Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-10
DOI:10.22034/cmde.2024.61552.2665

Study of p-Laplacian hybrid fractional differential equations involving the generalized Caputo proportional fractional derivative

Samira Zerbib*, **Najat Chefnaj, Khalid Hilal, and Ahmed Kajouni**

LMACS Laboratory, Sultan Moulay Slimane University, Beni Mellal, Morocco.

Abstract

In this paper, we investigate the existence of solutions for hybrid p-Laplacian differential equations involving the generalized fractional proportional Caputo derivative of order $1 < \vartheta < 2$, employing Schauder's fixed point theorem. To illustrate the practical application of our findings, we provide a concrete example.

Keywords. Generalized Caputo proportional fractional derivative, Hybrid differential equation, P-Laplacian operator, Schauder's fixed point theorem.

2010 Mathematics Subject Classification. 34A12, 32G34, 35B50.

1. Introduction

Over the past three decades, fractional calculus has garnered significant popularity and importance, primarily due to its established applications across various scientific and technical fields. It offers several potentially valuable techniques for solving integral and differential equations, as well as addressing other issues related to specific mathematical physics functions, their extensions, and generalizations involving one or more variables.

The laws governing physical dynamics are not always described by ordinary differential equations of standard order. In some cases, their behavior is governed by differential equations of fractional order, as documented in [11, 14]. Fractional derivatives have played a central role in engineering science and applied mathematics, as highlighted in [2, 15]. Notably, recent attention has been directed towards the study of fractional differential equations, with relevant results available in references [7, 17].

The p-Laplacian operator has a wide range of scientific and mathematical applications. As a generalization of the Laplace operator, it frequently emerges in the context of elliptic partial differential equations. Notably, it enhances edge-preserving smoothing in image processing. It is also employed to model diffusion in non-Newtonian fluids and electrical networks. In materials science, it helps define the conductivity of heterogeneous media. Additionally, it describes population dynamics in mathematical biology. The flexibility of the p-Laplacian allows it to be adapted to various phenomena. Fractional differential equations involving the p-Laplacian operator are utilized to model various phenomena in applied fields such as blood flow issues, biology, and turbulent filtration in porous media. Further insights into these equations can be found in references [4, 10, 13].

Quadratically perturbed equations are highly useful in studying nonlinear dynamical systems that are not easily solvable or analyzable. Nonlinear dynamical systems perturbed in this manner are referred to as hybrid differential equations. For a more comprehensive understanding of hybrid fractional differential equations, additional details can be found in [1, 7, 18].

The authors of [16] studied the existence, uniqueness, and the Hyers–Ulam stability results for the solution of the

Received: 06 May 2024 ; Accepted: 29 September 2024.
∗ Corresponding author. Email: samira.zerbib@usms.ma .

following fractional coupled hybrid pantograph system with a p-Laplacian operator

$$\begin{cases} {}^{C}D^{\beta}\Phi_p\left(\frac{\mathcal{B}_1(t)}{F_1(t,\mathcal{B}_1(t),\mathcal{B}_2(t))}\right) = \Pi_1(t,\mathcal{B}_1(\lambda t),\mathcal{B}_2(t)), \quad t\in I_1=[0,1], \\ {}^{C}D^{\gamma}\Phi_p\left(\frac{\mathcal{B}_2(t)}{F_2(t,\mathcal{B}_1(t),\mathcal{B}_2(t))}\right) = \Pi_2(t,\mathcal{B}_1(t),\mathcal{B}_2(\lambda t)), \\ \mathcal{B}_1(0)=0, \quad \mathcal{B}_1(1)=\alpha\mathcal{B}_1(\eta), \\ \mathcal{B}_2(0)=0, \quad \mathcal{B}_2(1)=\sigma\mathcal{B}_2(\xi), \end{cases} \tag{1.1}$$

where $\beta,\gamma\in(1,2]$, α, λ, σ, η, ξ, $\alpha\eta^{\beta-1}$, $\sigma\xi^{\gamma-1}\in(0,1)=I_2$, $\Pi_1,\Pi_2 : I_1\times\mathbb{R}^2\longrightarrow\mathbb{R}$, and $F_1,F_2\in(I_1\times\mathbb{R}^2\longrightarrow\mathbb{R},\quad \mathbb{R}-\{0\})$ are nonlinear continuous functions.
In [5], the authors established the existence and uniqueness results for the following p-Laplacian fractional equation:

$$\begin{cases} \left(\Phi_p\left({}^{C}D^{\alpha,\psi}_{0_+}w(t)\right)\right)' = f(t,w(t)), \quad t\in\Sigma=[0,T], \\ w(0)=\sigma_1 w(T), \quad w'(0)=\sigma_2 w'(T), \quad w_0\in\mathbb{R}, \end{cases} \tag{1.2}$$

where ${}^{C}D^{\alpha,\psi}_{0_+}$ is the ψ-Caputo fractional derivative of order $\alpha\in(1,2)$, and Φ_p is the p-Laplacian operator.

Inspired by the above mentioned works, this paper delves into the development of the theory concerning hybrid fractional differential equations involving the p-Laplacian operator. Specifically, we focus on investigating the existence of solutions for the following p-Laplacian hybrid fractional differential equation, which incorporates the generalized Caputo proportional fractional derivative.

$$\begin{cases} {}^{C}_{\delta}D^{\theta,g}_{0_+}\Phi_p\left({}^{C}_{\delta}D^{\vartheta,g}_{0_+}\left(\frac{w(t)}{\mathcal{F}(t,w(t))}\right)\right) = \mathcal{G}(t,w(t)), \quad t\in\Sigma=[0,b], \\ \left(\frac{w(t)}{\mathcal{F}(t,w(t))}\right)_{t=0} = w_0, \quad \left(\frac{w(t)}{\mathcal{F}(t,w(t))}\right)'_{t=0} = 0, \quad w_0\in\mathbb{R}, \end{cases} \tag{1.3}$$

where $0<\theta<1$, $1<\vartheta<2$, ${}^{C}_{\delta}D^{\vartheta,g}_{0^+}(.)$ is the generalized Caputo proportional fractional derivative of order ϑ, $\Phi_p(x)=|x|^{p-2}x$, $p>1$ is the p-Laplacian operator, $g:\Sigma\to\mathbb{R}$, $\mathcal{F}\in C(\Sigma\times\mathbb{R},\mathbb{R}^*)$, and $\mathcal{G}\in C(\Sigma\times\mathbb{R},\mathbb{R})$. To the best of our knowledge, this is the first time that the problem (1.3) is being studied.

This paper is organized as follows: In section 2, some definitions and notations are introduced. In section 3, we present the main results concerning the existence of solutions for Problem (1.3). In section 4, we provide an application to demonstrate the key points of this work, and finally, we formulate a conclusion in section 5.

2. Preliminaries

In this section, we present definitions and lemmas regarding the generalized Caputo proportional fractional derivative, as well as other characteristics of the p-Laplacian operator. These definitions and lemmas will be consistently employed in the subsequent sections of this work.

- Let $\Sigma=[0,b]$ be a finite interval of $\mathbb{R}$. We denote by $C(\Sigma,\mathbb{R})$ the Banach space of continuous functions with the norm $\|w\|=\sup\{|w(t)|: t\in\Sigma\}$.
- Throughout this paper, we consider the function $g:\Sigma\to\mathbb{R}$ to be a strictly positive, increasing, and differentiable function.

Definition 2.1. [8] Let $0<\delta<1$, $\vartheta>0$, $h\in L^1(\Sigma,\mathbb{R})$. The left-sided generalized proportional fractional integral with respect to g of order ϑ of the function h is given by

$$_{\delta}I^{\vartheta,g}_{0^+}h(t)=\frac{1}{\delta^{\vartheta}\Gamma(\vartheta)}\int_0^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g'(s)(g(t)-g(s))^{\vartheta-1}h(s)ds,$$

where $\Gamma(\vartheta)=\displaystyle\int_0^{+\infty}e^{-\tau}\tau^{\vartheta-1}d\tau$ is the Euler gamma function.

Definition 2.2. [8] Let $0<\delta<1$, ζ, $\rho:[0,1]\times\mathbb{R}\to[0,\infty)$ be continuous functions such that $\lim\limits_{\delta\to0^+}\zeta(\delta,t)=0$, $\lim\limits_{\delta\to1^-}\zeta(\delta,t)=1$, $\lim\limits_{\delta\to0^+}\rho(\delta,t)=1$, $\lim_{\delta\to1^-}\rho(\delta,t)=0$, and $\zeta(\delta,t)\neq0$, $\rho(\delta,t)\neq0$ for each $\delta\in[0,1]$, $t\in\mathbb{R}$. Then the proportional derivative of order δ with respect to g of the function h is given by

$$_{\delta}D^{g}h(t) = \rho(\delta,t)h(t) + \zeta(\delta,t)\frac{h^{'}(t)}{g^{'}(t)}.$$

In particular, if $\zeta(\delta,t) = \delta$ and $\rho(\delta,t) = 1-\delta$, then we have

$$_{\delta}D^{g}h(t) = (1-\delta)h(t) + \delta\frac{h^{'}(t)}{g^{'}(t)}.$$

Definition 2.3. [8] Let $\delta \in (0,1]$. The left-sided generalized Caputo proportional fractional derivative of order $n-1 < \vartheta < n$ is defined by

$$\begin{aligned} {}^{C}_{\delta}D^{\vartheta;g}_{0^+}h(t) &= {}_{\delta}I^{n-\vartheta;g}_{0^+}({}_{\delta}D^{n,g}h(t)) \\ &= \frac{1}{\delta^{n-\vartheta}\Gamma(n-\vartheta)}\int_0^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))}g^{'}(s)(g(t)-g(s))^{n-\vartheta-1}({}_{\delta}D^{n,g}h)(s)ds, \end{aligned}$$

where $n = [\vartheta]+1$ and ${}_{\delta}D^{n,g} = \underbrace{{}_{\delta}D^{g}.{}_{\delta}D^{g}...{}_{\delta}D^{g}}_{n-times}$.

Lemma 2.4. *[9] Let $t \in \Sigma$, $\delta \in (0,1]$, $(\vartheta,\theta > 0)$, and $h \in L^1(\Sigma,\mathbb{R})$. Then, we have*

$${}_{\delta}I^{\vartheta;g}_{0^+}({}_{\delta}I^{\theta;g}_{0^+}h(t)) = {}_{\delta}I^{\theta;g}_{0^+}({}_{\delta}I^{\vartheta;g}_{0^+}h(t)) = {}_{\delta}I^{\vartheta+\theta;g}_{0^+}h(t).$$

Throughout this paper, as a simplification, we set

$$\Omega^{\vartheta-1}_{g}(t,0) = e^{\frac{\delta-1}{\delta}(g(t)-g(0))}(g(t)-g(0))^{\vartheta-1}.$$

Lemma 2.5. *[9] Let $\vartheta > 0$, $\theta > 0$, and $\delta \in (0,1]$. Then, we have*

(i) $({}_{\delta}I^{\vartheta,g}_{0^+}e^{\frac{\delta-1}{\delta}(g(\tau)-g(0))}(g(\tau)-g(0))^{\theta-1})(t) = \frac{\Gamma(\theta)}{\delta^{\vartheta}\Gamma(\vartheta+\theta)}\Omega^{\vartheta+\theta-1}_{g}(t,0)$.

(ii) ${}^{C}_{\delta}D^{\vartheta,g}_{0^+}e^{\frac{\delta-1}{\delta}(g(\tau)-g(0))}(g(\tau)-g(0))^{\theta-1})(t) = \frac{\delta^{\vartheta}\Gamma(\theta)}{\Gamma(\theta-\vartheta)}\Omega^{\theta-\vartheta-1}_{g}(t,0)$.

Lemma 2.6. *[9] Let $\vartheta > 0$, $\delta \in (0,1]$, and $h \in L^1(\Sigma,\mathbb{R})$. Then, we have*

$$\lim_{t\to 0}({}_{\delta}I^{\vartheta;g}_{0^+}h(t)) = 0.$$

Lemma 2.7. *[12] Let $\delta \in (0,1]$, $n-1 < \vartheta < n$, $n = [\vartheta]+1$. Then, we have*

$${}_{\delta}I^{\vartheta;g}_{0^+}({}^{C}_{\delta}D^{\vartheta;g}_{0^+}h(t)) = h(t) - \sum_{k=0}^{n-1}\frac{({}_{\delta}D^{k,g}h)(0)}{\delta^{k}\Gamma(k+1)}\Omega^{k}_{g}(t,0).$$

Lemma 2.8. *[3] The p-Laplacian operator $\Phi_p : \mathbb{R} \to \mathbb{R}$ defined by $\Phi_p(w) = |w|^{p-2}w$ satisfies the following proprieties:*

(1) *The p-Laplacian operator Φ_p is invertible, moreover we have $\Phi_p^{-1}(w) = \Phi_q$ such that $\frac{1}{p}+\frac{1}{q} = 1$.*

(2) *If $p \in (1,2)$, $|w|,|v| \geq C_1 > 0$, and $wv > 0$. Then, we get*

$$|\Phi_p(w) - \Phi_p(v)| \leq (p-1)C_1^{p-2}|w-v|.$$

(3) *If $p > 2$, $|w|,|v| \leq C_2$. Then, we obtain*

$$|\Phi_p(w) - \Phi_p(v)| \leq (p-1)C_2^{p-2}|w-v|.$$

Theorem 2.9. *(Schauder's Fixed-Point Theorem [6]) Let D be a nonempty bounded convex and closed subset of a Banach space X, and let $\Delta : D \to D$ be a continuous and compact map. Then, Δ has at least one fixed point in D.*

3. Results

In this section, we present the proof of several lemmas and introduce certain assumptions. These lemmas and assumptions will be instrumental in establishing the existence theorem for the solution to the p-Laplacian hybrid fractional differential Equation (1.3).

Lemma 3.1. *Let $\Sigma = [0, b]$, $\mathcal{F} \in C(\Sigma \times \mathbb{R}^*, \mathbb{R})$, and $\mathcal{G} \in C(\Sigma \times \mathbb{R}, \mathbb{R})$. Then the problem (1.3) has a solution given by*

$$w(t) = \mathcal{F}(t, w(t)) \left[w_0 e^{\frac{\delta-1}{\delta}(g(t)-g(0))} + \frac{(1-\delta)w_0}{\delta}\Omega_g^1(t,0) +_\delta I_{0^+}^{\vartheta,g}\Phi_q\left({}_\delta I_{0^+}^{\theta,g}\mathcal{G}(t, w(t))\right)\right]. \tag{3.1}$$

Proof. Let $w(t)$ be a solution of the problem (1.3). Applying the operator ${}_\delta I_{0^+}^{\theta,g}(.)$ on both sides of the fractional differential Equation (1.3), using Lemma 2.7, and the fact that $\Phi_p\left({}_\delta^C D_{0^+}^{\vartheta,g}\left(\frac{w(t)}{\mathcal{F}(t,w(t))}\right)\right)_{t=0} = 0$, we get

$$\Phi_p\left({}_\delta^C D_{0_+}^{\vartheta,g}\left(\frac{w(t)}{\mathcal{F}(t, w(t))}\right)\right) =_\delta I_{0^+}^{\theta,g}\mathcal{G}(t, w(t)).$$

Using the inverse operator of Φ_p, we obtain

$${}_\delta^C D_{0_+}^{\vartheta,g}\left(\frac{w(t)}{\mathcal{F}(t, w(t))}\right) = \Phi_q\left({}_\delta I_{0^+}^{\theta,g}\mathcal{G}(t, w(t))\right). \tag{3.2}$$

Applying the operator ${}_\delta I_{0^+}^{\vartheta,g}(.)$ on both sides of the fractional differential Equation (3.2) and using Lemma 2.7. Then, we have

$$\begin{aligned}\frac{w(t)}{\mathcal{F}(t, w(t))} &= \lambda_0 e^{\frac{\delta-1}{\delta}(g(t)-g(0))} + \lambda_1\frac{\Omega_g^1(t,0)}{\delta}\\ &\quad+\frac{1}{\delta^\vartheta\Gamma(\vartheta)}\int_0^t \Omega_g^{\vartheta-1}(t,s)g'(s)\Phi_q\left({}_\delta I_{0^+}^{\theta,g}\mathcal{G}(s, w(s))\right)ds,\end{aligned} \tag{3.3}$$

with $\lambda_0, \lambda_1 \in \mathbb{R}$.
Putting $t = 0$ in the above integral Equation (3.3), we get $\lambda_0 = \left(\frac{w(t)}{\mathcal{F}(t,w(t))}\right)_{t=0} = w_0$.
We have

$$\begin{aligned}&\left(\frac{w(t)}{\mathcal{F}(t, w(t))}\right)' = \frac{d}{dt}\left(\frac{w(t)}{\mathcal{F}(t, w(t))}\right)\\ &= \frac{w_0(\delta-1)g'(t)}{\delta}e^{\frac{\delta-1}{\delta}(g(t)-g(0))} + \frac{\lambda_1}{\delta}\left(g'(t)e^{\frac{\delta-1}{\delta}(g(t)-g(0))} + \frac{g'(t)(\delta-1)\Omega_g^1(t,0)}{\delta}\right)\\ &+\frac{1}{\delta^\vartheta\Gamma(\vartheta)}\frac{d}{dt}\left(\int_0^t \Omega_g^{\vartheta-1}(t,s)g'(s)\Phi_q\left({}_\delta I_{0^+}^{\theta,g}\mathcal{G}(s, w(s))\right)ds\right)\\ &= \frac{w_0(\delta-1)g'(t)}{\delta}e^{\frac{\delta-1}{\delta}(g(t)-g(0))} + \frac{\lambda_1}{\delta}\left(g'(t)e^{\frac{\delta-1}{\delta}(g(t)-g(0))} + \frac{g'(t)(\delta-1)\Omega_g^1(t,0)}{\delta}\right)\\ &+\frac{(\delta-1)(g'(t))}{\delta^{\vartheta+1}\Gamma(\vartheta)}\int_0^t \Omega_g^{\vartheta-1}(t,s)g'(s)\Phi_q\left({}_\delta I_{0^+}^{\theta,g}\mathcal{G}(s, w(s))\right)ds\\ &+\frac{(g'(t))}{\delta^\vartheta\Gamma(\vartheta-1)}\int_0^t \Omega_g^{\vartheta-2}(t,s)g'(s)\Phi_q\left({}_\delta I_{0^+}^{\theta,g}\mathcal{G}(s, w(s))\right)ds.\end{aligned} \tag{3.4}$$

Putting $t = 0$ in the integral Equation (3.4) and using the initial condition $\left(\frac{w(t)}{\mathcal{F}(t,w(t))}\right)'_{t=0} = 0$, we get $\lambda_1 = (1-\delta)w_0$.
Substituting λ_0 and λ_1 in (3.3) we obtain

$$w(t) = \mathcal{F}(t, w(t)) \left[w_0 e^{\frac{\delta-1}{\delta}(g(t)-g(0))} + \lambda_1\frac{\Omega_g^1(t,0)}{\delta} +_\delta I_{0^+}^{\vartheta,g}\Phi_q\left({}_\delta I_{0^+}^{\theta,g}\mathcal{G}(t, w(t))\right)\right].$$

This completes the proof. □

Next, we introduce the following assumptions:

(A_1) There are constants $L_0,\ L_1 > 0$ such that for each $w, v \in C(\Sigma, \mathbb{R})$ and for each $t \in \Sigma$, we have

$$|\mathcal{G}(t, w) - \mathcal{G}(t, v)| \le L_0|p - q| \quad and \quad |\mathcal{G}(t, w)| \le L_1.$$

(A_2) There exists a constant $L_2 > 0$ such that for each $w \in C(\Sigma, \mathbb{R})$ and $t \in \Sigma$, we have

$$|\mathcal{F}(t, w)| \le L_2.$$

(A_3) There exists $C > 0$ such that for all $t \in \Sigma$ and $w \in C(\Sigma, \mathbb{R})$, we have

$$\left|{}_{\delta}I_{0^+}^{\theta,g}\mathcal{G}(t, w(t))\right| > C.$$

Lemma 3.2. *Let $1 < q < 2$, and $v, w \in C(\Sigma, \mathbb{R})$, then we have*

$$\left|\Phi_q\left({}_{\delta}I_{0^+}^{\theta,g}\mathcal{G}(t, w(t))\right) - \Phi_q\left({}_{\delta}I_{0^+}^{\theta,g}\mathcal{G}(t, v(t))\right)\right| \le \frac{(q-1)L_0C^{q-2}(g(b) - g(0))^{\theta}}{\delta^{\theta}\Gamma(\theta + 1)} \parallel w - v \parallel .$$

Proof. Let $1 < q < 2$ and $v, w \in C(\Sigma, \mathbb{R})$, then by using the assumptions A_1, A_3, the fact that $e^{\frac{\delta-1}{\delta}(g(t)-g(0))} < 1$, and Lemma 2.8, we get

$$\begin{aligned}
&\left|\Phi_q\left({}_{\delta}I_{0^+}^{\theta,g}\mathcal{G}(t, w(t))\right) - \Phi_q\left({}_{\delta}I_{0^+}^{\theta,g}\mathcal{G}(t, v(t))\right)\right| \\
&\le (q-1)C^{q-2}\left|{}_{\delta}I_{0^+}^{\theta,g}\mathcal{G}(t, w(t)) -{}_{\delta} I_{0^+}^{\theta,g}\mathcal{G}(t, v(t))\right| \\
&\le \frac{(q-1)C^{q-2}}{\delta^{\theta}\Gamma(\theta)} \int_0^t e^{\frac{\delta-1}{\delta}(g(t)-g(s))} g'(s)(g(t) - g(s))^{\theta-1} \left|\mathcal{G}(s, w(s)) - \mathcal{G}(s, v(s))\right| ds \\
&\le \frac{(q-1)L_0C^{q-2}}{\delta^{\theta}\Gamma(\theta)} \int_0^t g'(s)(g(t) - g(s))^{\theta-1} \parallel w - v \parallel ds \\
&\le \frac{(q-1)L_0C^{q-2}(g(b) - g(0))^{\theta}}{\delta^{\theta}\Gamma(\theta + 1)} \parallel w - v \parallel .
\end{aligned}$$

Then, the proof is completed. □

Let $w \in C(\Sigma, \mathbb{R})$ and $t \in \Sigma$, we consider the operator Δ defined by:

$$(\Delta w)(t) = \mathcal{F}(t, w(t)) \left[w_0 e^{\frac{\delta-1}{\delta}(g(t)-g(0))} + \lambda_1 \frac{\Omega_g^1(t, 0)}{\delta} +_{\delta} I_{0^+}^{\vartheta,g}\Phi_q\left({}_{\delta}I_{0^+}^{\theta,g}\mathcal{G}(t, w(t))\right)\right],$$

where $\lambda_1 = (1 - \delta)w_0$.

Lemma 3.3. *The operator $\Delta : C(\Sigma, \mathbb{R}) \to C(\Sigma, \mathbb{R})$ satisfies the following inequality*

$$\parallel (\Delta w)(t) \parallel \le L_2 \left(|w_0| + \frac{|\lambda_1|(g(b) - g(0))}{\delta} + \frac{L_1^{q-1}(g(b) - g(0))^{\theta(q-1)+\vartheta}}{\delta^{\theta(q-1)+\vartheta} \left(\Gamma(\theta + 1)\right)^{(q-1)} \Gamma(\vartheta + 1)}\right).$$

Proof. Let $t \in \Sigma$ and $w \in C(\Sigma, \mathbb{R})$. By using the assumptions A_1, A_2, and the fact that $e^{\frac{\delta-1}{\delta}(g(t)-g(0))} < 1$, we have

$$
\begin{aligned}
&|(\Delta w)(t)| \\
&\leq L_2\left(|w_0|e^{\frac{\delta-1}{\delta}(g(b)-g(0))} + \frac{|\lambda_1|(g(b)-g(0))}{\delta}e^{\frac{\delta-1}{\delta}(g(b)-g(0))} + {}_\delta I_{0^+}^{\vartheta,g}\left|\Phi_q\left({}_\delta I_{0^+}^{\theta,g}\mathcal{G}(t,w(t))\right)\right|\right) \\
&\leq L_2\left(|w_0| + \frac{|\lambda_1|(g(b)-g(0))}{\delta} + \frac{L_1^{q-1}}{\delta^{\vartheta}\Gamma(\vartheta)}\int_0^t (g(t)-g(s))^{\vartheta-1}g'(s)\left|\frac{1}{\delta^{\theta}\Gamma(\theta)}\int_0^s (g(s)-g(\tau))^{\theta-1}g'(\tau)d\tau\right|^{q-1}ds\right) \\
&\leq L_2\left(|w_0| + \frac{|\lambda_1|(g(b)-g(0))}{\delta} + \frac{L_1^{q-1}(g(b)-g(0))^{\theta(q-1)}}{\delta^{\theta(q-1)}\left(\Gamma(\theta+1)\right)^{(q-1)}}\left(\frac{1}{\delta^{\vartheta}\Gamma(\vartheta)}\int_0^t (g(t)-g(s))^{\vartheta-1}g'(s)ds\right)\right) \\
&\leq L_2\left(|w_0| + \frac{|\lambda_1|(g(b)-g(0))}{\delta} + \frac{L_1^{q-1}(g(b)-g(0))^{\theta(q-1)+\vartheta}}{\delta^{\theta(q-1)+\vartheta}\left(\Gamma(\theta+1)\right)^{(q-1)}\Gamma(\vartheta+1)}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|(\Delta w)(t)| &\leq L_2\left(|w_0| + \frac{|\lambda_1|(g(b)-g(0))}{\delta} + \frac{L_1^{q-1}(g(b)-g(0))^{\theta(q-1)+\vartheta}}{\delta^{\theta(q-1)+\vartheta}\left(\Gamma(\theta+1)\right)^{(q-1)}\Gamma(\vartheta+1)}\right) \\
&:= r. \qquad (3.5)
\end{aligned}
$$

Then, the proof is completed. □

Let the Banach space $X = (C(\Sigma, \mathbb{R}), \| \, . \, \|)$. Then we consider B_r defined as:

$$B_r = \{w \in X : \|w\| \leq r\}.$$

It is easy to see that B_r is a convex, closed, bounded, and nonempty subset of the Banach space X.
Now, we have all the arguments to show the existence result for problem (1.3). Subsequently, we present the following existence theorem.

Theorem 3.4. *Assume that $p > 2$ and all assumptions (A_1)-(A_3) hold. Then, the p-Laplacian hybrid fractional differential Equation (1.3) has a solution $w \in C(\Sigma, \mathbb{R})$.*

Proof. To show that the problem (1.3) has a solution $w \in C(\Sigma, \mathbb{R})$ is equivalent to showing that the operator $\Delta : B_r \to B_r$ satisfies all the conditions of Theorem 2. Then, the proof is given in the two steps:
Step 1 The operator Δ is continuous.
Let $(w_n)_{n\in\mathbb{N}}$ be a sequence in B_r such that $w_n \to w$ as $n \to \infty$ in B_r. By using assumptions A_1 and A_2, Lemma 3.2, the fact that $\Omega_g^1(t,0) = e^{\frac{\delta-1}{\delta}(g(t)-g(0))}(g(t)-g(0))$, $e^{\frac{\delta-1}{\delta}(g(t)-g(0))} < 1$, and based on the same arguments as in Lemma

3.3, we get

$$
\begin{aligned}
&|(\Delta w_n)(t) - (\Delta w(t))| \\
&= \Bigg| \mathcal{F}(t, w_n(t)) \left[w_0 e^{\frac{\delta-1}{\delta}(g(t)-g(0))} + \lambda_1 \frac{\Omega_g^1(t,0)}{\delta} +_\delta I_{0^+}^{\vartheta,g} \Phi_q \left({}_\delta I_{0^+}^{\theta,g} \mathcal{G}(t, w_n(t)) \right) \right] \\
&- \mathcal{F}(t, w(t)) \left[w_0 e^{\frac{\delta-1}{\delta}(g(t)-g(0))} + \lambda_1 \frac{\Omega_g^1(t,0)}{\delta} +_\delta I_{0^+}^{\vartheta,g} \Phi_q \left({}_\delta I_{0^+}^{\theta,g} \mathcal{G}(t, w_n(t)) \right) \right] \\
&+ \mathcal{F}(t, w(t)) \left[w_0 e^{\frac{\delta-1}{\delta}(g(t)-g(0))} + \lambda_1 \frac{\Omega_g^1(t,0)}{\delta} +_\delta I_{0^+}^{\vartheta,g} \Phi_q \left({}_\delta I_{0^+}^{\theta,g} \mathcal{G}(t, w_n(t)) \right) \right] \\
&- \mathcal{F}(t, w(t)) \left[w_0 e^{\frac{\delta-1}{\delta}(g(t)-g(0))} + \lambda_1 \frac{\Omega_g^1(t,0)}{\delta} +_\delta I_{0^+}^{\vartheta,g} \Phi_q \left({}_\delta I_{0^+}^{\theta,g} \mathcal{G}(t, w(t)) \right) \right] \Bigg| \\
&\leq |\mathcal{F}(t, w_n(t)) - \mathcal{F}(t, w(t))| \left[|w_0| e^{\frac{\delta-1}{\delta}(g(t)-g(0))} + |\lambda_1| \frac{\Omega_g^1(t,0)}{\delta} + {}_\delta I_{0^+}^{\vartheta,g} \left| \Phi_q \left({}_\delta I_{0^+}^{\theta,g} \mathcal{G}(t, w_n(t)) \right) \right| \right] \\
&+ |\mathcal{F}(t, w(t))| \, {}_\delta I_{0^+}^{\vartheta,g} \left| \Phi_q \left({}_\delta I_{0^+}^{\theta,g} \mathcal{G}(t, w_n(t)) \right) - \Phi_q \left({}_\delta I_{0^+}^{\theta,g} \mathcal{G}(t, w(t)) \right) \right| \\
&\leq |\mathcal{F}(t, w_n(t)) - \mathcal{F}(t, w(t))| \left[|w_0| + \frac{|\lambda_1|(g(b)-g(0))}{\delta} + \frac{L_1^{q-1}(g(b)-g(0))^{\theta(q-1)+\vartheta}}{\delta^{\theta(q-1)+\vartheta} (\Gamma(\theta+1))^{(q-1)} \Gamma(\vartheta+1)} \right] \\
&+ \frac{(q-1)L_0 L_2 C^{q-2}(g(b)-g(0))^{\theta}}{\delta^{\theta}\Gamma(\theta+1)} \frac{1}{\delta^{\vartheta}\Gamma(\vartheta)} \int_0^t (g(t)-g(s))^{\vartheta-1} g'(s) \parallel w_n - w \parallel ds \\
&\leq |\mathcal{F}(t, w_n(t)) - \mathcal{F}(t, w(t))| \left[|w_0| + \frac{|\lambda_1|(g(b)-g(0))}{\delta} + \frac{L_1^{q-1}(g(b)-g(0))^{\theta(q-1)+\vartheta}}{\delta^{\theta(q-1)+\vartheta} (\Gamma(\theta+1))^{(q-1)} \Gamma(\vartheta+1)} \right] \\
&+ \frac{(q-1)L_0 L_2 C^{q-2}(g(b)-g(0))^{\theta+\vartheta}}{\delta^{\theta+\vartheta}\Gamma(\theta+1)\Gamma(\vartheta+1)} \parallel w_n - w \parallel . \qquad (3.6)
\end{aligned}
$$

By using the continuity of the function $\mathcal{F}$, from the above inequality (3.6), we get

$$\|(\Delta w_n)(t) - (\Delta w)(t)\| \to 0 \quad as \quad n \to \infty.$$

This implies that the operator Δ is continuous.
Step 2: We prove that the operator Δ is compact.
(i) We show that $\Delta(B_r) = \{\Delta w : w \in B_r\}$ is uniformly bounded.
From Lemma 3.3, we have

$$
\begin{aligned}
|(\Delta w)(t)| &\leq L_2 \left(|w_0| + \frac{|\lambda_1|(g(b)-g(0))}{\delta} + \frac{L_1^{q-1}(g(b)-g(0))^{\theta(q-1)+\vartheta}}{\delta^{\theta(q-1)+\vartheta} (\Gamma(\theta+1))^{(q-1)} \Gamma(\vartheta+1)} \right) \\
&:= r.
\end{aligned}
$$

This proves that $\Delta(B_r) = \{\Delta w : w \in B_r\}$ is uniformly bounded.
(ii) $\Delta(B_r)$ is equicontinuous.

Let $t_1, t_2 \in \Sigma$, $t_1 < t_2$, and $w \in B_r$, by using our assumptions, we get

$$
\begin{aligned}
&|(\Delta w)(t_2) - (\Delta w)(t_1)| \\
&= \Bigg| \mathcal{F}(t_2, w(t_2)) \left[w_0 e^{\frac{\delta-1}{\delta}(g(t_2)-g(0))} + \lambda_1 \frac{\Omega_g^1(t_2,0)}{\delta} + \frac{1}{\delta^{\vartheta}\Gamma(\vartheta)} \int_0^{t_2} \Omega_g^{\vartheta-1}(t_2,s) g^{'}(s) \Phi_q \left({}_{\delta}I_{0^+}^{\theta;g} \mathcal{G}(s, w(s)) ds \right) \right] \\
&- \mathcal{F}(t_1, w(t_1)) \left[w_0 e^{\frac{\delta-1}{\delta}(g(t_1)-g(0))} + \lambda_1 \frac{\Omega_g^1(t_1,0)}{\delta} + \frac{1}{\delta^{\vartheta}\Gamma(\vartheta)} \int_0^{t_1} \Omega_g^{\vartheta-1}(t_1,s) g^{'}(s) \Phi_q \left({}_{\delta}I_{0^+}^{\theta;g} \mathcal{G}(s, w(s)) ds \right) \right] \Bigg| \\
&= \Bigg| \mathcal{F}(t_2, w(t_2)) \left[w_0 e^{\frac{\delta-1}{\delta}(g(t_2)-g(0))} + \lambda_1 \frac{\Omega_g^1(t_2,0)}{\delta} + \frac{1}{\delta^{\vartheta}\Gamma(\vartheta)} \int_0^{t_2} \Omega_g^{\vartheta-1}(t_2,s) g^{'}(s) \Phi_q \left({}_{\delta}I_{0^+}^{\theta;g} \mathcal{G}(s, w(s)) ds \right) \right] \\
&- \mathcal{F}(t_1, w(t_1)) \left[w_0 e^{\frac{\delta-1}{\delta}(g(t_2)-g(0))} + \lambda_1 \frac{\Omega_g^1(t_2,0)}{\delta} + \frac{1}{\delta^{\vartheta}\Gamma(\vartheta)} \int_0^{t_2} \Omega_g^{\vartheta-1}(t_2,s) g^{'}(s) \Phi_q \left({}_{\delta}I_{0^+}^{\theta;g} \mathcal{G}(s, w(s)) ds \right) \right] \\
&+ \mathcal{F}(t_1, w(t_1)) \left[w_0 e^{\frac{\delta-1}{\delta}(g(t_2)-g(0))} + \lambda_1 \frac{\Omega_g^1(t_2,0)}{\delta} + \frac{1}{\delta^{\vartheta}\Gamma(\vartheta)} \int_0^{t_2} \Omega_g^{\vartheta-1}(t_2,s) g^{'}(s) \Phi_q \left({}_{\delta}I_{0^+}^{\theta;g} \mathcal{G}(s, w(s)) ds \right) \right] \\
&- \mathcal{F}(t_1, w(t_1)) \left[w_0 e^{\frac{\delta-1}{\delta}(g(t_1)-g(0))} + \lambda_1 \frac{\Omega_g^1(t_1,0)}{\delta} + \frac{1}{\delta^{\vartheta}\Gamma(\vartheta)} \int_0^{t_1} \Omega_g^{\vartheta-1}(t_1,s) g^{'}(s) \Phi_q \left({}_{\delta}I_{0^+}^{\theta;g} \mathcal{G}(s, w(s)) ds \right) \right] \Bigg| \\
&\leq |\mathcal{F}(t_2, w(t_2)) - \mathcal{F}(t_1, w(t_1))| \Bigg| w_0 e^{\frac{\delta-1}{\delta}(g(t_2)-g(0))} + \lambda_1 \frac{\Omega_g^1(t_2,0)}{\delta} \\
&+ \frac{1}{\delta^{\vartheta}\Gamma(\vartheta)} \int_0^{t_2} \Omega_g^{\vartheta-1}(t_2,s) g^{'}(s) \Phi_q \left({}_{\delta}I_{0^+}^{\theta;g} \mathcal{G}(s, w(s)) \right) ds \Bigg| \\
&+ |\mathcal{F}(t_1, w(t_1))| \Bigg| w_0 e^{\frac{\delta-1}{\delta}(g(t_2)-g(0))} + \lambda_1 \frac{\Omega_g^1(t_2,0)}{\delta} + \frac{1}{\delta^{\vartheta}\Gamma(\vartheta)} \int_0^{t_1} \Omega_g^{\vartheta-1}(t_2,s) g^{'}(s) \Phi_q \left({}_{\delta}I_{0^+}^{\theta;g} \mathcal{G}(s, w(s)) \right) ds \\
&+ \frac{1}{\delta^{\vartheta}\Gamma(\vartheta)} \int_{t_1}^{t_2} \Omega_g^{\vartheta-1}(t_2,s) g^{'}(s) \Phi_q \left({}_{\delta}I_{0^+}^{\theta;g} \mathcal{G}(s, w(s)) \right) ds - w_0 e^{\frac{\delta-1}{\delta}(g(t_1)-g(0))} - \lambda_1 \frac{\Omega_g^1(t_1,0)}{\delta} \\
&- \frac{1}{\delta^{\vartheta}\Gamma(\vartheta)} \int_0^{t_1} \Omega_g^{\vartheta-1}(t_1,s) g^{'}(s) \Phi_q \left({}_{\delta}I_{0^+}^{\theta;g} \mathcal{G}(s, w(s)) \right) ds \Bigg| \\
&\leq |\mathcal{F}(t_2, w(t_2)) - \mathcal{F}(t_1, w(t_1))| \left(|w_0| + |\lambda_1| \frac{(g(b)-g(0))}{\delta} + \frac{L_1^{q-1} (g(b)-g(0))^{\theta(q-1)+\vartheta}}{\delta^{\theta(q-1)+\vartheta} \left(\Gamma(\theta+1)\right)^{(q-1)} \Gamma(\vartheta+1)} \right) \\
&+ L_2 \Bigg(\left| w_0 e^{\frac{\delta-1}{\delta}(g(t_2)-g(0))} - w_0 e^{\frac{\delta-1}{\delta}(g(t_1)-g(0))} \right| + \left| \lambda_1 \frac{\Omega_g^1(t_2,0)}{\delta} - \lambda_1 \frac{\Omega_g^1(t_1,0)}{\delta} \right| \\
&+ \frac{1}{\delta^{\vartheta}\Gamma(\vartheta)} \int_0^{t_1} \left| \Omega_g^{\vartheta-1}(t_2,s) - \Omega_g^{\vartheta-1}(t_1,s) \right| g^{'}(s) \left| \Phi_q \left({}_{\delta}I_{0^+}^{\theta;g} \mathcal{G}(s, w(s)) \right) \right| ds \\
&+ \frac{1}{\delta^{\vartheta}\Gamma(\vartheta)} \int_{t_1}^{t_2} \Omega_g^{\vartheta-1}(t_2,s) g^{'}(s) \left| \Phi_q \left({}_{\delta}I_{0^+}^{\theta;g} \mathcal{G}(s, w(s)) \right) \right| ds \Bigg).
\end{aligned}
\tag{3.7}
$$

By using the continuity of the functions $\mathcal{F}, g, e$, and by Lebesgue dominated convergence theorem, from the above inequality (3.7), we get $|(\Delta w)(t_2) - (\Delta w)(t_1)| \to 0$ as $t_1 \to t_2$.
Therefore, the operator $\Delta(B_r)$ is equicontinuous.
From (i), (ii), and by applying the Arzela-Ascoli theorem, we deduce that $\Delta(B_r)$ is relatively compact. According to steps 1 and 2, it follows that the operator $\Delta : B_r \to B_r$ is continuous and compact. Hence, from Theorem 2 the operator Δ has a fixed point in B_r. This implies that the p-Laplacian hybrid fractional differential Equation (1.3) has

a solution $w \in C(\Sigma, \mathbb{R})$.
Then the proof is completed. □

4. Example

In this section, we give an illustrative example to demonstrate the practical applications of the main results of this work. In particular, our results can be reduced to the example in [5].
Let $\Sigma = [0,1]$, $p = \frac{7}{3}$, $\theta = \delta = \frac{1}{2}$, $\vartheta = \frac{3}{2}$, $g(t) = t$, $\mathcal{G}(t, w(t)) = \frac{1}{4} + \cos\left(\frac{\pi w(t)}{7}\right)$, and $\mathcal{F}(t, w(t)) = \frac{3}{2} + \frac{t^3}{9}\sin^2\left(\frac{\pi w(t)}{3}\right)$.
we Consider the following p-Laplacian hybrid fractional differential equation:

$$\begin{cases} {}^{C}_{\frac{1}{2}}D^{\frac{1}{2},t}_{0^+}\Phi_{\frac{7}{3}}\left({}^{C}_{\frac{1}{2}}D^{\frac{3}{2},t}_{0^+}\left(\frac{w(t)}{\frac{3}{2}+\frac{t^3}{9}\sin^2\left(\frac{\pi w(t)}{3}\right)}\right)\right) = \frac{1}{4} + \cos\left(\frac{\pi w(t)}{7}\right), & t \in \Sigma = [0,1], \\ \left(\frac{w(t)}{\frac{3}{2}+\frac{t^3}{9}\sin^2\left(\frac{\pi w(t)}{3}\right)}\right)_{t=0} = w_0, \quad \left(\frac{w(t)}{\frac{3}{2}+\frac{t^3}{9}\sin^2\left(\frac{\pi w(t)}{3}\right)}\right)'_{t=0} = 0, & w_0 \in \mathbb{R}. \end{cases} \tag{4.1}$$

It is easy to see that $1 < q = \frac{7}{4} < 2$, $\mathcal{G} \in C(\Sigma \times \mathbb{R}, \mathbb{R})$, and $\mathcal{F} \in C(\Sigma \times \mathbb{R}, \mathbb{R}^*)$. Now, we check for assumptions A_1, A_2, and A_3. We have

$$\begin{aligned} |\mathcal{G}(t, w(t)) - \mathcal{G}(t, v(t))| &\leq \frac{\pi}{7}|w(t) - v(t)| \\ &\leq \frac{\pi}{7} \parallel w - v \parallel, \end{aligned}$$

and

$$|\mathcal{G}(t, w(t))| \leq \frac{5}{4}.$$

Therefore, the assumption A_1 holds, with $L_0 = \frac{2\pi}{7}$ and $L_1 = \frac{5}{4}$.
It is clear that $|\mathcal{F}(t, w(t))| \leq \frac{29}{18}$, then the assumption A_2 holds with $L_2 = \frac{29}{18}$.
After calculations, we found:

$$\begin{aligned} &\mathcal{G}(t, w(t)) = \frac{1}{4} + \cos\left(\frac{\pi w(t)}{7}\right) > \frac{1}{4} \\ &\Rightarrow {}_{\frac{1}{2}}I^{\frac{1}{2},t}_{0^+}\mathcal{G}(t, w(t)) > \frac{1}{4}\frac{1}{\frac{1}{2}^{\frac{1}{2}}\Gamma(\frac{1}{2})}\int_0^t \Omega_t^{\frac{1}{2}-1}(t,s)t'\,ds \\ &\Rightarrow {}_{\frac{1}{2}}I^{\frac{1}{2},t}_{0^+}\mathcal{G}(t, w(t)) > \frac{\Omega_t^{\frac{1}{2}}(t,0)}{2.828}\mathbb{E}_{1,\frac{3}{2}}(t), \end{aligned}$$

where $\mathbb{E}_{1,\frac{3}{2}}(t)$ is the Mittag-Leffler function.

Hence, the assumption A_3 holds, with $C = \frac{\Omega_t^{\frac{1}{2}}(t,0)}{2.828}\mathbb{E}_{1,\frac{3}{2}}(t) > 0$ for all $t \in [0,1]$.
We remark that all assumptions are satisfied. Then we deduce that the p-Laplacian hybrid fractional differential equation (4.1) has a solution $w \in C(\Sigma, \mathbb{R})$.

5. Conclusion

This study investigated the existence of solutions for p-Laplacian fractional differential equations with quadratic perturbations, involving the generalized Caputo proportional fractional derivative of order $1 < \vartheta < 2$. To establish the existence of the solutions, we invoked Schauder's famous fixed-point theorem. Finally, an illustrative example is presented to demonstrate the key results of this work.

References

[1] N. Chefnaj, K. Hilal, and A. Kajouni, *Impulsive Ψ-Caputo hybrid fractional differential equations with non-local conditions*, Journal of Mathematical Sciences, *280* (2024), 168–179.

[2] N. Chefnaj, A. Taqbibt, K. Hilal, and S. Melliani, *Study of nonlocal boundary value problems for hybrid differential equations involving Ψ-Caputo Fractional Derivative with measures of noncompactness*, Journal of Mathematical Sciences, *271* (2023), 458–467.

[3] T. Chen and W. Liu, *An anti-periodic boundary value problem for the fractional differential equation with a p-Laplacian operator*, Applied Mathematics Letters, *25* (2012), 1671–1675.

[4] Y. Dong and W. Zhang, *Existence and multiplicity of solutions for fractional p-Laplacian equation involving critical concave-convex nonlinearities*, Advanced Nonlinear Studies, (2024).

[5] A. El Mfadel, S. Melliani, and M. Elomari, *Existence and uniqueness results for Ψ-Caputo fractional boundary value problems involving the p-Laplacian operator*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys, *84*(1) (2022), 37-46.

[6] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, *14* (2003).

[7] K. Hilal, A. Kajouni, and S. Zerbib, *Hybrid fractional differential equation with nonlocal and impulsive conditions*, Filomat, *37* (2023), 3291–3303.

[8] F. Jarad, M. A. Alqudah, and T. Abdeljawad, *On more general forms of proportional fractional operators*, Open Math., *18* (2020), 167—176.

[9] F. Jarad, T. Abdeljawad, S. Rashid, and Z. Hammouch, *More properties of the proportional fractional integrals and derivatives of a function with respect to another function*, Adv. Differ. Equ., *2020* (2020), 1–16.

[10] K. Kirti and A. Kumar, *Higher-ordered hybrid fractional differential equations with fractional boundary conditions: Stability analysis and existence theory*, Chaos, Solitons & Fractals, *185* (2024), 115127.

[11] J. Klafter, S. C. Lim, and R. Metzler, *Fractional dynamics*, Recent advances, (2012).

[12] I. Mallah, I. Ahmed, A. Akgul, F. Jarad, and S. Alha, *On ψ-Hilfer generalized proportional fractional operators*, AIMS Math., *7* (2021), 82-103.

[13] S. Mengqiu, *Existence and multiplicity of solutions to p-Laplacian equations on graphs*, Revista Matemática Complutense, *37*(1) (2024), 185-203.

[14] Y. A. Rossikhin and M. V. Shitikova, *Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids*, (1997).

[15] J. Tariboon, A. Samadi, and S. K. Ntouyas, *Nonlocal boundary value problems for Hilfer generalized proportional fractional differential equations*, Fractal and Fractional, *6*(3) (2022), 154.

[16] A. WafaF, H. Khan, and J. Alzabut, *Stability analysis for a fractional coupled Hybrid pantograph system with p-Laplacian operator*, Results in Control and Optimization, *14* (2024), 100333.

[17] S. Zerbib, K. Hilal, and A. Kajouni, *Some new existence results on the hybrid fractional differential equation with variable order derivative*, Results in Nonlinear Analysis, *6* (2023), 34–48 .

[18] S. Zerbib, K. Hilal, and A. Kajouni, *On the hybrid fractional semilinear evolution equations*, International Journal of Nonlinear Analysis and Applications, (2024).