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Fitted mesh cubic spline tension method for singularly perturbed delay differential equations with integral boundary condition

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Abstract

The cubic spline in tension method is taken into consideration to solve the singularly perturbed delay differential equations of convection diffusion type with integral boundary condition. Simpson's 1/3 rule is used to the non-local boundary condition and three model problems are examined for numerical treatment and are addressed using a variety of values for the perturbation parameter ϵ and the mesh size to verify the scheme's applicability. The computational results and rate of convergence are given in tables, and it is seen that the proposed method is more precise and improves the methods used in the literature.

Keywords. Singular perturbation problems, Delay differential equations, Cubic spline in tension, Integral boundary conditions.
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1. Introduction

The non-local boundary conditions in the boundary value problem have been uncovered to be fascinating and the significant category of problems in recent years. There are several applications for these problems in the fields of science and engineering. The presence of small perturbation parameter ϵ causes the solution of singularly perturbed boundary value problems to show a multiscale trend. The solution to these problems exhibit extremely steep gradients in thin region known as layer region and the gradients are minor elsewhere, where the region is known to be outer or regular region. As a result, both asymptotic and numerical approaches to the problems provide significant challenges. In recent years, asymptotic and numerical studies have been done on these topics. Also, these problems arise in the mathematical modelling of a variety of realistic situations such as HIV infection models [2], control theory [11], microscale heat transfer [30], etc.

Some authors used the hybrid difference scheme [8], fitted mesh the B-spline collocation method [12] and the finite difference scheme [14, 26] to solve the boundary-value problem of singular perturbation problems with small delay while others looked into various concepts of singularly perturbed problems(SPPs) involving delay and advanced parameters [13, 23]. The researchers in [9] studied on the existence of periodic solutions of third order delay differential equations. Many different numerical systems including the iterative scheme [25], the finite and hybrid difference approach [28], and the finite element method [22] have been developed in recent years to address the singularly perturbed delay differential equations with a large delay variable and boundary conditions. The authors in [3] dealt with singularly perturbed delay parabolic convection-diffusion equations arising in the modeling of neuronal variability. For the singularly perturbed delay differential equation with the non-local boundary condition, the fitted finite difference technique is taken into consideration by Debela and Duressa [4]. Using the Simpson's rule, the non-local boundary condition is dealt by Kumar and Rao [19]. They used the central difference method to solve singularly perturbed delay differential equations (SPDDEs) with the large delay. Lalu, Phaneendra, and Emineni [20] suggested a non polynomial spline method for solving this type of problems. Sekar and Tamilselvan [24] treated the problem using the finite difference method with the piecewise Shishkin mesh. Debela and Duressa [5] suggested an exponentially fitted

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finite difference method and the integral boundary condition is solved by applying the Simpson's rule. The works in [1, 7, 10, 15, 16, 18, 21, 29] also give the approximate solution of SPDDEs with different numerical approaches.

Researchers are constantly striving to design the non-standard or non-classical numerical schemes to solve these kinds of problems for small epsilon since standard or classical numerical methods are inappropriate and deliver results distant from the expectations for the singularly perturbed boundary value problems. There hasn't been much research done on the related singularly perturbed delay differential equations with integral boundary conditions. A strategy must be developed in order to investigate the error analysis and identify the approximate solution for this type of problem. The main goal of this study is to come up with an improved numerical technique, which is uniformly convergent for solving SPDDEs with integral boundary conditions. In this work, we propose a second order numerical approach for solving SPDDEs of the convection diffusion type with the integral boundary condition.

2. Statement of the problem

As a means of explaining the procedure, we consider the following singularly perturbed delay differential equation of convection diffusion type with integral boundary condition:

$$Lz \equiv -\epsilon z''(u) + a(u)z'(u) + b(u)z(u) + c(u)z(u-1) = d(u),\ u \in \Omega = (0, 2), \tag{2.1}$$

with

$$\begin{aligned} z(u) &= f(u),\ u \in [-1, 0], \\ z(2) &= k + \epsilon \int_0^2 g(u)z(u)du = \beta, \end{aligned} \tag{2.2}$$

where $0 < \epsilon \leq 1$ and β is a constant.

All the related function in (2.1) and (2.2) are taken to be smooth and bounded and ϵ independent. Also, $a(u), b(u), c(u)$ are such that $a(u) \geq a^* > 0$, $b(u) \geq b^* > 0$, $c(u) \leq c^* < 0$, $b(u) + c(u) \geq \delta > 0$, where $a^* + b^* + c^* > 0$ and $b^* - c^* \geq 0$. For ϵ near to 0, the solution of (2.1) and (2.2) has an interior layer and boundary layer. (2.1) and (2.2) can be written as:

$$Lz \equiv h(u),\ u \in \Omega, \tag{2.3}$$

where,

$$Lz = \begin{cases} L_1 z(u) = -\epsilon z''(u) + a(u)z'(u) + b(u)z(u), & u \in \Omega_1, \\ L_2 z(u) = -\epsilon z''(u) + a(u)z'(u) + b(u)z(u) + c(u)z(u-1), & u \in \Omega_2, \end{cases}$$

$$h(u) = \begin{cases} d(u) - c(u)f(u-1), & u \in \Omega_1, \\ d(u), & u \in \Omega_2, \end{cases}$$

with $z(1^-) = z(1^+)$, $z'(1^-) = z'(1^+)$, $z(2) = \beta$. Also, let $Kz(2) = z(2) - \epsilon \int_0^2 g(u)z(u)du = k$, $\Omega^* = \Omega_1 \cup \Omega_2$, and $U = C^0(\Omega) \cup C^1(\Omega) \cup C^2(\Omega)$.

3. Properties of Solution

Lemma 3.1. *(Maximum Principle) Let $\xi(u)$ be any function in U such that $\xi(0) \geq 0$, $K\xi(2) \geq 0$, $L_1\xi(2) \geq 0$ for all $u \in \Omega_1$, $L_2\xi(2) \geq 0$ for all $u \in \Omega_2$ and $[\xi'](1) \leq 0$, then $\xi(u) \geq 0$ for all $u \in \bar{\Omega}$.*

Proof. Define the function $t(u)$ as:

$$t(u) = \begin{cases} \frac{1}{8} + \frac{u}{2}, & u \in [0, 1], \\ \frac{3}{8} + \frac{u}{4}, & u \in [1, 2]. \end{cases}$$

Then $t(u)$ is positive for all $u \in \bar{\Omega}$. Also $Lt(u) > 0$ for all $u \in \Omega_1 \cup \Omega_2$, $t(0) > 0$, $Kt(2) > 0$ and $[t]'(1) < 0$. Let

$$\bar{\mu} = \max\left\{\frac{-\xi(u)}{t(u)} : u \in \bar{\Omega}\right\}.$$

Then $\exists u_0 \in \bar{\Omega}$ satisfying $\xi(u_0) + \bar{\mu}t(u_0) = 0$ and $\xi(u) + \bar{\mu}t(u) \geq 0$, for all $u \in \bar{\Omega}$. Hence $\xi + \bar{\mu}t(u_0)$ gives minimum value.
Now, suppose that the theorem is not true then $\bar{\mu} > 0$. So, we analyze the possible cases for u0 and in all cases we arrive at a contradiction.

Case 1: $u_0 = 0$,

$$0 < (\xi + \bar{\mu}t)(0) = \xi(0) + \bar{\mu}t(0) = 0.$$

Case 2: $u_0 \in \Omega_1$,

$$0 < L(\xi + \bar{\mu}t)(u_0) = -\epsilon(\xi + \bar{\mu}t)''(u_0) + a(u_0)(\xi + \bar{\mu}t)'(u_0) + b(u_0)(\xi + \bar{\mu}t)(u_0) \leq 0.$$

Case 3: $u_0 = 1$,

$$0 \leq [(\xi + \bar{\mu}t)'](1) = [\xi'](1) + \bar{\mu}[t'](1) < 0.$$

Case 4: $u_0 \in \Omega_2$,

$$0 < L(\xi + \bar{\mu}t)(u_0) = -\epsilon(\xi + \bar{\mu}t)''(u_0) + a(u_0)(\xi + \bar{\mu}t)'(u_0) + b(u_0)(\xi + \bar{\mu}t)(u_0) + c(u_0)(\xi + \bar{\mu}t)(u_0 - 1) \leq 0.$$

Case 5: $u_0 = 2$,

$$0 \leq K(\xi + \bar{\mu}t)'(2) = (\xi + \bar{\mu}t)'(2) - \epsilon\int_0^2 g(u)(\xi + \bar{\mu}t)(u_0)du \leq 0.$$

Hence the proof is completed. □

Lemma 3.2. *(Stability Result) The solution $z(u)$ for (2.1) and (2.2) satisfies:*

$$| z(u) | \leq \bar{C} \max\{| z(0) |, | Kz(2) |, \sup_{u\in\Omega^*} | Lz(u) |\}, u \in \bar{\Omega}.$$

Proof. This is proved using the above lemma with the functions:

$$\theta^{\pm}(u) = \bar{C}.\bar{M}.t(u) \pm z(u), u \in \bar{\Omega},$$

where $\bar{M} = \max\{| z(0) |, | Kz(2) |, \sup_{u\in\Omega^*} | Lz(u) |\}$ and $t(u)$ are the test functions in the above lemma. □

Lemma 3.3. *Let $z(u)$ be the solution for the problem (2.1) and (2.2). Then*

$$\|z^{(k)}(u)\|_{\Omega^*} \leq C\epsilon^{-k}, \text{ for } k = 1, 2, 3.$$

4. Description of the method

Consider $u_0 = 0$, $u_{2N} = 2$, $u_i = ih$ and $h = 1/N$. A function $T(u,\tau) = T(u)$ satisfying the following differential equation in $[u_i, u_{i+1}]$,

$$T''(u) - \tau T(u) = [T''(u_i) - \tau T(u_i)]\frac{(u_{i+1} - u)}{h} - [T''(u_{i+1}) - \tau T(u_{i+1})]\frac{(u_i - u)}{h}, \tag{4.1}$$

where $T(u_i) = z_i$ and $\tau > 0$ is called cubic spline in tension. Equation (4.1) is solved as a linear second order differential equation to get,

$$T(u) = Pe^{\frac{\lambda u}{h}} + Qe^{\frac{-\lambda u}{h}} + \left(\frac{R_i - \tau z_i}{\tau}\right)\left[\frac{u - u_{i+1}}{h}\right] + \left(\frac{R_{i+1} - \tau z_{i+1}}{\tau}\right)\left[\frac{u_i - u}{h}\right].$$

By using the conditions $T(u_{i+1}) = z_{i+1}$, $T(u_i) = z_i$, we can find the constants P and Q. Let $\lambda = h\tau^{1/2}$ and $R_i = T''(u_i)$, we get

$$T(u) = \frac{h^2}{\lambda^2 \sinh\lambda}\left[R_{i+1}\sinh\frac{\lambda(u - u_i)}{h} + R_i \sinh\frac{\lambda(u_{i+1} - u)}{h}\right]$$
$$- \frac{h^2}{\lambda^2}\left[\frac{(u - u_i)}{h}\left(R_{i+1} - \frac{\lambda^2}{h^2}z_{i+1}\right) + \left(\frac{u_{i+1} - u}{h}\right)\left(R_i - \frac{\lambda^2}{h^2}z_i\right)\right].$$

Differentiating the above equation and finding the limit $u \to u_i$, we get

$$T'(u_i^+) = \frac{z_{i+1} - z_i}{h} - \frac{h}{\lambda^2}\left[\left(1 - \frac{\lambda}{\sinh\lambda}\right) R_{i+1} - (1 - \lambda\coth\lambda)R_i\right]. \tag{4.2}$$

Taking (u_{i-1}, u_i) as the interval and continuing in the same manner, we obtain,

$$T'(u_i^-) = \frac{z_i - z_{i-1}}{h} - \frac{h}{\lambda^2}\left[(\lambda\coth\lambda - 1)R_i + \left(1 - \frac{\lambda}{\sinh\lambda}\right) R_{i+1}\right]. \tag{4.3}$$

Equations (4.2) and (4.3) are equalized at u_i to get

$$\begin{aligned}\frac{z_{i+1} - z_i}{h} - \frac{h}{\lambda^2}&\left[\left(1 - \frac{\lambda}{\sinh\lambda}\right) R_{i+1} - (1 - \lambda\coth\lambda)R_i\right]\\ &= \frac{z_i - z_{i-1}}{h} + \frac{h}{\lambda^2}\left[(\lambda\coth\lambda - 1)R_i + \left(1 - \frac{\lambda}{\sinh\lambda}\right) R_{i+1}\right].\end{aligned}$$

As a result, we have a tridiagonal system:

$$h^2(\lambda_1 R_{i-1} + 2\lambda_2 R_i + \lambda_1 R_{i+1}) = z_{i+1} - 2z_i + z_{i-1},\ i = 1(1)2N - 1, \tag{4.4}$$

where,

$$\begin{aligned}\lambda_1 &= \frac{-1}{\lambda^2}\left(\frac{\lambda}{\sinh\lambda} - 1\right),\\ \lambda_2 &= \frac{-1}{\lambda^2}\left(1 - \lambda\coth\lambda\right),\\ R_i &= T''(u_i), \quad i = 1, 2, ..., 2N - 1.\end{aligned}$$

The first order derivatives of the spline $T(u, \tau)$ at interior nodes are guaranteed to be continuous by the condition of continuity and the system given in (4.4). If equation is consistent, it is appropriate for solving the given differential equation. This condition is satisfied, if $\lambda_1 + \lambda_2 = \frac{1}{2}$.

At $u = u_i$,

$$\epsilon z''(u_i) = a(u_i)z_i' + b(u_i)z_i + c(u_i)z(u_i - 1) - d(u_i).$$

The conditions can be expressed as:

$$\begin{aligned}&z_i = f_i,\ for\ -N \le i \le 0,\\ &z_{2N} = \beta,\end{aligned}$$

where $f_i = f(u_i)$. Let $a(u_i) = a_i, b(u_i) = b_i, c(u_i) = c_i$ and $d(u_i) = d_i$.

Then we have

$$\epsilon R_i = a_i z_i' + b_i z_i + c_i z(u_i - 1) - d_i. \tag{4.5}$$

The first derivative of z can be replaced using following Taylor series approximations and these are used after substituting (4.5) into (4.4) to obtain the scheme.

$$\begin{aligned}z_{i-1}' &\simeq \frac{-z_{i+1} + 4z_i - 3z_{i-1}}{2h},\\ z_i' &\simeq \frac{z_{i+1} - z_{i-1}}{2h},\\ z_{i+1}' &\simeq \frac{3z_{i+1} - 4z_i + z_{i-1}}{2h}.\end{aligned}$$

Following is the scheme obtained for $i = 1(1)2N - 1$.

$$\left(\epsilon + \frac{3\lambda_1 a_{i-1}h}{2} - h^2\lambda_1 b_{i-1} + \lambda_2 b_i h - \frac{\lambda_1 a_{i+1}h}{2}\right) z_{i-1} + \left(-2\epsilon - 2\lambda_1 a_{i-1}h - 2\lambda_2 b_i h^2 + 2\lambda_1 a_{i+1}h\right) z_i$$

$$+\left(\epsilon+\frac{\lambda_1 a_{i-1}h}{2}-h^2\lambda_1 b_{i+1}-\lambda_2 a_i h-\frac{3\lambda_1 a_{i+1}h}{2}\right)z_{i+1}$$
$$=h^2\left[\lambda_1 c_{i-1}z(u_{i-1-N})+2\lambda_2 c_i z(u_{i-N})+\lambda_1 c_{i+1}z(u_{i+1-N})\right]-h^2(\lambda_1 d_{i-1}+2\lambda_2 d_i+\lambda_1 d_{i+1}). \quad (4.6)$$

5. Numerical Algorithm

Step 1: To get the reduced problem, put $\epsilon=0$ in (2.1), then we have

$$a(u)z_0'+b(u)z_0+c(u)z_0(u-1)=d(u),\ \ u\in[0,1],$$

with $z(u)=f(u)$, $u\in[-1,0]$.

Since $z(u)=f(u)$ in $[-1,0]$, $z_0(u-1)=f(u-1)$. This implies

$$z_0'=\frac{1}{a(u)}\left[d(u)-b(u)z_0-c(u)f(u-1)\right]\ \text{ with } z_0(0)=f(0).$$

We use Runge-Kutta method to solve this IVP to get the solution at $u=1$, say γ [ie $z_0(1)=\gamma$].

Step 2: To find the solution in Ω_1, we employ the scheme in (4.6) with the fitting factor

$$\sigma_\rho=a(1)\rho\coth\left(\frac{a(1)\rho}{2}\right)(\lambda_1+\lambda_2),$$

where $h=\epsilon\rho$.

Now, the scheme in Ω_1 can be rewritten as:

$$K_i z_{i-1}+L_i z_i+M_i z_{i+1}=N_i,\ \ 1<i<N-1,$$

where

$$\begin{aligned}
K_i &= \epsilon\sigma_\rho+\frac{3\lambda_1 a_{i-1}h}{2}-h^2\lambda_1 b_{i-1}+\lambda_2 b_i h-\frac{\lambda_1 a_{i+1}h}{2},\\
L_i &= -2\epsilon\sigma_\rho-2\lambda_1 a_{i-1}h-2\lambda_2 b_i h^2+2\lambda_1 a_{i+1}h,\\
M_i &= \epsilon\sigma_\rho+\frac{\lambda_1 a_{i-1}h}{2}-h^2\lambda_1 b_{i+1}-\lambda_2 a_i h-\frac{3\lambda_1 a_{i+1}h}{2},\\
N_i &= h^2\left[\lambda_1 c_{i-1}f_{i-1-N}+2\lambda_2 c_i f_{i-N}+\lambda_1 c_{i+1}f_{i+1-N}\right]-h^2\left[\lambda_1 d_{i-1}+2\lambda_2 d_i+\lambda_1 d_{i+1}\right].
\end{aligned}$$

Using Gauss elimination method and the conditions $z_0=f(0)$ and $z_N=\gamma$, we solve the system.

Step 3: We need to know the value of $z(2)$ to proceed for the solution in $(1,2)$. Given that,

$$z(2)=k+\epsilon\int_0^2 g(u)z(u)du. \quad (5.1)$$

We employ the Simpson's rule to find the value of the integral in (2.2).

$$\int_0^2 g(u)z(u)du=\frac{h}{3}\left[2\sum_{i=1}^{2N-1}g(u_{2i})z(u_{2i})+4\sum_{i=1}^{2N}g(u_{2i-1})z(u_{2i-1})\right]+\frac{h}{3}\left[g(0)z(0)+g(2)z(2)\right].$$

Using (5.1), we can write as

$$z(2)-\frac{h}{3}\left[2\sum_{i=1}^{2N-1}g(u_{2i})z(u_{2i})+4\sum_{i=1}^{2N}g(u_{2i-1})z(u_{2i-1})\right]-\frac{h}{3}\left[g(0)z(0)+g(2)z(2)\right]=k.$$

Use $z(0) = f(0)$, we get

$$z(2) = \frac{h\epsilon}{(3 - h\epsilon g(2))}\left[g(0)f(0) + 2\sum_{i=1}^{2N-1} g(u_{2i})z(u_{2i}) + 4\sum_{i=1}^{2N} g(u_{2i-1})z(u_{2i-1})\right] + \frac{k}{\left(1 - \frac{\epsilon h}{3}g(2)\right)}.$$

Let $z_{2N} = \beta$, then,

$$\beta = \frac{1}{\left(1 - \frac{\epsilon h}{3}g(2)\right)}\left[\frac{2\epsilon h}{3}\sum_{i=1}^{2N-1} g(u_{2i})z(u_{2i}) + \frac{4\epsilon h}{3}\sum_{i=1}^{2N} g(u_{2i-1})z(u_{2i-1})\right]$$
$$+ \left[\frac{k}{\left(1 - \frac{\epsilon h}{3}g(2)\right)} + \frac{\epsilon h}{3 - \epsilon h g(2)}g(0)f(0)\right].$$

Step 4: To find the solution in Ω_2, we introduce the fitting factor and rewrite the scheme as:

$$K_i z_{i-1} + L_i z_i + M_i z_{i+1} = N_i, \quad N + 1 < i < 2N - 1,$$

where

$$\begin{aligned}
K_i &= \epsilon\sigma_\rho + \frac{3\lambda_1 a_{i-1}h}{2} - h^2\lambda_1 b_{i-1} + \lambda_2 b_i h - \frac{\lambda_1 a_{i+1}h}{2},\\
L_i &= -2\epsilon\sigma_\rho - 2\lambda_1 a_{i-1}h - 2\lambda_2 b_i h^2 + 2\lambda_1 a_{i+1}h,\\
M_i &= \epsilon\sigma_\rho + \frac{\lambda_1 a_{i-1}h}{2} - h^2\lambda_1 b_{i+1} - \lambda_2 a_i h - \frac{3\lambda_1 a_{i+1}h}{2},\\
N_i &= h^2\left[\lambda_1 c_{i-1}z_{i-1-N} + 2\lambda_2 c_i z_{i-N} + \lambda_1 c_{i+1}z_{i+1-N}\right] - h^2\left[\lambda_1 d_{i-1} + 2\lambda_2 d_i + \lambda_1 d_{i+1}\right].
\end{aligned}$$

This system can again be solved by the Gauss Elimination method with the condition $z_N = \gamma$ and $z_{2N} = \beta$.

6. Convergence Analysis

We write the system in matrix form as:

$$VZ = W, \tag{6.1}$$

where $V = (v_{ij})$ is a matrix of order $2N - 1$.

Then, for $i = 1(1)2N - 2$,

$$\begin{aligned}
v_{ii-1} &= \epsilon\sigma_\rho + \frac{3\lambda_1 a_{i-1}h}{2} - h^2\lambda_1 b_{i-1} + \lambda_2 b_i h - \frac{\lambda_1 a_{i+1}h}{2},\\
v_{ii} &= -2\epsilon\sigma_\rho - 2\lambda_1 a_{i-1}h - 2\lambda_2 b_i h^2 + 2\lambda_1 a_{i+1}h,\\
v_{ii+1} &= \epsilon\sigma_\rho + \frac{\lambda_1 a_{i-1}h}{2} - h^2\lambda_1 b_{i+1} - \lambda_2 a_i h - \frac{3\lambda_1 a_{i+1}h}{2}.
\end{aligned}$$

Now, for $i = 2N - 1$,

$$v_{2n-1,i} = \begin{cases}
\frac{4g_i M_{2N-1}\epsilon h}{(3-\epsilon h)g(2)}, & i = 1(2)N - 1,\\
\frac{2g_i M_{2N-1}\epsilon h}{(3-\epsilon h)g(2)}, & i = 2(2)N,\\
\frac{4g_i\epsilon h}{(3-\epsilon h)g(2)}, & i = N + 1(2)2N - 3,\\
\frac{2g_i\epsilon h}{(3-\epsilon h)g(2)}, & i = N + 2(2)2N - 4,\\
\frac{2g_{2N-2}\epsilon h}{(3-\epsilon h)g(2)} - K_{2N-1}, & i = 2N - 2,\\
\frac{4g_{2N-1}\epsilon h}{(3-\epsilon h)g(2)} - L_{2N-1}, & i = 2N - 1,
\end{cases}$$

and $W = (w_i)$ is a column vector, where

$$w_i = \begin{cases} N_1 - K_1 f_0, & i = 1, \\ h^2 \left[\lambda_1 c_{i-1} f_{i-1-N} + 2\lambda_2 c_i f_{i-N} + \lambda_1 c_{i+1} f_{i+1-N}\right] - h^2(\lambda_1 d_{i-1} + 2\lambda_2 d_i + \lambda_1 d_{i+1}), & i = 2(1)N-1, \\ h^2 \left[\lambda_1 c_{i-1} z_{i-1-N} + 2\lambda_2 c_i z_{i-N} + \lambda_1 c_{i+1} z_{i+1-N}\right] - h^2(\lambda_1 d_{i-1} + 2\lambda_2 d_i + \lambda_1 d_{i+1}), & i = N(1)2N-2, \\ N_{2n-1} - \frac{3M_{2N-1}}{(3-\epsilon h)g(2)} \left[k + \frac{\epsilon h}{3} g(0) f_0\right], & i = 2N-1. \end{cases}$$

Also, truncation error is $Y_i(h) = \frac{h^4}{2} L^* + O(h^5)$, where $L^* = \frac{z_i^{(3)}}{\epsilon} \left[\lambda_1 a_{i-1} - \frac{2\lambda_2 a_i}{3} + \lambda_1 a_{i+1}\right]$.

In Error form, (6.1) can also be written as:

$$V\overline{Z} - Y(h) = W, \tag{6.2}$$

where $\overline{Z} = \begin{pmatrix} \overline{z}_1 & \overline{z}_2 & . & . & . & \overline{z}_{2N-1} \end{pmatrix}^t$ and $Y(h) = \begin{pmatrix} Y_1(h) & Y_2(h) & . & . & . & Y_{2N-1}(h) \end{pmatrix}^t$. Here $\overline{Z}$ denote the exact solution and $Y(h)$ is the truncation error.

From (6.1) and (6.2), we get $V(\overline{Z} - Z) = Y(h)$. This can be written as:

$$V\overline{E} = Y(h), \tag{6.3}$$

where $\overline{E} = \overline{Z} - Z = \begin{pmatrix} e_1 & e_2 & . & . & . & e_{2N-1} \end{pmatrix}^t$. Now we find each row sum of the matrix V. Let its $i-$th row sum be denoted by S_j.

$$\begin{aligned} S_1 &= \epsilon\sigma_\rho - \frac{3\lambda_1 a_{i-1} h}{2} + \frac{\lambda_1 a_{i+1} h}{2} - 2h^2\lambda_2 b_i - \lambda_2 a_i h - \lambda_1 h^2 b_{i+1}, \\ S_j &= h^2\left[-\lambda_1 b_{i-1} - \lambda_2 b_i - \lambda_1 b_{i+1}\right] = h^2(B_i), \quad j = 2(1)2N-2, \\ S_{2N-1} &= \frac{2\epsilon h}{(3-\epsilon h)g(2)} M_{2N-1}\left[2(g_1 + g_3 + ... + g_{N-1}) + (g_2 + g_4 + ... + g_N)\right] \\ &+ \frac{2\epsilon h}{(3-\epsilon h)g(2)}\left[2(g_{N+1} + g_{N+3} + ... + g_{2N-3}) + (g_{N+2} + g_{N+4} + ... + g_{2N-4})\right] \\ &+ \frac{2\epsilon h}{(3-\epsilon h)g(2)}\left[g_{2N-2} + 2g_{2N-1} - K_{2N-1} - L_{2N-1}\right]. \end{aligned}$$

Also, $h \to 0$, V is monotone and irreducible, which implies that the matrix is invertible and its elements are not less than zero. From (6.3), it follows that:

$$\overline{E} \quad = \quad V^{-1}Y(h). \tag{6.4}$$

Hence

$$\|\overline{E}\| \quad \leq \quad \|V^{-1}\|\|Y(h)\|. \tag{6.5}$$

Let $\overline{v}_{j,i}$ denote the (j,i)-th element of V^{-1}.

Then $\sum_{i=1}^{2N-1} \overline{v}_{j,i} S_i = 1$, $j = 1(1)2N-1$. This gives

$$\sum_{i=1}^{2N-1} \overline{v}_{j,i} \leq \frac{1}{\min_{1 \leq i \leq 2N-1} S_i} \leq \frac{1}{h^2 \mid B_i \mid}. \tag{6.6}$$

From Equations (6.1), (6.4), (6.5), and (6.6), we have,

$$e_i = \sum_{j=1}^{2N-1} \overline{v}_{j,i} Y_i(h) \qquad , i = 1(1)2N-1,$$

which implies

$$\begin{aligned} e_i &\leq \left(\sum_{j=1}^{2N-1} \overline{v}_{j,i}\right) \max_{1\leq i\leq 2N-1} \mid Y_i(h) \mid, \\ &\leq \frac{1}{h^2 \mid B_i \mid} \times \frac{h^4 L^*}{2} = O(h^2), \end{aligned}$$

where L^* is a constant which does not depend on h. So $\|\overline{E}\| = O(h^2)$. Hence the proposed approach is of second order convergence.

7. Numerical Experiments

In order to demonstrate that the method being proposed is applicable, we look at the results of three different numerical experiments and the results are presented for $\lambda_1 = 1/12$ and $\lambda_2 = 5/12$. The computed answer is presented in the form of tables that demonstrate the solution for a variety of ϵ values. The double-mesh principle is used to the instances that are provided in order to arrive at a determination of the maximum absolute errors.

$$E_\epsilon^N = \max_i \mid z_i^N - z_{2i}^{2N} \mid . \tag{7.1}$$

The $\epsilon-$uniform maximum absolute error for N is calculated by

$$E^N = \max_\epsilon E_\epsilon^N. \tag{7.2}$$

In this work, MATLAB R2022a mathematical software has been used to obtain the numerical results and plots and these are compared with some published works.

Rate of Convergence. The numerical rate of convergence ρ is also computed using the double mesh principle and is defined as:

$$\rho = \frac{\log(E_h) - \log\left(E_{h/2}\right)}{\log 2}.$$

Example 7.1. Consider

$$-\epsilon z''(u) + 3z'(u) + z(u) - z(u-1) = 1,\ u \in (0,2),$$

with

$$\begin{aligned} z(u) &= 1,\ u \in [-1,0], \\ z(2) &= 2 + \frac{\epsilon}{3}\int_0^2 uz(u)du. \end{aligned}$$

As the perturbation parameter is varied, the Table 1 display the maximum absolute inaccuracy that can occur. Figure 1 depicts the numerical solution and Figure 2 gives the point-wise absolute errors of this example for different N values. In addition, Figure 3 shows the maximum absolute error for various ϵ values. The rate of convergence table is also presented in Table 2.

Example 7.2. Consider

$$-\epsilon z''(u) + (1+u)z'(u) + (u+10)z(u) - e^u z(u-1) = \frac{4}{\pi^2}u(1-u),\ u \in (0,2),$$

with,

$$\begin{aligned} z(u) &= 2+u,\ u \in [-1,0], \\ z(2) &= 2 + \frac{\epsilon}{3}\int_0^2 ue^u \sin uz(u)du. \end{aligned}$$

As the perturbation parameter is varied, the Table 4 display the maximum absolute inaccuracy that can occur. Figure 6 depicts the estimated solution and Figure 4 gives the point-wise absolute errors of this example for different N values.

TABLE 1. The maximum absolute error of Example 7.1 for different values of ϵ.

ϵ	N						
	32	64	128	256	512	1024	2048
2^{-4}	4.55e-04	1.70e-04	9.75e-05	5.21e-05	2.69e-05	1.37e-05	6.88e-06
2^{-8}	1.12e-03	5.65e-04	2.66e-04	1.02e-04	2.90e-05	6.45e-06	1.61e-06
2^{-12}	1.12e-03	5.67e-04	2.85e-04	1.43e-04	7.14e-05	3.56e-05	1.67e-05
2^{-16}	1.12e-03	5.67e-04	2.85e-04	1.43e-04	7.14e-05	3.57e-05	1.79e-05
2^{-20}	1.12e-03	5.67e-04	2.85e-04	1.43e-04	7.14e-05	3.57e-05	1.79e-05
2^{-24}	1.12e-03	5.67e-04	2.85e-04	1.43e-04	7.14e-05	3.57e-05	1.79e-05
2^{-28}	1.12e-03	5.67e-04	2.85e-04	1.43e-04	7.14e-05	3.57e-05	1.79e-05
2^{-32}	1.12e-03	5.67e-04	2.85e-04	1.43e-04	7.14e-05	3.57e-05	1.79e-05
E^N	1.12e-03	5.67e-04	2.85e-04	1.43e-04	7.14e-05	3.57e-05	1.79e-05
CPU time	0.0938 s	0.1094 s	0.2031 s	0.4531 s	1.2500 s	6.5312 s	35.2812 s
Results in [17]							
2^{-4}	4.04e-01	1.19e-01	2.62e-02	6.83e-03	1.94e-03	6.09e-04	2.18e-04
2^{-8}	6.71e-01	3.17e-01	1.18e-01	3.51e-02	1.00e-02	2.89e-03	1.06e-03
2^{-12}	6.70e-01	3.18e-01	1.21e-01	3.73e-02	1.13e-02	3.26e-03	1.56e-03
2^{-16}	6.70e-01	3.17e-01	1.20e-01	3.73e-02	1.25e-02	4.29e-03	1.29e-03
2^{-20}	6.70e-01	3.17e-01	1.20e-01	3.69e-02	1.17e-02	4.20e-03	2.08e-03
2^{-24}	6.70e-01	3.17e-01	1.20e-01	3.69e-02	1.16e-02	3.87e-03	1.43e-03
2^{-28}	6.70e-01	3.17e-01	1.20e-01	3.69e-02	1.16e-02	3.84e-03	1.35e-03
2^{-32}	6.70e-01	3.17e-01	1.20e-01	3.69e-02	1.16e-02	3.84e-03	1.32e-03
E^N	6.70e-01	3.17e-01	1.20e-01	3.69e-02	1.16e-02	3.84e-03	1.32e-03
Results in [24]							
2^{-4}	5.61e-04	2.85e-04	1.44e-04	7.21e-05	3.61e-05	1.81e-05	-
2^{-8}	3.82e-03	1.63e-03	6.80e-04	2.83e-04	1.22e-04	1.93e-05	-
2^{-12}	5.57e-03	2.74e-03	1.34e-03	6.60e-04	3.26e-04	1.64e-04	-
2^{-16}	5.99e-03	3.00e-03	1.50e-03	7.47e-04	3.73e-04	1.87e-04	-
2^{-20}	6.10e-03	3.07e-03	1.54e-03	7.69e-04	3.85e-04	1.92e-04	-
E^N	6.10e-03	3.07e-03	1.54e-03	7.69e-04	3.85e-04	1.92e-04	-
Results in [6]							
2^{-4}	3.83e-03	1.18e-03	3.28e-04	8.66e-05	2.23e-05	-	-
2^{-8}	5.18e-03	2.71e-03	1.46e-04	6.67e-04	2.42e-05	-	-
2^{-12}	5.17e-03	2.59e-03	1.30e-03	6.50e-04	3.26e-04	-	-
2^{-16}	5.17e-03	2.59e-03	1.30e-03	6.50e-04	3.25e-04	-	-
2^{-20}	5.17e-03	2.59e-03	1.30e-03	6.50e-04	3.25e-04	-	-
2^{-24}	5.17e-03	2.59e-03	1.30e-03	6.50e-04	3.25e-04	-	-
2^{-28}	5.17e-03	2.59e-03	1.30e-03	6.50e-04	3.25e-04	-	-
2^{-32}	5.17e-03	2.59e-03	1.30e-03	6.50e-04	3.25e-04	-	-
E^N	5.17e-03	2.59e-03	1.30e-03	6.50e-04	3.25e-04	-	-

In addition, Figure 5 shows the maximum absolute error for various ϵ values. The rate of convergence table is also presented in Table 3.

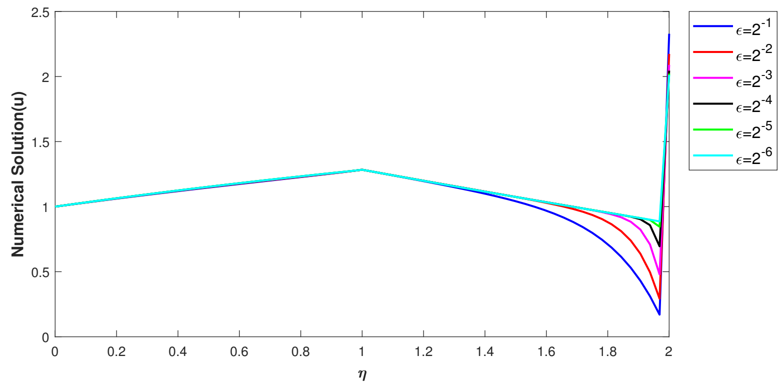

FIGURE 1. The numerical solution of Example 7.1 for different ϵ values.

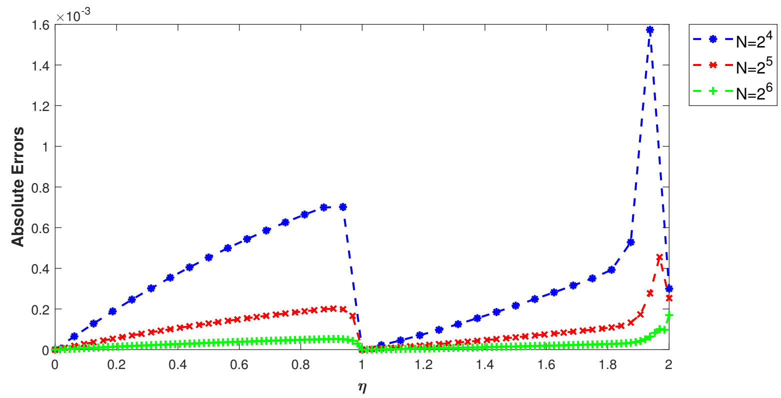

FIGURE 2. The point-wise absolute errors of Example 7.1 for different values of N.

TABLE 2. Rate of convergence ρ of Example 7.1 for $\epsilon = 2^{-6}$.

h	$\frac{h}{2}$	E_h	$\frac{h}{4}$	$E_{\frac{h}{2}}$	ρ
1/16	1/32	2.0629e-03	1/64	8.0068e-04	1.3778
1/32	1/64	8.0068e-04	1/128	2.3016e-04	1.8036
1/64	1/128	2.3016e-04	1/256	5.1075e-05	2.1711

TABLE 3. Rate of convergence ρ of Example 7.2 for $\epsilon = 2^{-6}$.

h	$\frac{h}{2}$	E_h	$\frac{h}{4}$	$E_{\frac{h}{2}}$	ρ
1/16	1/32	8.1678e-02	1/64	2.6983e-02	1.5979
1/32	1/64	2.6983e-02	1/128	7.6138e-03	1.8253
1/64	1/128	7.6138e-03	1/256	3.0963e-03	1.4787

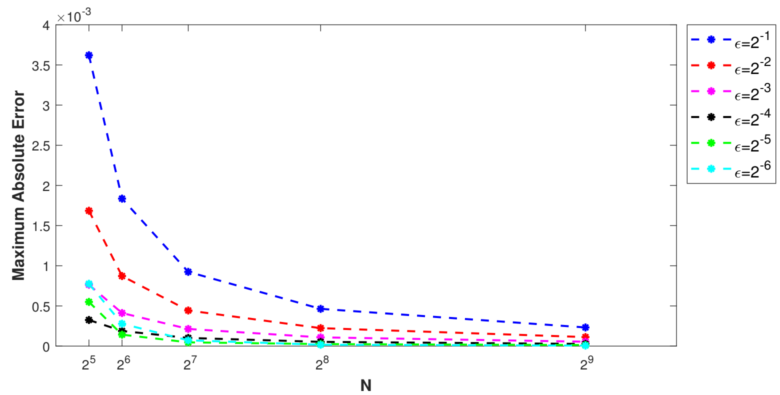

FIGURE 3. The maximum absolute error of Example 7.1 for different ϵ values.

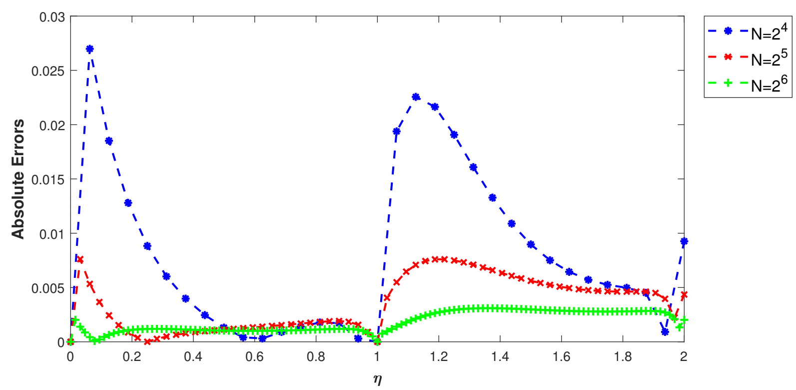

FIGURE 4. The point-wise absolute errors of Example 7.2 for different values of N.

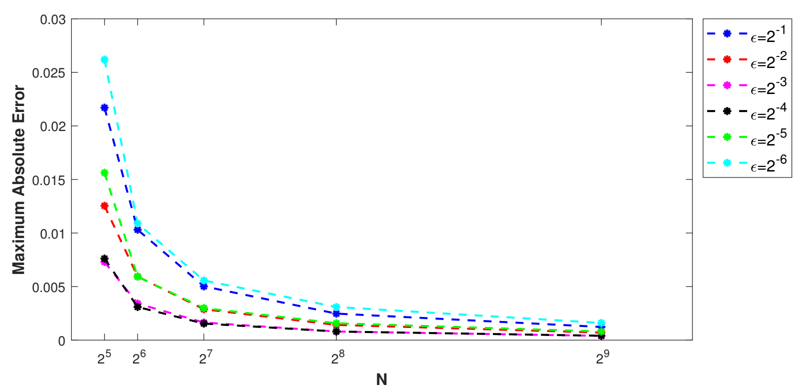

FIGURE 5. The maximum absolute error of Example 7.2 for different ϵ values.

Example 7.3. Consider

$$-\epsilon z''(u) + 3z'(u) + z(u) - z(u-1) = 1, \;\; u \in (0,2),$$

with,

$$z(u) \;\; = \;\; 0, \;\; u \in [-1,0],$$

Table 4. The maximum absolute error of Example 7.2 for different values of ϵ.

ϵ	N						
	32	64	128	256	512	1024	2048
2^{-4}	7.61e-03	3.10e-03	1.51e-03	8.03e-04	4.09e-04	2.06e-04	1.03e-04
2^{-8}	5.04e-02	1.81e-02	1.19e-03	1.94e-03	5.86e-03	3.19e-03	1.64e-03
2^{-12}	5.42e-02	2.91e-02	1.51e-02	7.39e-03	3.19e-03	8.93e-03	9.60e-03
2^{-16}	5.42e-02	2.91e-02	1.51e-02	7.61e-03	3.95e-03	1.99e-03	9.98e-04
2^{-20}	5.42e-02	2.91e-02	1.51e-02	7.61e-03	3.95e-03	1.99e-03	9.98e-04
2^{-24}	5.42e-02	2.91e-02	1.51e-02	7.61e-03	3.95e-03	1.99e-03	9.98e-04
2^{-28}	5.42e-02	2.91e-02	1.51e-02	7.61e-03	3.95e-03	1.99e-03	9.98e-04
2^{-32}	5.42e-02	2.91e-02	1.51e-02	7.61e-03	3.95e-03	1.99e-03	9.98e-04
E^N	5.42e-02	2.91e-02	1.51e-02	7.61e-03	3.95e-03	1.99e-03	9.98e-04
CPU time	0.1094 s	0.1406 s	0.2500 s	0.5000 s	1.4375 s	7.7344 s	36.4375 s
Results in [17]							
2^{-4}	6.65e-01	1.76e-01	3.55e-02	7.70e-03	1.55e-03	8.75e-04	4.24e-04
2^{-8}	8.77e-01	4.15e-01	1.51e-01	4.19e-02	1.27e-02	3.25e-03	1.07e-03
2^{-12}	8.80e-01	4.11e-01	1.51e-01	4.36e-02	1.25e-02	4.63e-03	2.17e-03
2^{-16}	8.80e-01	4.11e-01	1.51e-01	4.39e-02	1.29e-02	3.85e-03	1.15e-03
2^{-20}	8.80e-01	4.11e-01	1.51e-01	4.39e-02	1.29e-02	3.88e-03	1.15e-03
2^{-24}	8.80e-01	4.11e-01	1.51e-01	4.39e-02	1.29e-02	3.89e-03	1.17e-03
2^{-28}	8.80e-01	4.11e-01	1.51e-01	4.39e-02	1.29e-02	3.89e-03	1.17e-03
2^{-32}	8.80e-01	4.11e-01	1.51e-01	4.39e-02	1.29e-02	3.89e-03	1.17e-03
E^N	8.80e-01	4.11e-01	1.51e-01	4.39e-02	1.29e-02	3.89e-03	1.17e-03
Results in [6]							
2^{-4}	1.42e-02	7.86e-03	4.13e-03	2.11e-03	1.07e-03	-	-
2^{-8}	4.41e-02	1.49e-02	4.26e-03	1.97e-03	1.14e-03	-	-
2^{-12}	5.53e-02	2.95e-02	1.52e-02	7.53e-03	3.11e-03	-	-
2^{-16}	5.53e-02	2.95e-02	1.52e-02	7.74e-03	3.90e-03	-	-
2^{-20}	5.53e-02	2.95e-02	1.52e-02	7.74e-03	3.90e-03	-	-
2^{-24}	5.53e-02	2.95e-02	1.52e-02	7.74e-03	3.90e-03	-	-
2^{-28}	5.53e-02	2.95e-02	1.52e-02	7.74e-03	3.90e-03	-	-
2^{-32}	5.53e-02	2.95e-02	1.52e-02	7.74e-03	3.90e-03	-	-
E^N	5.53e-02	2.95e-02	1.52e-02	7.74e-03	3.90e-03	-	-

$$z(2) = 2 + \frac{\epsilon}{3}\int_0^2 uz(u)du.$$

As the perturbation parameter is varied, the Table 5 displays the maximum absolute inaccuracy that can occur. Figure 8 depicts the numerical solution and Figure 9 gives the point-wise absolute errors of this example for different N values. In addition, Figure 7 shows the maximum absolute error for various ϵ values. The rate of convergence table is also presented in Table 6.

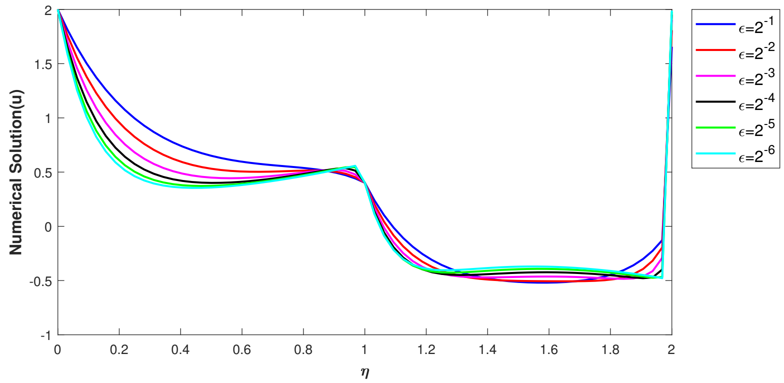

FIGURE 6. The numerical solution of Example 7.2 for different ϵ values.

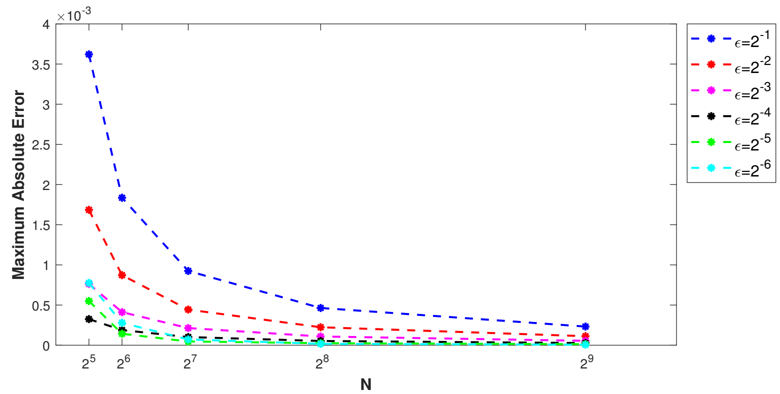

FIGURE 7. The maximum absolute error of Example 7.3 for different ϵ values.

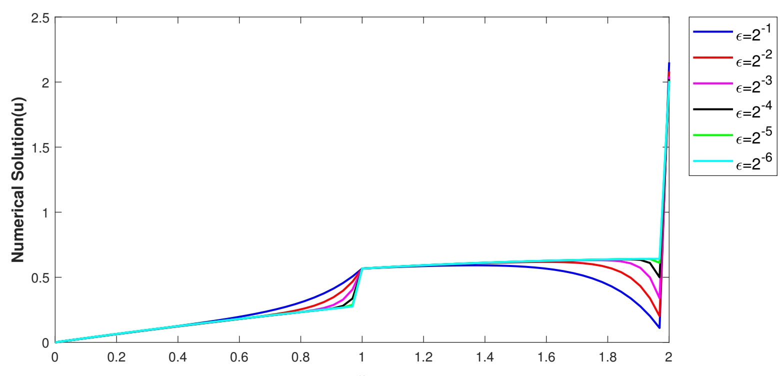

FIGURE 8. The numerical solution of Example 7.3 for different ϵ values.

8. Conclusion

In this study, we use cubic spline in tension method to solve a second order singularly perturbed delay differential equation with integral boundary condition. We applied the current methodology to three examples using different values of ϵ and the resulting computational results are shown in Tables 1, 4, and 5. According to the graphs given

TABLE 5. The maximum absolute error of Example 7.3 for different values of ϵ.

ϵ	N						
	32	64	128	256	512	1024	2048
2^{-4}	3.252e-04	1.882e-04	1.017e-04	5.282e-05	2.691e-05	1.358e-05	6.821e-06
2^{-8}	8.466e-04	4.268e-04	1.976e-04	7.033e-05	1.805e-05	4.324e-06	1.082e-06
2^{-12}	8.466e-04	4.287e-04	2.157e-04	1.082e-04	5.417e-05	2.699e-05	1.012e-05
2^{-16}	8.466e-04	4.287e-04	2.157e-04	1.082e-04	5.417e-05	2.711e-05	1.016e-05
2^{-20}	8.466e-04	4.287e-04	2.157e-04	1.082e-04	5.417e-05	2.711e-05	1.016e-05
2^{-24}	8.466e-04	4.287e-04	2.157e-04	1.082e-04	5.417e-05	2.711e-05	1.016e-05
2^{-28}	8.466e-04	4.287e-04	2.157e-04	1.082e-04	5.417e-05	2.711e-05	1.016e-05
2^{-32}	8.466e-04	4.287e-04	2.157e-04	1.082e-04	5.417e-05	2.711e-05	1.016e-05
E^N	8.466e-04	4.287e-04	2.157e-04	1.082e-04	5.417e-05	2.711e-05	1.016e-05
CPU time	0.1406 s	0.1562 s	0.2031 s	0.5469 s	1.2500 s	6.4375 s	35.5156 s
Results in [27]							
2^{-4}	9.680e-03	3.280e-03	1.105e-03	3.662e-04	1.182e-04	3.726e-05	1.151e-05
2^{-8}	8.763e-03	2.966e-03	9.967e-04	3.302e-04	1.065e-04	3.356e-05	1.036e-05
2^{-12}	8.710e-03	2.949e-03	9.910e-04	3.283e-04	1.059e-04	3.335e-05	1.030e-05
2^{-16}	8.707e-03	2.948e-03	9.906e-04	3.282e-04	1.059e-04	3.335e-05	1.029e-05
2^{-20}	8.707e-03	2.948e-03	9.906e-04	3.282e-04	1.059e-04	3.335e-05	1.029e-05
2^{24}	8.707e-03	2.948e-03	9.906e-04	3.282e-04	1.059e-04	3.335e-05	1.029e-05
2^{-28}	8.707e-03	2.948e-03	9.906e-04	3.282e-04	1.059e-04	3.335e-05	1.029e-05
2^{-32}	8.707e-03	2.948e-03	9.906e-04	3.282e-04	1.059e-04	3.335e-05	1.029e-05
E^N	9.680e-03	3.280e-03	1.105e-03	3.662e-04	1.182e-04	3.726e-05	1.151e-05

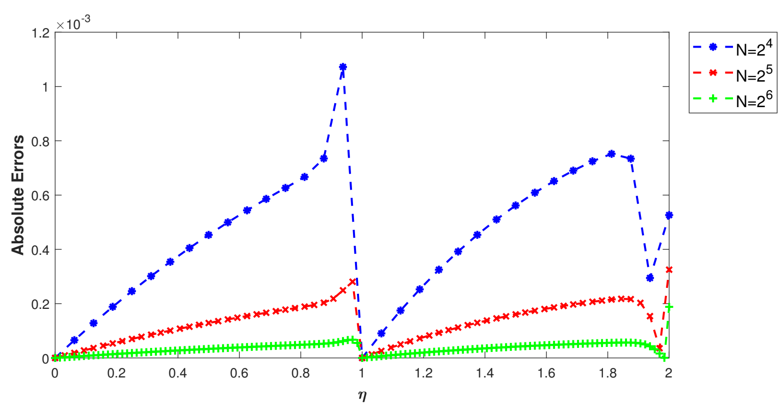

FIGURE 9. The point-wise absolute errors of Example 7.3 for different values of N.

in 3, 5, and 7, it is possible to deduce that the maximum absolute errors go smaller as the grid size h gets smaller, which demonstrates the convergence to the computed solution. The rate of convergence tables are also presented in Tables 2, 3, and 6. Our results are compared to those of previously developed numerical methods found in published works [6, 17, 24, 27]. The suggested approach yields more precise, consistent and convergent numerical results.

Table 6. Rate of convergence ρ of Example 7.3 for $\epsilon = 2^{-6}$

h	$\frac{h}{2}$	E_h	$\frac{h}{4}$	$E_{\frac{h}{2}}$	ρ
1/16	1/32	7.75834-04	1/64	2.7864e-04	1.4773
1/32	1/64	2.7864e-04	1/128	7.1786e-05	1.9566
1/64	1/128	7.1786e-05	1/256	1.7195e-05	2.0617

9. Future Recommendations

The approach we used in addressing singularly perturbed large delay differential equations with integral boundary conditions can be extended to partial differential equation problems as well as the problems with delay and advanced parameters. These equations also find applications in various fields such as chemical engineering, mechanical systems, neuroscience, environmental sciences, climate modelling and so on. This also helps in modelling biochemical reactions with delays, such as gene expression and they also pay a major role in designing control systems with delayed feedback for stability analysis and examining systems with long communication delays in the networked control. By using this work, analyzing communication networks with propagation delays and studying signal processing in networks with large delays can also be done. These applications demonstrate the versatility of SPDDEs in capturing complex phenomena across diverse domains.

Acknowledgment

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