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ADI numerical method to modeling stock insurance based on spread options

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Abstract

This paper introduces a spread option model based on two underlying assets, namely Bandar Abbas oil refining (Shebandar) and Tehran oil refining (Shatran) companies. Regarding the available data of the former, we propose the jump-diffusion model for its dynamic. After constructing our portfolio, we first consider a partial integro-differential equation (PIDE) for the spread option model. Then by making some alterations to the literature of the problem and parameters of the model, it is demonstrated that the assumed option can be considered as insurance, hedging the aforementioned stocks. The PIDE is solved by the well-known ADI numerical method. Finally, we use the real data extracted from the Tehran Stock Exchange and a reliable result is obtained by using MATLAB software.

Keywords. Spread option pricing, Jump-diffusion model, Alternating Direction Implicit, Insurance, Numerical method.

2010 Mathematics Subject Classification. 91B30, 97M30, 35R60.

1. Introduction

A spread option is a type of option which derives its value from the difference between the price of two or more assets. It is important to know how spread options work. They are used in stocks, currencies, and bonds, and some of them are traded on large exchange markets. The most notable examples of these spread options are crack, crush, and spark which measure profits in the oil, soybean, and electricity markets.

The underlying assets of the above examples are considered from different commodities and spread options which cover the difference between the prices of the same commodities traded in two different locations (location spreads) or different grades (quality spreads). Furthermore, spreads can be the difference between the prices of two identical goods at two different geographical locations. One of the applications of spread options is in the energy market where the crack spread is the difference between the price of refined oil and gas products and the input price of crude oil. When a trader expects that the crack spread will grow, he believes that the refining market will grow because crude oil is cheap and there is a high demand for refined products. So, he simply buys a call option on the crack spread instead of buying refined products and selling crude oil.

All of the options allow the holder a right, but not the obligation, to buy or sell a specific underlying asset on a specific date at a specified price. The most important thing here is the price difference between two or more assets. Although fractional models have been introduced in papers [9] and [2], the Black-Scholes model in papers [1, 3] is more interesting than others. In order to hedge the stocks in our portfolio, assume that you hold 3030000 Tomans which enables you to purchase 1050 shares of Shebandar, each at the price of 2000 Tomans, and 310 shares of Shatran, each at the price of 3000 Tomans. One possible way to construct our portfolio is to make it only consisting of various shares, saying Shebandar and Shatran. In this case, all the money is used to purchase different shares. Clearly, it imposes a great risk to the holder of the portfolio. However, we can handle this risk by diversifying our portfolio. This can be done by buying shares and their put options. For example, consider a portfolio consisting of 1000 Shebandar's share

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and 1000 Shebandar's put options each cost 100 Tomans, also 300 Shatran's share, and 300 its put options each cost 100 Tomans. Now, in case a decline occurs, our shares will be automatically sold and so our loss will be hedged. On the other hand, if the price of our shares increases, we only lose 130000 Tomans, which was paid for the put options. One can see the loss in the second case is entirely worthwhile, when we consider the possible loss in the first case.
The payoff function for a call spread is $C = max(0, S_1 - S_2 - K)$ where S_1 and S_2 are the prices of the two assets and K is a fixed strike price. The payoff function for a put spread is $P = max(0, K - S_1 + S_2)$. When $K = 0$, the spread option is the same as exchange contracts.
In 1995, Kirk's approximation which is a valid formula for small and non-zero K, was published. This value led to a modification to the standard Black Scholes formula, which was used with a specific expression for volatility based on the volatility and correlation of the two assets. Deng et al. (2001) studied the spread option in the energy market [4]. Zhou et al. (2015) proposed spread option pricing models that depended on time, credit spread logarithm, and interest rates [12]. Vida and Christina (2017) studied spread pricing, which has no jump term in their equation, and the differentials of these assets were based on the Wiener process and time [6]. Vasquez et al (2012) studied spread option pricing based on the Libor interest rate without any jumps [13]. The above modeling equations did not consider sudden changes in currency, stocks, and so on. In 2020, Mohamadinejad et al examined spread pricing based on Libor interest rates and considered these sudden changes of political, economic, and natural disasters such as floods and earthquakes [10, 11].
In this paper, we use two underlying assets S_1 and S_2 (one of them has the jump term) to build a financial portfolio then the model with random data is implemented in MATLAB. In the paper [10], the spread option pricing was based on two Libor interest rates where one of the interest rates had jumped, and we converted the obtaining PIDE to PDE after using efficient change of variables. Finally, this partial differential equation was solved by the ADI method, but with $\frac{1}{6}$ step size. In this paper, the obtaining model is based on two assets, and the PDE is solved by the ADI method with $\frac{1}{2}$ step size [6].
The remainder of this paper is structured as follows: In Section 2, we describe the model and the boundary and initial conditions, the obtaining PIDE depends on two stock assets and time. In Section 3, this problem is numerically solved by using the ADI method with $\frac{1}{2}$ step size. We implement the model in MATLAB software in section 4 and finally section 5 includes a conclusion and some recommendations for further researches.

2. Modelling Framework (Model Formulation)

In this section, we study the spread option pricing model based on asset prices of S_1 and S_2. Suppose S_1 and S_2 are the Shatran and Shebandar refining stock prices, respectively, which have the following dynamics:

$$dS_1 = (r - q_1)S_1 dt + \sigma_1 S_1 dW_1, \; dS_2 = (r - q_2)S_2 dt + \sigma_2 S_2 dW_2, \tag{2.1}$$

where r is the risk-free interest rate, σ_1 and σ_2 are the volatility of S_1 and S_2 respectively, q_1 and q_2 are the dividends paid by S_1 and S_2 respectively, ρ is the correlation between S_1 and S_2 and W_1 and W_2 are two standard Brownian motions where $Corr(W_1, W_2) = \rho$. Now assume that S_2 faces with some jumps caused by unknown reasons due to different situations so the dynamic of S_2 in equation (1) is rewritten as follows

$$dS_2 = (r - q_2)S_2 dt + \sigma_2 S_2 dW_2 + d(\sum_{i=1}^{N_t} L_i),$$

where N_t is a Poisson process with intensity rate $\lambda > 0$ (λ shouldn't be zero or close to zero, otherwise the integral part of the PIDE is eliminated and there will be no specific jump) and sequence L_i denotes the magnitude of jumps which are independent and identically distributed ($i.i.d$) and the distribution function is as follows

$$f_L(z) = \begin{cases} \lambda e^{-\lambda z} & , z > 0, \\ 0 & , z \leq 0. \end{cases}$$

Note that W and N_t processes are independent. Let Ω be the spread option price, according to the Equation (2.1) and by applying Ito Lemma, the dynamic of the option price is obtained as follows

$$\begin{aligned} d\Omega = &(\frac{\partial \Omega}{\partial t} + (r - q_1)S_1\frac{\partial \Omega}{\partial S_1} + (r - q_2)S_2\frac{\partial \Omega}{\partial S_2} + \sigma_1^2 S_1^2 \frac{\partial^2 \Omega}{\partial S_1^2} + \sigma_2^2 S_2^2 \frac{\partial^2 \Omega}{\partial S_2^2} \\ &+ \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 \Omega}{\partial S_1 \partial S_2})dt + (S_1\sigma_1 + S_2\sigma_2)dW + (\Omega(t, S_1, S_2 + x) \\ &- \Omega(t, S_1, S_2))dN_t. \end{aligned}$$

In the above equation, x is a jump that occurs due to a sudden catastrophe or disaster. Now, we build a financial portfolio consisting of two spread options Ω_1 and Ω_2 as follows [14]:

$$\Pi = x_1\Omega_1 + x_2\Omega_2,$$

where x_1 and x_2 are the volume of the spread options. Now, by substituting variations of Π in dynamics of Ω_1 and Ω_2, the following PIDE is obtained [10]:

$$\begin{aligned} &\frac{\partial \Omega}{\partial t} + (r - q_1 - q\sigma_1)S_1\frac{\partial \Omega}{\partial S_1} + (r - q_2 - q\sigma_2)S_2\frac{\partial \Omega}{\partial S_2} + \sigma_1^2 S_1^2 \frac{\partial^2 \Omega}{\partial S_1^2} + \sigma_2^2 S_2^2 \frac{\partial^2 \Omega}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 \Omega}{\partial S_1 \partial S_2} \\ &\quad - (\lambda + r)\Omega + \int_{-\infty}^{+\infty} \Omega(t, S_1, S_2 + z)f(z)dz = 0. \end{aligned} \tag{2.2}$$

We consider τ as a backward time variable, in other words $t = T - \tau$ where $t \in [0, T]$ is a time variable and T is the maturity. Consider the following changes of variables, too:

$$S_1 = e^x, \ \ S_2 = e^y.$$

By replacing the change of variables and their derivatives in Equation (2.2) and using the second-order Taylor approximation in the integral part that is estimated in [10], we obtain the following PDE. Note that $\Omega(t, S_1, S_2)$ is the option price at the time t where it satisfies in the PIDE (2.2). It also can be represented as the expectation of discounted future prices under the risk-neutral measure.

$$\begin{aligned} \Omega_\tau = &(r - q_1 - q\sigma_1 - \sigma_1^2)\frac{\partial \Omega}{\partial x} + (r - q_2 - q\sigma_2 - \sigma_2^2 + \frac{1}{\lambda}e^{-y} - \frac{1}{\lambda^2}e^{-2y})\frac{\partial \Omega}{\partial y} \\ &+ \sigma_1^2\frac{\partial^2 \Omega}{\partial x^2} + (\sigma_2^2 + \frac{1}{\lambda^2}e^{-2y})\frac{\partial^2 \Omega}{\partial y^2} + \rho\sigma_1\sigma_2\frac{\partial^2 \Omega}{\partial x \partial y} - (\lambda + r - 1)\Omega. \end{aligned}$$

Equation (2.2) is defined in a semi-finite domain $[0, +\infty) \times [0, +\infty)$. In order to formulate the boundary condition, we constraint the semi-finite domain for large enough of e^x and e^y. Therefore we let $x \in [x_{min}, x_{max}]$ and $y \in [y_{min}, y_{max}]$. If K is the strike price of the payoff function for a European put option, the boundary conditions for Equation (2.2) are considered as follows

$$\begin{aligned} &\Omega(\tau, e^{x_{min}}, e^y) = \max\{0, e^y e^{(-q_2\tau)} + Ke^{(-r\tau)}\}, \\ &\Omega(\tau, e^x, e^{y_{min}}) = \max\{0, -e^x e^{(-q_1\tau)} + Ke^{(-r\tau)}\}, \\ &\Omega(\tau, e^{x_{max}}, e^y) = \max\{0, -Me^{(-q_1\tau)} + e^y e^{(-q_2\tau)} + Ke^{(-r\tau)}\}, \\ &\Omega(\tau, e^x, e^{y_{max}}) = \max\{0, -e^x e^{(-q_1\tau)} + Ne^{(-q_2\tau)} + Ke^{(-r\tau)}\}. \end{aligned}$$

Moreover, the initial and final conditions are as below

$$\begin{aligned} &\Omega(0, e^x, e^y) = \max\{0, -e^x + e^y + K\}, \\ &\Omega(T, e^x, e^y) = \max\{0, -e^x e^{(-q_1 T)} + e^y e^{(-q_2 T)} + Ke^{(-rT)}\}. \end{aligned}$$

3. ADI method with $\frac{1}{2}$ step size for solving spread option pricing model

Consider the PDE in Equation (2.2). We define the $L\Omega$ operator as

$$L\Omega = L^x\Omega + L^y\Omega + L^{xx}\Omega + L^{yy}\Omega + L^{xy}\Omega. \tag{3.1}$$

The coefficients of the $L\Omega$ operator are determined by using the coefficients of Equation (2.2). These coefficients can be determined by different methods. But their sum must be consistent with the sum of the coefficients of Equation (2.2). Consider the following terms:

$$\begin{aligned}
L^x\Omega &= (r - q_1 - q\sigma_1 - \sigma_1^2)\frac{\partial\Omega}{\partial x} - (\lambda + r - 1)\Omega,\\
L^y\Omega &= (r - q_2 - q\sigma_2 - \sigma_2^2 + \frac{1}{\lambda}e^{-y} - \frac{1}{\lambda^2}e^{-2y})\frac{\partial\Omega}{\partial y},\\
L^{xx}\Omega &= \sigma_1^2\frac{\partial^2\Omega}{\partial x^2},\ L^{yy}\Omega = (\sigma_2^2 + \frac{1}{\lambda^2}e^{-2y})\frac{\partial^2\Omega}{\partial y^2},\\
L^{xy}\Omega &= \rho\sigma_1\sigma_2\frac{\partial^2\Omega}{\partial x\partial y}.
\end{aligned}$$

Now substitute the finite differences in the above terms, obtain the system of equations and solve it step by step and recursively.
We use the following differences to solve the above equations numerically [8].

$$\begin{aligned}
L^x\Omega_{ij}^n &= \frac{\Omega_{i+\frac{1}{2}j}^n - \Omega_{ij}^n}{h},\\
L^y\Omega_{ij}^n &= \frac{\Omega_{ij+\frac{1}{2}}^n - \Omega_{ij}^n}{k},\\
L^{xx}\Omega_{ij}^n &= \frac{\Omega_{i+1j}^n - 2\Omega_{i+\frac{1}{2}j}^n + \Omega_{ij}^n}{h^2},\\
L^{yy}\Omega_{ij}^n &= \frac{\Omega_{ij+1}^n - 2\Omega_{ij+\frac{1}{2}}^n + \Omega_{ij}^n}{k^2},\\
L^{xy}\Omega_{ij}^n &= \frac{\Omega_{i+\frac{1}{2}j+\frac{1}{2}}^n - \Omega_{ij+\frac{1}{2}}^n - \Omega_{i+\frac{1}{2}j}^n + \Omega_{ij}^n}{hk},
\end{aligned} \tag{3.2}$$

where h, and k are step length of x, and y divisions respectively. After replacing the above differences in Equations (3.2), we obtain the following equations where $i = 0, \frac{1}{2}, \cdots, m$ and $j = 0, \frac{1}{2}, \cdots, m$ (m is constant number). To summarize the calculations, we only write the obtained equations [5] as

$$(C - B)\Omega_{i+\frac{1}{2}j}^n = \Omega_{ij}^{n+1} - (1 - C)\Omega_{ij}^n,$$

where $A = (r - q_1 - q\sigma_1 - \sigma_1^2)$, $C = \frac{\Delta\tau}{h}A$, $B = (r + \lambda - 1)$ and we should replace $i = 0, \frac{1}{2}, \cdots, m$. So, by using the finite differences in the second equation of (3.2), we have:

$$D\Omega_{ij+\frac{1}{2}}^n = \Omega_{ij}^{n+1} - (1 - D)\Omega_{ij}^n,$$

where $D = \frac{\Delta\tau}{k}(r - q_2 - q\sigma_2 - \sigma_2^2 + \frac{1}{\lambda}e^{-y} - \frac{1}{\lambda^2}e^{-2y})$ and we should replace $j = 0, \frac{1}{2}, \cdots, m$.
By replacing the finite differences in the 3rd equation of (3.2), we have:

$$E\Omega_{i+1j}^n = \Omega_{ij}^{n+1} + 2E\Omega_{i+\frac{1}{2}j}^n - (1 - E)\Omega_{ij}^n,$$

where $E = \frac{\Delta\tau\sigma_1^2}{h^2}$ and $i = 0, \frac{1}{2}, \cdots, m$.
The fourth equation is also obtained as follows:

$$F\Omega_{ij+1}^n = \Omega_{ij}^{n+1} + 2F\Omega_{ij+\frac{1}{2}}^{n+\frac{1}{2}} - (1 - F)\Omega_{ij}^n,$$

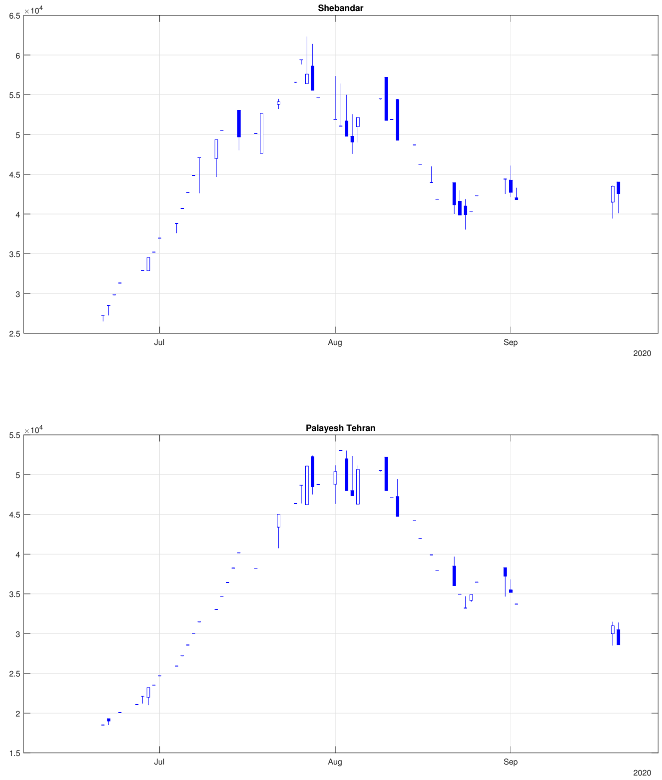

FIGURE 1. Candlestick charts of Shebandar and Shatran.

where $F = \frac{\Delta\tau}{k^2}(\sigma_2^2 + \frac{1}{\lambda^2}e^{-2y})$ and $j = 0, \frac{1}{2}, \cdots, m$.
With respect to the finite differences, the 5th equation is rewritten as follows:

$$G\Omega^n_{i+\frac{1}{2}j+\frac{1}{2}} = \Omega^n_{ij} + G(\Omega^n_{ij+\frac{1}{2}} + \Omega^n_{i+\frac{1}{2}j}) - (1-G)\Omega^n_{ij},$$

where $G = \frac{\Delta\tau}{hk}\rho\sigma_1\sigma_2$ and $j = 0, \frac{1}{2}, \cdots, m$.

4. Implementation

As mentioned in the previous sections, we considered two shares from the refining stocks groups, called Shebandar and Shatran. We extracted the prices of these two stocks from the website http://www.tsetmc.com in the quarterly period from 06/21/2020 to 09/21/2020. Also, the candlestick charts of these two stocks in the same period is as follows, which shows that these prices satisfy in SDEs (See Figure 1).
The parameters and values used in MATLAB are introduced in Table 1 [6]. We found the price of the spread option, the first $(\Omega(S_1, S_2, 0))$ and the last step $(\Omega(S_1, S_2, T))$ of the solution are shown in Figure 2 (due to $\frac{1}{2}$ step size, we have 10 steps of solution)

5. Conclusion

In this paper, we studied the spread option pricing based on the two-dimensional Black-Scholes model. One of these two underlying equations had a jump term that originated from economic conditions and natural factors such as floods, earthquakes, etc. If we assume that the strike price is non-zero, then there will be no exact formula for European spread option, but there are numerical methods with convergence properties such as ADI that approximate PDE with

TABLE 1. Data.

Parameters	Definitions	Values
r	Interest rate	0.03
λ	Intensity rate of Poisson process	0.1
ρ	$corr(W_1, W_2)$	0.5
q	The market price of the risk	1
q_1	the dividends paid by s_1	0
q_2	the dividends paid by s_2	0
T	Max(t): maximum value of t	3
h	step length of x	0.1
k	step length of y	0.1

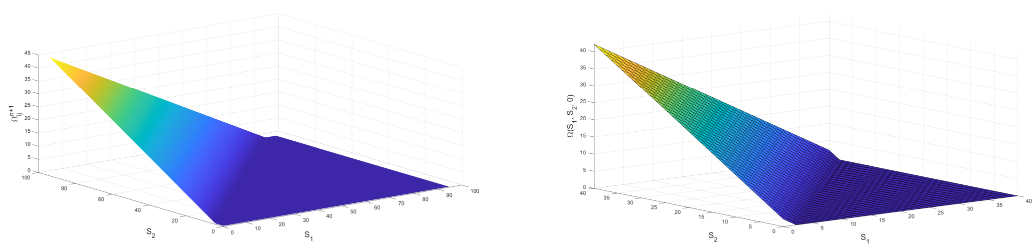

FIGURE 2. Numerical solution of $\Omega(S_1, S_2, T)$ (leftside) and $\Omega(S_1, S_2, 0)$ (rightside) by using ADI method with $\frac{1}{2}$ step size.

higher accuracy and greater computational power. Note that other models such as Heston can be used instead of the Black-Scholes model, and other methods such as Radial Basis Function (RBF) and Machin Learning can be applied to solve the equation, and we can compare the solutions of these different methods. These recommendations can be used as suitable options to find better results [7].

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