0+20hP_0+5h^2Q_0+h^3R_0\right)C_{-2}+\left(120A_0+40hP_0+50h^2Q_0+26h^3R_0\right)C_{-1}\\&+\left(-120hP_0+66h^3R_0\right)C_0+\left(-120A_0+40hP_0-50h^2Q_0+26h^3R_0\right)C_1\\&+\left(60A_0+20hP_0-5h^2Q_0+h^3R_0\right)C_2=f_0'h^3,\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}&\left(-60A_N+20hP_N+5h^2Q_N+h^3R_N\right)C_{N-2}+\left(120A_N+40hP_N+50h^2Q_N+26h^3R_N\right)C_{N-1}\\&+\left(-120hP_N+66h^3R_N\right)C_N+\left(-120A_N+40hP_N-50h^2Q_N+26h^3R_N\right)C_{N+1}\\&+\left(60A_N+20hP_N-5h^2Q_N+h^3R_N\right)C_{N+2}=f_0'h^3.\end{aligned} \tag{3.16}$$

Here, we obtain a system of $(N+5)$ equations in $(N+5)$ unknowns. We convert this system into a diagonally dominant system by eliminating C_{-2},C_{-1},C_{N+1}, and C_{N+2}. Eliminating C_{-2} from the Equations (3.12), (3.15), and equation obtained by putting $i=0$ in Equation (3.11), we get

$$\begin{aligned}&\left(-480A_0-80ha_0\right)C_{-1}+\left(-1440A_0-330ha_0\right)C_0+\left(-480A_0-180ha_0\right)C_1-10ha_0C_2\\&=f_0h^2-\phi(0)\left(20A_0+ha_0+h^2b_0\right),\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}&\left(1680A_0-480hP_0-80h^2Q_0\right)C_{-1}+\left(3960A_0-1440hP_0-330h^2Q_0\right)C_0\\&+\left(1440A_0-480hP_0-180h^2Q_0\right)C_1+\left(120A_0-10h^2Q_0\right)C_2\\&={f_0}'h^3-\phi(0)\left(-60A_0+20hP_0+5h^2Q_0+h^3R_0\right).\end{aligned} \tag{3.18}$$

By using Equations (3.13) and (3.16), eliminate C_{N+2} from the equation obtained by putting $i=N+2$ in Equation (3.11), we get

$$\begin{aligned}&-10ha_0C_{N-2}+\left(-480A_N-180ha_N\right)C_{N-1}+\left(-1440A_N-330ha_N\right)C_N\\&+\left(-480A_N-80ha_N\right)C_{N+1}=f_Nh^2-\gamma\left(20A_N+5ha_N+h^2b_N\right),\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}&\left(-120A_N-10h^2Q_N\right)C_{N-2}+\left(-1440A_N-480hP_N-180h^2Q_N\right)C_{N-1}\\&+\left(-3960A_N-1440hP_N-330h^2Q_N\right)C_N+\left(-1680A_N-480hP_N\right.\\&\left.+-80h^2Q_N\right)C_{N+1}=f_N'h^3-\gamma\left(60A_N+20hP_N+5h^2Q_N+h^3R_N\right).\end{aligned} \tag{3.20}$$

Eliminating C_{-1} from Equations (3.17), (3.18), and the equation obtained by taking $i=1$ in Equation (3.11), we get

$$\begin{aligned}&\left(9600A_0A_1+16800hA_0a_1-11040h^2A_0b_1-3400hA_1a_0-2350h^2a_0a_1+1750h^3a_0b_1\right)C_0\\&+\left(67200A_0A_1-2400hA_0a_1-31200h^2A_0b_1-13200hA_1a_0+900h^2a_0a_1+5100h^3a_0b_1\right)C_1\\&+\left(-19200A_0A_1-24000hA_0a_1-12480h^2A_0b_1+3000hA_1a_0+4050h^2a_0a_1+2070h^3a_0b_1\right)C_2\\&+\left(-9600A_0A_1-2400hA_0a_1-480h^2A_0b_1+1600hA_1a_0+400h^2a_0a_1+80h^3a_0b_1\right)C_3\\&=h^2f_1\left(-480A_0+80ha_0\right)-\left(h^2f_0-\phi(0)\left(20A_0-5ha_0-h^2b_0\right)\right)\left(20A_1-5ha_1+h^2b_1\right),\end{aligned} \tag{3.21}$$

$$
\begin{aligned}
&\left(518400A_0^2 - 237600hA_0a_0 + 43200h^2a_0P_0 - 43200h^2A_0Q_0\right)C_0 \\
&+\left(11500A_0^2 - 187200hA_0a_0 + 48000h^2a_0P_0 - 48000h^2A_0Q_0\right)C_1 \\
&+\left(-57600A_0^2 - 7200hA_0a_0 + 4800h^2a_0P_0 - 4800h^2A_0Q_0\right)C_2 \\
&=\left[h^3{f_0}' - \phi(0)\left(-60A_0 + 20hP_0 - 5h^2Q_0 + h^3R_0\right)\right]\left(-480A_0 + 80ha_0\right) \\
&-\left(h^2f_0 - \phi(0)\left(20A_0 - 5ha_0 - h^2b_0\right)\right)\left(1680A_0 - 480hP_0 + 80h^2Q_0\right).
\end{aligned} \tag{3.22}
$$

Similarly, by using Equations (3.19), (3.20), eliminate C_{N+1} from the second last equation of (3.11), we get

$$
\begin{aligned}
&\left(-9600A_NA_{N-1} - 1600hA_{N-1}a_N + 2400hA_Na_{N-1} + 400h^2a_Na_{N-1} - 480h^2A_Nb_{N-1}\right. \\
&\left.-80h^3a_Nb_{N-1}\right)C_{N-3} + \left(-19200A_NA_{N-1} - 300hA_{N-1}a_N + 24000hA_Na_{N-1} + 400h^2a_Na_{N-1}\right. \\
&\left.-12480h^2A_Nb_{N-1} - 2070h^3a_Nb_{N-1}\right)C_{N-2} + \left(67200A_NA_{N-1} + 13200hA_{N-1}a_N + 2400hA_Na_{N-1}\right. \\
&\left.+900h^2a_Na_{N-1} - 31200h^2A_Nb_{N-1} - 5100h^3a_Nb_{N-1}\right)C_{N-1} + \left(9600A_NA_{N-1} + 3400hA_{N-1}a_N\right. \\
&\left.+48000hA_Na_{N-1} + 12500h^2a_Na_{N-1} - 11040h^2A_Nb_{N-1} - 1750h^3a_Nb_{N-1}\right)C_N \\
&= h^2f_{N-1}\left(-480A_N + 80ha_N\right) - \left(h^2f_N - \gamma\left(20A_N + 5ha_N + h^2b_0\right)\right)\left(20A_{N-1} + 50ha_{N-1} + h^2b_{N-1}\right),
\end{aligned} \tag{3.23}
$$

$$
\begin{aligned}
&\left(57600A_N^2 - 7200hA_Na_N + 4800h^2A_NQ_N - 4800h^2a_NP_N\right)C_{N-2} \\
&+\left(-115200A_N^2 - 187200hA_Na_N + 48000h^2A_NQ_N - 48000h^2a_NP_N\right)C_{N-1} \\
&+\left(-518400A_N^2 - 237600hA_Na_N + 43200h^2A_NQ_N - 43200h^2a_NP_N\right)C_N \\
&=\left[h^3{f_N}' - \gamma\left(60A_N + 20hP_N + 5h^2Q_N + h^3R_N\right)\right]\left(-480A_N - 80ha_N\right) \\
&+\left(h^2f_N - \gamma\left(20A_N + 5ha_N + h^2b_N\right)\right)\left(1680A_N + 480hP_N + 80h^2Q_N\right).
\end{aligned} \tag{3.24}
$$

The $(N-3)$ equations obtained by putting $i = 2, 3, \ldots, N-2$ in Equation (3.11) along with Equations (3.21)-(3.24) form a $(N+1)\times(N+1)$system $AC = F$ of linear equations, where the matrix A is a penta-diagonal matrix given by

$$
A = \left[\begin{array}{ccccccc}
\gamma_5 & \gamma_6 & \gamma_7 & 0 & 0 & 0 & \cdots \\
\gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 & 0 & 0 & \cdots \\
\mu_1(w_2) & \mu_2(w_2) & \mu_3(w_2) & \mu_4(w_2) & \mu_5(w_2) & 0 & \cdots \\
0 & \mu_1(w_3) & \mu_2(w_3) & \mu_3(w_3) & \mu_4(w_3) & \mu_4(w_3) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \cdots & \cdots & \cdots & \cdots
\end{array}\right.
$$

$$
\left.\begin{array}{ccccccc}
\cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\cdots & \mu_1(w_{N-3}) & \mu_2(w_{N-3}) & \mu_3(w_{N-3}) & \mu_4(w_{N-3}) & \mu_5(w_{N-3}) & 0 \\
\cdots & 0 & \mu_1(w_{N-2}) & \mu_2(w_{N-2}) & \mu_3(w_{N-2}) & \mu_4(w_{N-2}) & \mu_5(w_{N-2}) \\
\cdots & & 0 & \gamma_8 & \gamma_9 & \gamma_{10} & \gamma_{11} \\
\cdots & & 0 & 0 & \gamma_{12} & \gamma_{13} & \gamma_{14}
\end{array}\right]
$$

where

$$\gamma_1 = 9600A_0A_1 + 16800hA_0a_1 - 11040h^2A_0b_1 - 3400hA_1a_0 - 2350h^2a_0a_1 + 1750h^3a_0b_1,$$
$$\gamma_2 = 67200A_0A_1 - 2400hA_0a_1 - 31200h^2A_0b_1 - 13200hA_1a_0 + 900h^2a_0a_1 + 5100h^3a_0b_1,$$
$$\gamma_3 = -19200A_0A_1 - 24000hA_0a_1 - 12480h^2A_0b_1 + 3000hA_1a_0 + 4050h^2a_0a_1 + 2070h^3a_0b_1,$$
$$\gamma_4 = -9600A_0A_1 - 2400hA_0a_1 - 480h^2A_0b_1 + 1600hA_1a_0 + 400h^2a_0a_1 + 80h^3a_0b_1,$$
$$\gamma_5 = 518400A_0^2 - 237600hA_0a_0 + 43200h^2a_0P_0 - 43200h^2A_0Q_0,$$

$$\begin{aligned}
\gamma_6 &= 11500A_0^2 - 187200hA_0a_0 + 48000h^2a_0P_0 - 48000h^2A_0Q_0,\\
\gamma_7 &= -57600A_0^2 - 7200hA_0a_0 + 4800h^2a_0P_0 - 4800h^2A_0Q_0,\\
\gamma_8 &= -9600A_NA_{N-1} - 1600hA_{N-1}a_N + 2400hA_Na_{N-1} + 400h^2a_Na_{N-1} - 480h^2A_Nb_{N-1}\\
&\quad -80h^3a_Nb_{N-1},\\
\gamma_9 &= -19200A_NA_{N-1} - 300hA_{N-1}a_N + 24000hA_Na_{N-1} + 400h^2a_Na_{N-1} - 12480h^2A_Nb_{N-1}\\
&\quad -2070h^3a_Nb_{N-1},\\
\gamma_{10} &= 67200A_NA_{N-1} + 13200hA_{N-1}a_N + 2400hA_Na_{N-1} + 900h^2a_Na_{N-1} - 31200h^2A_Nb_{N-1}\\
&\quad -5100h^3a_Nb_{N-1},\\
\gamma_{11} &= 9600A_NA_{N-1} + 3400hA_{N-1}a_N + 48000hA_Na_{N-1} + 12500h^2a_Na_{N-1} - 11040h^2A_Nb_{N-1}\\
&\quad -1750h^3a_Nb_{N-1},\\
\gamma_{12} &= 57600A_N^2 - 7200hA_Na_N + 4800h^2A_NQ_N - 4800h^2a_NP_N,\\
\gamma_{13} &= -115200A_N^2 - 187200hA_Na_N + 48000h^2A_NQ_N - 48000h^2a_NP_N,\\
\gamma_{14} &= -518400A_N^2 - 237600hA_Na_N + 43200h^2A_NQ_N - 43200h^2a_NP_N.
\end{aligned}$$

and $C = \begin{bmatrix} C_0 \\ C_1 \\ C_2 \\ \vdots \\ C_{N-2} \\ C_{N-1} \\ C_N \end{bmatrix}$, $F = \begin{bmatrix} F_0 \\ F_1 \\ f_2h^2 \\ f_3h^2 \\ \vdots \\ f_{N-1}h^2 \\ F_{N-1} \\ F_N \end{bmatrix}$,

where

$$\begin{aligned}
F_0 &= \left[h^3f_0' - \phi(0)\left(-60A_0 + 20hP_0 - 5h^2Q_0 + h^3R_0\right)\right](-480A_0 + 80ha_0)\\
&\quad - \left(h^2f_0 - \phi(0)\left(20A_0 - 5ha_0 - h^2b_0\right)\right)\left(1680A_0 - 480hP_0 + 80h^2Q_0\right),\\
F_1 &= h^2f_1(-480A_0 + 80ha_0) - \left(h^2f_0 - \phi(0)\left(20A_0 - 5ha_0 - h^2b_0\right)\right)\left(20A_1 - 5ha_1 + h^2b_1\right),\\
F_{N-1} &= -\left(h^2f_{N-1} - \gamma\left(20A_{N-1} + 5ha_{N-1} + h^2b_{N-1}\right)\right)\left(20A_{N-1} + 50ha_{N-1} + h^2b_{N-1}\right)\\
&\quad + h^2f_{N-1}(-480A_{N-1} + 80ha_{N-1}),\\
F_N &= \left[h^2f_N' - \gamma\left(60A_N + 20hP_N + 5h^2Q_N + h^3R_N\right)\right](-480A_N - 80ha_N)\\
&\quad + \left(h^2f_N - \gamma\left(20A_N + 5ha_N + h^2b_N\right)\right)\left(1680A_N + 480hP_N + 80h^2Q_N\right).
\end{aligned}$$

As the coefficient matrix A is non-singular and penta-diagonal, the existence of the matrix's inverse is ensured leading to a unique solution to the system $AC = F$, and this system of equations has been solved to obtain the values of unknowns $C_i's$, where $i = 0, 1, \ldots, N$. Substituting the values of $C_i's$, for $i = 0, 1, \ldots, N$ in Equations (3.12), (3.13), (3.17), and (3.19), we obtain the values of C_{-2}, C_{-1}, C_{N+1} and C_{N+2}. The approximate solution of Equations (2.1)-(2.2) is obtained using these values and basis function in Equation (3.1).

4. Existence and Uniqueness of Solution

This section discusses the existence and uniqueness of the solution for the BVP. For the left boundary layer, we assume that $a(w) > H \geq 0, \quad \forall w \in (0,1)$, H is positive integer; we also assume that $\varepsilon - \delta H \geq 0$. Consider the case when $b(w) \le -\theta < 0, \quad \forall w \in (0,1)$, where the constant $\theta > 0$. Define a linear operator, using the Equation (3.9), we have

$$\begin{aligned} L(C_i) = \tfrac{A_i}{h^2}\left(20C_{i-2} + 40C_{i-1} - 120C_i + 40C_{i+1} + 20C_{i+2}\right) + b_i\left(C_{i-2} + 26C_{i-1}\right. \\ \left.+66C_i + 26C_{i+1} + C_{i+2}\right) = f_i - \tfrac{a_i}{h}\left(-5C_{i-2} - 50C_{i-1} + 50C_{i+1} + 5C_{i+2}\right). \end{aligned} \tag{4.1}$$

Lemma 4.1. *If $G_0 \ge 0$ and $G_N \ge 0$ then $L(G_i) \le 0, \forall i = 1, 2, \ldots, N-1$ implies that $G_i \ge 0$ for all $i = 0, 1, 2, \ldots, N$.*

Proof. Let $k \in \{0,\ 1,\ 2, \ldots,\ N\}$ be such that $G_k = \min_{0 \le i \le N} G_i$. We prove the result by contradiction. So, assume that $G_k < 0$. Then

$$\begin{aligned} L(G_k) &= \tfrac{A_i}{h^2}\left(20G_{k-2} + 40G_{k-1} - 120G_k + 40G_{k+1} + 20G_{k+2}\right) \\ &\quad + b_k\left(G_{k-2} + 26G_{k-1} + 66G_k + 26G_{k+1} + G_{k+2}\right) \\ &= \tfrac{A_i}{h^2}\left[20\left(G_{k-2} - G_k\right) + 40\left(G_{k-1} - G_k\right) + 40\left(G_{k+1} - G_k\right) + 20\left(G_{k+2} - G_k\right)\right] \\ &\quad + b_k\left[\left(G_{k-2} - G_k\right) + 26\left(G_{k-1} - G_k\right) + 40\left(G_{k+1} - G_k\right) + 20\left(G_{k+2} - G_k\right) + 12G_k\right]. \end{aligned}$$

Since $(G_j - G_k) > 0$, $b_k < 0$, and $G_k < 0$, we have $L(G_k) > 0$ for $1 \le k < N$ which is a contradiction to our assumption that $L(G_i) \le 0$ for all $i = 0, 1, \ldots, N-1$. Thus $G_k \ge 0$, but $G_k = \min_{0 \le i \le N} G_i$, thus $G_i \ge 0$ for all $i = 0, 1, \ldots, N-1$. □

Theorem 4.2. *With the assumptions $a(w) > H \ge 0$ and $b(w) \le -\theta < 0$, where the constants H and θ are positive, the solution to the discretised problem (3.11), (3.12)-(3.13) exists, is unique, and satisfies the inequality*

$$\|y\|_\infty \le \theta^{-1}\|f\|_\infty + D\left(\|\phi\|_\infty + |\gamma|\right), \tag{4.2}$$

where $D \ge 1$ is constant. The $\|\,.\,\|_\infty$ is defined as $\|t\|_\infty = \max\limits_{0 \le i \le N} |t_i|$.

Proof. We prove the uniqueness of the solution using contradiction. Assume that u_{1i} and u_{2i} be two different solutions of the problem (3.11), (3.12)-(3.13). Then the mesh function, defined by $u_i = u_{1i} - u_{2i}$ satisfies $u_0 = 0$ and $u_N = 0$, also $L(u_i) = L(u_{1i}) - L(u_{2i})$. Since u_{1i} and u_{2i} satisfies (4.1), we have

$$L(u_i) = 0, \quad 1 \le i \le N-1.$$

This shows that the function $u_i = u_{1i} - u_{2i}$ satisfies the conditions in Lemma 4.1, thus we have

$$u_i = u_{1i} - u_{2i} \ge 0, \quad 0 \le i \le N. \tag{4.3}$$

Again, if we define $u_i = -(u_{1i} - u_{2i})$, then using the same arguments, we get

$$u_i = -(u_{1i} - u_{2i}) \le 0, \quad 0 \le i \le N. \tag{4.4}$$

Equations (4.3) and (4.4) imply that $u_{1i} - u_{2i} = 0$. This proves the uniqueness of the solution, and the linearity of the BVP assures the existence of the solution. To prove the inequality (4.2), define two functions as follows:

$$\Lambda_i^{\pm} = \theta^{-1}\|f\|_\infty + K_1\left(\|\phi\|_\infty + |\gamma|\right) \pm y_i, \quad 0 \le i \le N,$$

where $K_1 > 0$ is an arbitrary constant. Observe that

$$\begin{aligned} \Lambda_0^{\pm} &= \theta^{-1}\|f\|_\infty + K_1\left(\|\phi\|_\infty + |\gamma|\right) \pm y_0 \\ &= \theta^{-1}\|f\|_\infty + K_1\left(\|\phi\|_\infty \pm \phi_0\right) + K_1|\gamma|, \text{ since } y_0 = \phi_0 \\ &> 0, \quad as\ \|\phi\|_\infty \ge \phi_0 \quad and \quad K_1 \ge 1. \end{aligned}$$

$$\begin{aligned}\Lambda_N^{\pm} &= \theta^{-1}\|f\|_\infty + K_1\left(\|\phi\|_\infty + |\gamma|\right) \pm y_N \\ &= \theta^{-1}\|f\|_\infty + K_1\|\phi\|_\infty + \left(K_1|\gamma| \pm \gamma\right), \text{ since } \quad y_N = \gamma \\ &\geq 0, \quad \text{as} \quad |\gamma| \geq \gamma \quad \text{and} \quad K_1 \geq 1, \quad \forall \quad 0 \leq i \leq N.\end{aligned}$$

$$\begin{aligned}L\left(\Lambda_i^{\pm}\right) &= \tfrac{A_i}{h^2}{\Lambda_i^{\pm}}'' + \tfrac{a_i}{h}{\Lambda_i^{\pm}}' + b_i\Lambda_i^{\pm} = \tfrac{A_i}{h^2}{y_i}'' + \tfrac{a_i}{h}{y_i}' + b_i\left(\theta^{-1}\|f\|_\infty + K_1\left(\|\phi\|_\infty + |\gamma|\right) \pm y_i\right) \\ &= b_i\left(\theta^{-1}\|f\|_\infty + K_1\left(\|\phi\|_\infty + |\gamma|\right)\right) \pm L\left(y_i\right) \\ &= b_i\left(\theta^{-1}\|f\|_\infty + K_1\left(\|\phi\|_\infty + |\gamma|\right)\right) \pm f_i\,, \quad \text{since } L\left(y_i\right) = f_i \\ &\leq -\|f\|_\infty \pm f_i + K_1 b_i\left(\|\phi\|_\infty + |\gamma|\right), \quad \text{since } b_i \leq -\theta < 0.\end{aligned}$$

This shows that

$$L\left(\Lambda_i^{\pm}\right) \leq 0, \quad 0 \leq i \leq N.$$

Implies that, $\Lambda_i^{\pm}$satisfies the hypothesis of 4.1. Thus, we have

$$\Lambda_i^{\pm} = \theta^{-1}\|f\|_\infty + K_1\left(\|\phi\|_\infty + |\gamma|\right) \pm y_i \geq 0, \quad 0 \leq i \leq N,$$

i.e.,

$$y_i \leq \theta^{-1}\|f\|_\infty + K_1\left(\|\phi\|_\infty + |\gamma|\right).$$

□

5. Derivation of Convergence

This part of the article establishes the truncation error of the proposed scheme. The following relations can be easily derived from Equations (3.3)-(3.6). One can refer to [2] for more details.

$$\begin{aligned}&\Im'\left(w_{i-2}\right) + 26\Im'\left(w_{i-1}\right) + 66\Im'\left(w_i\right) + 26\Im'\left(w_{i+1}\right) + \Im'\left(w_{i+2}\right) \\ &= \tfrac{1}{h}\left\{-5y\left(w_{i-2}\right) - 50y\left(w_{i-1}\right) + 50y\left(w_{i+1}\right) + 5y\left(w_{i+2}\right)\right\},\end{aligned} \tag{5.1}$$

$$\begin{aligned}&\Im''\left(w_{i-2}\right) + 26\Im''\left(w_{i-1}\right) + 66\Im''\left(w_i\right) + 26\Im''\left(w_{i+1}\right) + \Im''\left(w_{i+2}\right) \\ &= \tfrac{1}{h^2}\left\{20y\left(w_{i-2}\right) + 40y\left(w_{i-1}\right) - 120y\left(w_i\right) + 40y\left(w_{i+1}\right) + 20y\left(w_{i+2}\right)\right\},\end{aligned} \tag{5.2}$$

$$\begin{aligned}&\Im'''\left(w_{i-2}\right) + 26\Im'''\left(w_{i-1}\right) + 66\Im'''\left(w_i\right) + 26\Im'''\left(w_{i+1}\right) + \Im'''\left(w_{i+2}\right) \\ &= \tfrac{1}{h^3}\left\{-60y\left(w_{i-2}\right) + 120y\left(w_{i-1}\right) - 120y\left(w_{i+1}\right) + 60y\left(w_{i+2}\right)\right\},\end{aligned} \tag{5.3}$$

$$\begin{aligned}&\Im^{iv}\left(w_{i-2}\right) + 26\Im^{iv}\left(w_{i-1}\right) + 66\Im^{iv}\left(w_i\right) + 26\Im^{iv}\left(w_{i+1}\right) + \Im^{iv}\left(w_{i+2}\right) \\ &= \tfrac{1}{h^4}\left\{120y\left(w_{i-2}\right) - 480y\left(w_{i-1}\right) + 720y\left(w_i\right) - 480y\left(w_{i+1}\right) + 120y\left(w_{i+2}\right)\right\}.\end{aligned} \tag{5.4}$$

Applying operator notation, Equations (5.1)-(5.4) can be written as

$$\left(E^{-2} + 26E^{-1} + 66I + 26E + E^2\right)\Im'\left(w_i\right) = \frac{1}{h}\left(-5E^{-2} - 50E^{-1} + 50E + E^2\right)y\left(w_i\right), \tag{5.5}$$

$$\left(E^{-2} + 26E^{-1} + 66I + 26E + E^2\right)\Im''\left(w_i\right) = \frac{1}{h^2}\left(20E^{-2} + 40E^{-1} - 120I + 40E + 20E^2\right)\, y\left(w_i\right), \tag{5.6}$$

$$\left(E^{-2} + 26E^{-1} + 66I + 26E + E^2\right)\Im'''\left(w_i\right) = \frac{1}{h^3}\left(-60E^{-2} + 120E^{-1} - 120E + 60E^2\right)y\left(w_i\right), \tag{5.7}$$

$$\left(E^{-2} + 26E^{-1} + 66I + 26E + E^2\right)\Im^{iv}\left(w_i\right) = \frac{1}{h^4}\left(120E^{-2} - 480E^{-1} + 720I - 480E + 120E^2\right)y\left(w_i\right). \tag{5.8}$$

Operators are defined as $E^{-2}y(w_i) = y(w_{i-2}), E^{-1}y(w_i) = y(w_{i-1}), Iy(w_i) = y(w_i), Ey(w_i) = y(w_{i+1}),$ $E^2y(w_i) = y(w_{i+2})$ and $E = e^{hD}$. By Taylor's series, we have

$$e^{hD} = 1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \frac{h^4D^4}{4!} + \ldots$$

$$e^{-hD} = 1 - hD + \frac{h^2D^2}{2!} - \frac{h^3D^3}{3!} + \frac{h^4D^4}{4!} - \ldots$$

Expand Equations (5.5)-(5.8) in powers of hD,

$$\Im'(w_i) = y'(w_i) + \frac{1}{5040}h^6y^7(w_i) - \frac{1}{21600}h^8y^9(w_i) + \frac{1}{1036800}h^{10}y^{11}(w_i) + O\left(h^{11}\right), \tag{5.9}$$

$$\begin{aligned}\Im''(w_i) &= y''(w_i) + \tfrac{1}{720}h^4y^6(w_i) - \tfrac{1}{3360}h^6y^8(w_i) + \tfrac{1}{86400}h^8y^{10}(w_i)\\ &+ \tfrac{221}{239500800}h^{10}y^{12}(w_i) + O\left(h^{11}\right),\end{aligned} \tag{5.10}$$

$$\begin{aligned}\Im'''(w_i) &= y'''(w_i) - \tfrac{1}{240}h^4y^7(w_i) + \tfrac{11}{30240}h^6y^9(w_i) - \tfrac{1}{28800}h^8y^{11}(w_i)\\ &+ \tfrac{37}{11404800}h^{10}y^{13}(w_i) + O\left(h^{11}\right),\end{aligned} \tag{5.11}$$

$$\begin{aligned}\Im^{iv}(w_i) &= y^{iv}(w_i) - \tfrac{1}{12}h^2y^6(w_i) + \tfrac{1}{240}h^4y^8(w_i) - \tfrac{1}{7560}h^6y^{10}(w_i)\\ &- \tfrac{13}{907200}h^8y^{12}(w_i) + \tfrac{643}{159667200}h^{10}y^{14}(w_i) + O\left(h^{11}\right).\end{aligned} \tag{5.12}$$

Error in the approximate solution is given by $e(w) = y(w) - S(w)$. Taking the Taylor's series expansion of $e(w_i + \theta h)$, we obtain

$$\begin{aligned}e(w_i + th) = &\left(\tfrac{t^2}{1440} - \tfrac{5t^4}{1440}\right)h^6y^6(w_i) + \left(\tfrac{t}{5040} - \tfrac{t^2}{1440}\right)h^7y^7(w_i)\\ &+ \left(-\tfrac{t^2}{6720} + \tfrac{t^4}{5760}\right)h^8y^8(w_i) + O\left(h^9\right).\end{aligned} \tag{5.13}$$

Theorem 5.1 summarizes the above result.

Theorem 5.1. *If $y(w)$, and $\Im(w)$ are the exact and approximate quintic B-spline solutions of the SPDDE (2.4) -(2.5) respectively, then for sufficiently small values of h the truncation error is of $O\left(h^6\right)$ and order of convergence is $O\left(h^4\right)$.*

6. Numerical Results and Discussion

In this part of the paper, two numerical examples are solved to demonstrate the proposed method. Example one is an SPDDE with constant coefficients, and example two is with variable coefficients. Maximum absolute error (MAE) is calculated. Also, the obtained solution of example one is compared with the existing method [17], and example two is compared with the existing method [18]. Graphs are plotted to observe the effect of the delay on solutions. MATLAB is used for computational purposes.

Example 6.1. Consider the following SPDDE with constant coefficients

$$\varepsilon y''(w) + y(w-\delta) - y(w) = 0,\ y(w) = 1,\ -\delta \le w \le 0,\ y(1) = 1.$$

The exact solution of Example 6.1 is given by

$$y(w) = \left(\left(1 - \exp(r_2)\right)\exp(r_1 w) + \left(\exp(r_1) - 1\right)\exp(r_2 w)\right) / \left(\exp(r_1) - \exp(r_2)\right)$$

where $r_1 = \frac{-1-\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}$ and $r_2 = \frac{-1+\sqrt{1+4(\varepsilon-\delta)}}{2(\varepsilon-\delta)}$. The MAE for Example 6.1 is calculated by comparing the exact solution and the obtained solution at nodal points, using the formula:

$$Er^N = \max_{0\le i\le N}\left|y_i^N - \Im_i^N\right|, \tag{6.1}$$

TABLE 2. Comparison of MAE for Example 6.1 with $\varepsilon = 0.1$, and various values of δ and N.

$\delta \rightarrow$	0.03		0.05		0.08	
N	Proposed Method	Method [17]	Proposed Method	Method [17]	Proposed Method	Method [17]
100	1.5584E-07	1.7830E-02	5.7038E-09	2.5306E-02	1.9418E-05	3.5989E-02
200	9.8163E-09	9.5140E-03	3.6038E-08	9.5140E-03	1.2941E-06	1.9250E-03
300	1.9411E-09	9.2760E-03	7.1394E-09	9.2760E-03	2.5868E-07	1.3132E-03
400	6.1467E-10	4.9190E-03	2.2610E-09	7.0420E-03	8.2195E-08	9.9650E-03
500	2.5235E-10	3.9620E-03	9.2674E-10	5.6740E-03	3.3734E-08	8.0280E-03

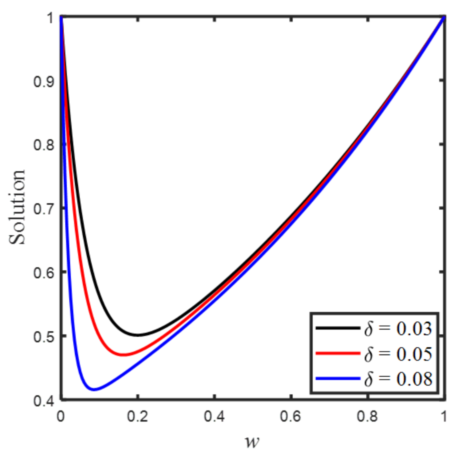

FIGURE 1. Numerical solution of Example 6.1 for $N = 100, \varepsilon = 0.1$ and different values of δ.

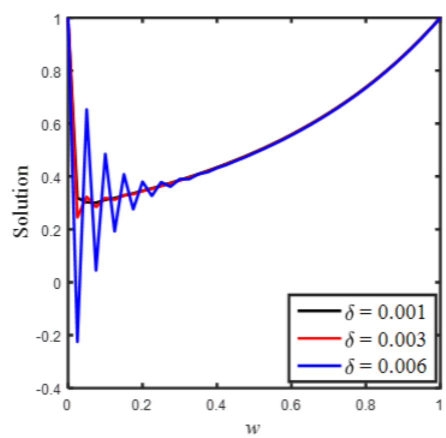

FIGURE 2. Numerical solution of Example 6.2 for $N = 20, \varepsilon = 0.01$ and different values of δ.

TABLE 3. Order of convergence for Example 6.1 with $\varepsilon = 0.1$ by proposed method.

$\delta \downarrow$ $N \rightarrow$	100	200	300	400	500
0.03	3.9887E+00	3.9973E+00	4.0023E+00	4.0222E+00	3.9886E+00
0.05	3.9843E+00	3.9945E+00	3.9973E+00	3.9977E+00	4.0385E+00
0.08	4.2301E+00	3.9768E+00	3.9895E+00	3.9939E+00	3.9958E+00

where y_i^N is the exact solution, and $\Im_i^N$ is the approximate solution. The numerical value of order of convergence Rc^N is determined using

$$Rc^N = \log_2 \left(Er^N / Er^{2N} \right) \tag{6.2}$$

The values of MAE for $\varepsilon = 0.1$, $\delta = 0.03, 0.05, 0.08$, and different values of N and the comparison of the same is presented in Table 2. Table 3 shows the order of convergence of the proposed method. Figure 1 shows the behavior of the solution for different values of δ.

Example 6.2. Consider the following SPDDE with variable coefficient:

$$\varepsilon y''(w) + exp\left(-0.5w\right) y(w-\delta) - y(w) = 0, \qquad y(w) = 1, \ -\delta \leq w \leq 0, \ y(1) = 1.$$

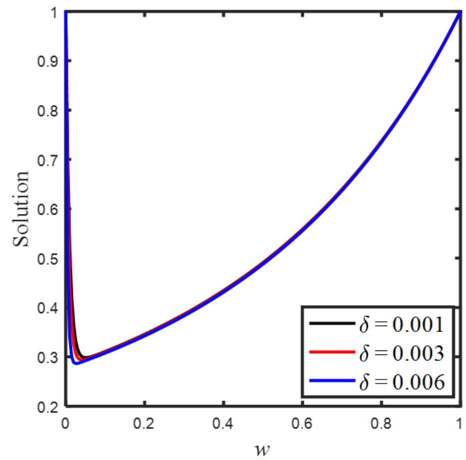

FIGURE 3. Numerical solution of Example 6.2 for $N = 100, \varepsilon = 0.01$ and different values of δ.

TABLE 4. Comparison of MAE for Example 6.2 with $\varepsilon = 0.01$, and various values of δ and N.

$N \rightarrow$	100		1000		10000	
δ	Proposed Method	Method [18]	Proposed Method	Method [18]	Proposed Method	Method [18]
0.001	3.1202E-04	9.0929E-02	2.6207E-07	1.0091E-02	1.1230E-09	1.2994E-03
0.003	5.9933E-04	1.0836E-01	3.6484E-07	1.5626E-02	1.5189E-09	1.6446E-03
0.006	7.0451E-03	1.2845E-01	1.0654E-06	2.6315E-02	3.0797E-09	2.8703E-03
0.008	2.6592E-01	1.0150E-01	1.9613E-05	4.8348E-02	7.8259E-09	5.6889E-03

The analytical solution for this problem does not exist. Thus, the MAE is determined by applying the double mesh principle. In this case, MAE is given by

$$Er_1{}^N = \max_{0 \le i \le N} \left| \Im_i^N - \Im_{2i}^{2N} \right|. \tag{6.3}$$

Table 4 compares the MAE obtained by the proposed method with the method in [18]. Figures 2 and 3 show the behavior of the solution for $\varepsilon = 0.01$, $N = 20$ and $N = 100$, respectively. One can easily observe that the perturbation is more for $N = 20$ and very less for $N = 100$.

7. Conclusion

In this paper, we have considered a linear SPDDEs with delay in the convection term. A numerical scheme based on QBSM has been applied to get the approximate solution. We have discussed the uniqueness and existence of the solution. Two numerical examples are solved by QBSM to support the theoretical discussion. The numerical order of convergence of the discussed method is identified as four. It is observed that the MAE decreases as N increases, and the graphs show that the thickness of the boundary layer decreases as δ increases. The comparison of the results obtained by QBSM with the existing results in the literature shows the superiority of the proposed method. We conclude that the QBSM is efficient in tackling second-order SPDDEs. It is computationally easy and further can be applied for a variety of SPDEs with different delay structures occurring in engineering, and scientific computing.

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