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Existence and Uniqueness Theorems for Fractional Differential Equations with Proportional Delay

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Abstract

In this paper, we applied successive approximation method (SAM) to deal with the solution of non-linear differential equations (DEs) with proportional delay. Utilizing SAM we derived the results about existence and uniqueness. The differential equations (DEs) with proportional delay are a particular case of the time-dependent delay differential equations (DDEs). In this sense, we demonstrated that the equilibrium solution of time-dependent DDEs is asymptotically stable on finite time intervals. We obtained a series solution of pantograph and Ambartsumian equations and proved its convergence. Further, we proved that the zero solution of pantograph and Ambartsumian equations are asymptotically stable. The outcomes of integer order obtained for DEs with proportional delay and time-dependent DDEs have been extended to initial value problem (IVP) for fractional DDEs and a system of fractional DDEs involving Caputo fractional derivative. Finally, we illustrate the efficacy of the SAM by considering particular non-linear DEs with proportional delay. The results obtained for non-linear DEs with proportional delay by SAM are compared with exact solutions and other iterative methods. It is noted that SAM is easier to use than other techniques and the solutions obtained using SAM are consistent with the exact solution.

Keywords. Successive approximation method, Lipschitz condition, Caputo derivative, Existence-uniqueness, Proportional delay, Pantograph equation, Ambartsumian equation.

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1. Introduction

The delay differential equations (DDE) contain the state variable term at a past time $t-\tau$. The inclusion of the delay τ makes the DDE an infinite dimensional dynamical system. Even if it is very difficult to analyze and solve such equations, this branch is popular among the applied scientists due to the applications in various fields.

On the other hand, if the order of the derivative in a differential equation is any arbitrary number (instead of a positive integer) then the equation is called as the fractional differential equation (FDE). Even though there are several inequivalent definitions of fractional derivative operator, one can select the derivative which is appropriate for the model under consideration. This flexibility is a key feature behind the popularity of fractional calculus.

Daftardar-Gejji and coworkers proposed numerical schemes [5, 15] for solving fractional order delay differential equations (FDDE). Modified Laguerre wavelets method [18], spectral collocation method [1], fractional-order fibonacci-hybrid functions [30] are few other methods for solving FDDEs. Stability analysis of FDDEs is proposed in [6–8, 20]. Applications of FDDE are presented in [9, 21, 28, 29].

In general, the delay τ in the DDE $x'(t) = f(t, x(t), x(t-\tau))$ is not constant. The analysis becomes more difficult when τ depends on time or state. The proportional delay differential equation $x'(t) = f(t, x(t), x(qt))$ or a pantograph equation is a particular case of time-dependent DDE with $\tau(t) = (1-q)t$. These equations are proposed by Ockendon and Tayler in the seminal work [22] to model the motion of an overhead trolley wire. Few other applications of these equations are discussed in [10, 12]. The Daftardar-Gejji and Jafari method (DJM) is applied in [11] to find analytical

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solutions of pantograph equation. Further, the authors presented the various relations of the solution series with the existing special functions. Patade and Bhalekar proposed the power series solution Ambartsumian equation [23] by using DJM. The analytical solution of pantograph equation are discussed in [24]. Vidhyaa *et al.* [34] have obtained oscillation conditions for non-canonical second-order nonlinear delay difference equations with a super linear neutral term. Asymptotic behavior of third order delay difference equations with a negative middle term are discussed in [31].

Solving nonlinear FDEs with proportional delay is an important task in mathematical analysis and applications. This motivates us to work on finding solutions of FDEs with proportional delay. In this paper, we derive the existence-uniqueness results for FDEs with proportional delay and find the solutions of FDEs with proportional delay using SAM in terms of power series.

The paper is organized as follows: The basic definition and results given in section 2 and the SAM is discussed in section 2.1. The existence and uniqueness results are described in section 2.3. The stability analysis is presented in section 3. The series solution of the pantograph equation and Ambartsumian equation are described in section 4. The results relevant to FDEs and the system of FDEs are derived in section 5 and section 6. Section 7 and section 8 deals with illustrative example. The conclusions are summarized in section 9.

2. Preliminaries and Notations

Definition 2.1. [19] The Riemann-Liouville fractional integral of order $\alpha > 0$ of $f \in C[0, \infty)$ is defined as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\zeta)^{\alpha-1} f(\zeta) d\zeta, \quad t > 0. \tag{2.1}$$

Definition 2.2. [19] The (left sided) Caputo fractional derivative of $f, f \in C^m_{-1}, m \in \mathbb{N} \cup \{0\}$, is defined as:

$$\begin{aligned} D^{\alpha} f(t) &= \frac{d^m}{dt^m} f(t), \quad \alpha = m, \\ &= I^{m-\alpha} \frac{d^m}{dt^m} f(t), \quad m-1 < \alpha < m, \quad m \in \mathbb{N}. \end{aligned} \tag{2.2}$$

Note that for $0 \le m-1 < \alpha \le m$ and $\beta > -1$

$$\begin{aligned} I^{\alpha} x^{\beta} &= \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha}, \\ \left(I^{\alpha} D^{\alpha} f\right)(t) &= f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) \frac{t^k}{k!}. \end{aligned} \tag{2.3}$$

Definition 2.3. [19] The Mittag-Leffler function is defined as

$$E_{\alpha}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0. \tag{2.4}$$

Definition 2.4. [19] The multi-parameter Mittag-Leffler function is defined as:

$$E_{(\alpha_1, \cdots, \alpha_n), \beta}(z_1, z_2, \cdots, z_n) = \sum_{k=0}^{\infty} \sum_{\substack{l_1 + \cdots + l_n = k \\ l_j \ge 0}} (k; l_1, \cdots, l_n) \left[\frac{\prod_{j=1}^{n} z_j^{l_j}}{\Gamma(\beta + \sum_{j=1}^{n} \alpha_j l_j)} \right].$$

where, $(k; l_1, l_2, \cdots, l_n)$ is the multinomial coefficient defined as

$$(k; l_1, l_2, \cdots, l_n) = \frac{k!}{l_1! l_2! \cdots l_n!}. \tag{2.5}$$

Definition 2.5. [16] Consider the DDE,

$$y'(t) = f(y(t), y(t-\tau(t))), \tag{2.6}$$

where $f : R \times R \to R$. The flow $\phi_t(t_0)$ is a solution $y(t)$ of Eq.(2.6) with initial condition $y(t) = t_0$, $t \leq 0$. The point y^* is called equilibrium solution of Eq. (2.6) if $f(y^*, y^*) = 0$.
(a) If, for any $\epsilon > 0$, there exist $\delta > 0$ such that $|t_0 - y^*| < \delta \Rightarrow |\phi_t(t_0) - y^*| < \epsilon$, then the system Eq. (2.6) is stable (in the Lyapunov sense) at the equilibrium y^*.
(b) If the system (2.6) is stable at y^* and moreover, $\lim_{t\to\infty} |\phi_t(t_0) - y^*| = 0$ then the system (2.6) is said to be asymptotically stable at y^*.

2.1. **Successive Approximation Method (SAM):.** The Successive Approximations Method (SAM) is a familiar classical technique for solving integral equations [13]. SAM has applications in various fields, including physics, engineering, and applied mathematics, especially in problems involving integral equations arising in initial value problems [14, 32, 33].
Consider the differential equation

$$y'(t) = f(t, y(t)), y(0) = y_0, \tag{2.7}$$

Let $\phi_0(t) = y_0$ be the first approximate solution of the IVP (2.7). Then

$$\begin{aligned} \phi_1(t) &= y_0 + \int_0^t f(x, \phi_0(x))dx, \\ \phi_2(t) &= y_0 + \int_0^t f(x, \phi_1(x))dx. \end{aligned}$$

Continuing in this way, we obtain

$$\phi_{k+1}(t) = y_0 + \int_0^t f(x, \phi_k(x))dx, \quad k = 0, 1, 2, \cdots. \tag{2.8}$$

2.2. **SAM for Differential Equations with Proportional Delay:** Consider the differential equations with proportional delay

$$y'(t) = f(t, y(t), y(qt)), y(0) = y_0, 0 < q < 1, \tag{2.9}$$

where f is a continuous function defined on some 3-dimensional rectangle

$$R = \{|t| \leq a, |y(t) - y_0| \leq b, |y(qt) - y_0| \leq b, a > 0, \ b > 0\}.$$

Let $\phi_0(t) = y_0$ be the first approximate solution of the IVP (2.9). Then

$$\begin{aligned} \phi_1(t) &= y_0 + \int_0^t f(x, \phi_0(x), \phi_0(qx))dx. \\ \phi_2(t) &= y_0 + \int_0^t f(x, \phi_1(x), \phi_1(qx))dx. \end{aligned}$$

Continuing in this way, we obtain

$$\phi_{k+1}(t) = y_0 + \int_0^t f(x, \phi_k(x), \phi_k(qx))dx. \quad k = 0, 1, 2, \cdots. \tag{2.10}$$

2.3. **Existence and Uniqueness Results.**

Theorem 2.6. *A function ϕ is a solution of the IVP (2.9) on an interval I if and only if it is a solution of the integral equation*

$$y(t) = y_0 + \int_0^t f(x, y(x), y(qx))dx, \quad on \quad I. \tag{2.11}$$

Proof. Let ϕ is a solution of the IVP (2.9) on an interval I. Then

$$\phi'(t) = f(t, \phi(t), \phi(qt)), \phi(0) = y_0, 0 < q < 1 \tag{2.12}$$

The equivalent integral Equation (2.12) is

$$\phi(t) = \phi(0) + \int_0^t f(x, \phi(x), \phi(qx))dx. \tag{2.13}$$

and $\phi(0) = y_0$. Thus ϕ is a solution of the IVP (2.11).
Conversely, suppose Equation (2.13) hold. Differentiate Equation (2.13) w.r.t. t, we get

$$\phi'(t) = f(t, \phi(t), \phi(qt)), 0 < q < 1 \quad \forall t \in I.$$

From Equation (2.11) $\phi(0) = y_0$.
Hence ϕ is a solution of the IVP (2.9). □

Theorem 2.7. *Let f is continuous and $|f| \leq M$ on R. The successive approximation (2.10) exist and continuous on the interval $I = [-\zeta, \zeta]$, where $\zeta = min\{a, \frac{b}{M}\}$. If $t \in I$ then $(t, y(t), y(qt)) \in R$ and $|\phi_k(t) - y_0| \leq M|t|$, $|\phi_k(qt) - y_0| \leq M|t|$.*

Proof. We prove the result by mathematical induction.
(i) Clearly $\phi(0) = y_0$ is continuous on I. Thus, theorem is true for $k = 0$.
(ii) For $k = 1$, we have

$$\begin{aligned} \phi_1(t) &= y_0 + \int_0^t f(x, \phi_0(x), \phi_0(qx))dx, \\ \phi_1(t) &= y_0 + \int_0^t f(x, y_0, y_0)dx. \end{aligned}$$

Since f is continuous and hence, $\phi_1(t)$ exist.

$$\begin{aligned} |\phi_1(t) - y_0| &= |\int_0^t f(x, \phi_0(x), \phi_0(qx))dx| \\ &\leq \int_0^t |f(x, \phi_0(x), \phi_0(qx))|dx \\ &\leq M|t| \\ &\leq b, \quad t \in I \\ \text{and} \quad |\phi_1(qt) - y_0| &\leq M|qt| \\ &\leq M|t|, \quad 0 < q < 1 \\ &\leq b, \quad t \in I \end{aligned}$$

Thus, for $t \in I$, $(t, y(t), y(qt)) \in R$ and $|\phi_1(t) - y_0| \leq M|t|$, $|\phi_1(qt) - y_0| \leq M|t|$.
The theorem is true for $k = 1$
(iii) Assume that theorem is true for $k = n$.
i.e. For $t \in I$, $(t, y(t), y(qt)) \in R$ and $|\phi_n(t) - y_0| \leq M|t|$, $|\phi_n(qt) - y_0| \leq M|t|$.

(iv) To prove the theorem for $k = n + 1$.
If $t \in I$, then

$$\phi_{n+1}(t) \quad = \quad y_0 + \int_0^t f(x, \phi_n(x), \phi_n(qx))dx.$$

Since f is continuous and hence, $\phi_{n+1}(t)$ exist on I.

$$\begin{aligned} |\phi_{n+1}(t) - y_0| &\leq M|t| \\ &\leq b, \quad t \in I \\ \text{and} \quad |\phi_{n+1}(qt) - y_0| &\leq M|qt| \\ &\leq M|t|, \quad 0 < q < 1 \\ &\leq b, \quad t \in I \end{aligned}$$

Thus, if $t \in I$, $(t, y(t), y(qt)) \in R$ and $|\phi_{n+1}(t) - y_0| \leq M|t|$, $|\phi_{n+1}(qt) - y_0| \leq M|t|$.
Hence by mathematical induction, the result is true for all positive integer n. □

Theorem 2.8. *(Existence Theorem) Let f is continuous and $|f| \leq M$ on the 3-dimensional rectangle*

$$R = \{|t| \leq a, |y(t) - y_0| \leq b, |y(qt) - y_0| \leq b, a > 0,\ b > 0\}.$$

Suppose f satisfies Lipschitz condition in second and third variable with Lipschitz constants L_1 and L_2 such that

$$|f(t, y_1(t), y_1(qt)) - f(t, y_2(t), y_2(qt))| \leq L_1|y_1(t) - y_2(t)| + L_2|y_1(qt) - y_2(qt)|.$$

Then the successive approximations (2.10) converges on the interval $I = [-\zeta, \zeta]$, where $\zeta = min\left\{a, \frac{b}{M}\right\}$ to a solution ϕ of the IVP (2.9) on I.

Proof. We have

$$\phi_k(t) = \phi_0(t) + \sum_{n=1}^{k}[\phi_n(t) - \phi_{n-1}(t)].$$

To prove the sequence $\{\phi_k\}$ converges, it is enough to prove the series

$$\phi_0(t) + \sum_{n=1}^{\infty}[\phi_n(t) - \phi_{n-1}(t)] \tag{2.14}$$

is convergent.
By Theorem 2.7 the function ϕ_k all exist and continuous on I.
Also, $|\phi_1(t) - \phi_0(t)| \leq M|t|$ and $|\phi_1(qt) - \phi_0(qt)| \leq M|t|$ for $t \in I$.
Now,

$$\begin{aligned} \phi_2(t) - \phi_1(t) &= \int_0^t [f(x, \phi_1(x), \phi_1(qx)) - f(x, \phi_0(x), \phi_0(qx))]dx \\ \therefore |\phi_2(t) - \phi_1(t)| &\leq \int_0^t |f(x, \phi_1(x), \phi_1(qx)) - f(x, \phi_0(x), \phi_0(qx))|dx \\ &\leq \int_0^t [L_1|\phi_1(x) - \phi_0(x)| + L_2|\phi_1(qx) - \phi_0(qx)|]dx \\ &\leq M(L_1 + L_2)\frac{|t|^2}{2}. \end{aligned}$$

We shall prove by mathematical induction

$$|\phi_n(t) - \phi_{n-1}(t)| \leq M(L_1 + L_2)^{n-1}\frac{|t|^n}{n!} \tag{2.15}$$

We have prove that Equation (2.15) true for $n = 1, 2$.
Assume that (2.15) true for $n = m$.
We have

$$\begin{aligned} \phi_{m+1}(t) - \phi_m(t) &= \int_0^t [f(x, \phi_m(x), \phi_m(qx)) - f(x, \phi_{m-1}(x), \phi_{m-1}(qx))]dx \\ \therefore |\phi_{m+1}(t) - \phi_m(t)| &\leq \int_0^t |f(x, \phi_m(x), \phi_m(qx)) - f(x, \phi_{m-1}(x), \phi_{m-1}(qx)|dx \\ &\leq \int_0^t [L_1|\phi_m(x) - \phi_{m-1}(x)| + L_2|\phi_m(qx) - \phi_{m-1}(qx)|]dx \\ &\leq M(L_1 + L_2)^m \frac{|t|^{m+1}}{(m+1)!}. \end{aligned}$$

Thus, the result is true for $n = m + 1$.
Hence, by the mathematical induction result is true for all $n = 1, 2, \cdots$.
Therefore, the infinite series (2.15) is absolutely convergent on I. This shows that the n^{th} term of the series $|\phi_0(t)| + \sum_{n=1}^{\infty} |\phi_n(t) - \phi_{n-1}(t)|$ is less than $\frac{M}{(L_1+L_2)}$ times the n^{th} term of the power series $e^{(L_1+L_2)|t|}$. Hence The series (2.15) is convergent. □

3. Stability Analysis

The differential equations with proportional delay

$$y'(t) = f\left(t, y(t), y(qt)\right), \tag{3.1}$$

is a special case of the time-dependent delay differential equation (DDE)

$$y'(t) = f\left(t, y(t), y\left(t - \tau(t)\right)\right) \quad \text{with} \quad \tau(t) = (1 - q)t,$$

The following results are similar to those in [16]

Theorem 3.1. *Suppose that the equilibrium solution y^* of the equation*

$$y' = f(y(t), y(t - \tau^*)), \quad \tau^* = \tau(t_0) \tag{3.2}$$

is stable and $\|f(y(t), y(t - \tau(t))) - f(y(t), y(t - \tau(t_1)))\| < \epsilon_1 |t - t_1|$, for some $\epsilon_1 > 0$ and $t, t_1 \in [t_0, t_0 + c)$, c is a positive constant, then there exists $\bar{t} > 0$ such that the equilibrium solution y^* of (2.6) is stable on finite time interval $[t_0, \bar{t})$.

Corollary 3.2. *If the real parts of all roots of $\lambda - a - be^{-\lambda\tau^*} = 0$ are negative, where $a = \partial_1 f, b = \partial_2 f$ evaluated at equilibrium. Then there exist $\epsilon_c, \bar{t}(> t_0)$, such that when $\epsilon_1 < \epsilon_c$, the solution $y^* = 0$ of (2.6) is stable on finite time interval $[t_0, \bar{t})$.*

4. Series Solution of Pantograph Equation

A pantograph is a device used in electric trains to collect current from overloaded lines. The pantograph equation was formulated by Ockendon and Taylor in 1971 and originates in electrodynamics [22].
Consider the pantograph equation,

$$y'(t) = ay(t) + by(qt), \quad y(0) = 1, \tag{4.1}$$

where $0 < q < 1$, $a, b \in R$. Integrating (4.1), we get

$$y(t) = 1 + \int_0^t \left(ay(x) + by(qx)\right) dt \tag{4.2}$$

Suppose $\phi_k(t)$ be the k^{th} approximate solution, where the initial approximate solution is taken as

$$\phi_0(t) = 1. \tag{4.3}$$

For $k \geq 1$, the recurrent formula as below:

$$\phi_k(t) = 1 + \int_0^t (a\phi_{k-1}(x) + b\phi_{k-1}(qx))\, dx. \tag{4.4}$$

From the recurrent formula, we have

$$\begin{aligned}
\phi_1(t) &= 1 + \int_0^t (a\phi_0(x) + by_0(qx))\, dx \\
&= 1 + (a+b)\frac{t}{1!}, \\
\phi_2(t) &= 1 + \int_0^t (a\phi_1(x) + b\phi_1(qx))\, dx \\
&= 1 + (a+b)\frac{t}{1!} + (a+b)(a+bq)\frac{t^2}{2!}, \\
\phi_3(t) &= 1 + \int_0^t (a\phi_2(x) + b\phi_2(qx))\, dx \\
&= 1 + (a+b)\frac{t}{1!} + (a+b)(a+bq)\frac{t^2}{2!} + (a+b)(a+bq)(a+bq^2)\frac{t^3}{3!}, \\
&\vdots \\
\phi_k(x) &= 1 + \frac{t^k}{k!}\prod_{j=0}^{k-1}\left(a + bq^j\right), \quad k = 1, 2, 3 \cdots . \\
&\text{As} \quad k \to \infty, \quad \phi_k(t) \to y(t) \\
y(t) &= 1 + \sum_{m=1}^{\infty} \frac{t^m}{m!}\prod_{j=0}^{m-1}\left(a + bq^j\right).
\end{aligned}$$

If we define $\prod_{j=0}^{m-1}\left(a + bq^j\right) = 1$, for $m = 0$, then

$$y(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!}\prod_{j=0}^{m-1}\left(a + bq^j\right). \tag{4.5}$$

Theorem 4.1. *For $0 < q < 1$, the power series (4.5) is convergent for $t \in R$.*

Corollary 4.2. *The power series (4.5) is absolutely convergent for all t and hence it is uniformly convergent on any compact interval on R.*

Theorem 4.3. *If $0 < q < 1$, $a, b \geq 0$, then*

$$e^{at} \leq y(t) = \sum_{m=0}^{\infty} \frac{t^m}{m!}\prod_{j=0}^{m-1}\left(a + bq^j\right) \leq e^{(a+b+c)t}, \quad 0 \leq t < \infty.$$

Theorem 4.4. *If $(a + b) < 0$ then zero solution of Eq. (4.1) is asymptotically stable.*

Proof.

$$\begin{aligned}
\text{Define} \qquad u(t) &= \max_{0\le x\le t} y^2(t) \\
\therefore \frac{1}{2}u'(t) &= \frac{1}{2}\frac{d}{dt}(y^2(t)) \\
&= y(t)y'(t) \\
&= y(t)(ay(t)+by(qt)) \\
&= ay^2(t)+by(t)y(qt) \\
&\le (a+b)u(t) \\
\Rightarrow u(t) &\le u(0)e^{2(a+b)t} \\
\therefore \lim_{x\to\infty} y(t) &= 0, \quad \text{if} \quad (a+b)<0.
\end{aligned}$$

□

4.1. **Series Solution of Ambartsumian Equation.** In [3] Ambartsumian derived a delay differential equation describing the fluctuations of the surface brightness in a milky way. The equation is described as:

$$y'(t) = -y(t) + \frac{1}{q}y\left(\frac{t}{q}\right), \tag{4.6}$$

where $q > 1$ and is constant for the given model.
The (4.6) with initial condition $y(0) = \lambda$ can be written equivalently as

$$y(t) = \lambda + \int_0^t \left(\frac{1}{q}y\left(\frac{x}{q}\right) - y(x)\right) dx. \tag{4.7}$$

Suppose $\phi_k(t)$ be the k^{th} approximate solution, where the initial approximate solution is taken as

$$\phi_0(t) = \lambda. \tag{4.8}$$

For $k \ge 1$, the recurrent formula as below:

$$\phi_k(t) = \lambda + \int_0^t \left(\frac{1}{q}\phi_{k-1}\left(\frac{x}{q}\right) - \phi_{k-1}(x)\right) dx. \tag{4.9}$$

From the recurrent formula, we have

$$\begin{aligned}
\phi_1(t) &= \lambda + \int_0^t \left(\frac{1}{q}\phi_0\left(\frac{x}{q}\right) - \phi_0(x)\right) dx \\
&= \lambda + \int_0^t \left(\frac{\lambda}{q} - \lambda\right) dx \\
&= \lambda + \left(\frac{\lambda}{q} - \lambda\right)\frac{t}{1!} \\
&= \left(1 + \left(\frac{1}{q} - 1\right)\frac{t}{1!}\right)\lambda, \\
\phi_2(t) &= \lambda + \int_0^t \left(\frac{1}{q}\phi_1\left(\frac{x}{q}\right) - \phi_1(x)\right) dx \\
&= \left(1 + \left(\frac{1}{q} - 1\right)\frac{t}{1!} + \left(\frac{1}{q} - 1\right)\left(\frac{1}{q^2} - 1\right)\frac{t^2}{2!}\right)\lambda, \\
&\vdots \\
\phi_k(t) &= \left(1 + \sum_{m=1}^{k} \frac{t^m}{m!}\prod_{j=1}^{m}\left(\frac{1}{q^j} - 1\right)\right)\lambda. \\
&\text{As} \quad k \to \infty, \quad \phi_k(t) \to y(t) \\
y(t) &= \left(1 + \sum_{m=1}^{\infty} \frac{t^m}{m!}\prod_{j=1}^{m}\left(\frac{1}{q^j} - 1\right)\right)\lambda.
\end{aligned}$$

If we define $\prod_{j=1}^{m}\left(\frac{1}{q^j} - 1\right) = 1$, for $m = 0$, then

$$y(t) = \left(\sum_{m=0}^{\infty} \frac{t^m}{m!}\prod_{j=1}^{m}\left(\frac{1}{q^j} - 1\right)\right)\lambda. \tag{4.10}$$

Theorem 4.5. *For $q > 1$, the power series (4.10) is convergent for $t \in R$.*

Corollary 4.6. *The power series (4.10) is absolutely convergent for all t and hence it is uniformly convergent on any compact interval on R.*

Theorem 4.7. *The zero solution of (4.6) is asymptotically stable.*

5. Fractional order differential equations with proportional delay

Consider the initial value problem (IVP)

$$\begin{aligned}
D^\alpha y(t) &= f(t, y(t), y(qt)), 0 < \alpha \le 1, 0 < q < 1 \\
y(0) &= y_0,
\end{aligned} \tag{5.1}$$

where D^α denotes Caputo fractional derivative and f is a continuous function defined on the 3-dimensional rectangle

$$R = \{|t| \le a, |y(t) - y_0| \le b, |y(qt) - y_0| \le b, a > 0, b > 0\}.$$

Theorem 5.1. *A function ϕ is a solution of the IVP (5.1) on an interval I if and only if it is a solution of the integral equation*

$$y(t) = y_0 + \int_0^t \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)} f(x, y(x), y(qx))dx \quad \textit{on} \quad I. \tag{5.2}$$

Theorem 5.2. *Let f is continuous and $|f| \leq M$ on R. The successive approximation*

$$\begin{aligned} \phi_{k+1}(t) &= y_0 \\ \phi_{k+1}(t) &= y_0 + \int_0^t \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)} f(x, \phi_k(x), \phi_k(qx))dx. \quad k = 0, 1, 2, \cdots . \end{aligned} \tag{5.3}$$

exist and continuous on the interval $I = [-\zeta, \zeta]$, where $\zeta = min\left\{a, (\frac{\Gamma(\alpha+1)b}{M})^{\frac{1}{\alpha}}\right\}$. If $t \in I$ then $(t, y(t), y(qt)) \in R$ and $|\phi_k(t) - y_0| \leq M\frac{|t|^\alpha}{\Gamma(\alpha+1)}$, $|\phi_k(qt) - y_0| \leq M\frac{|t|^\alpha}{\Gamma(\alpha+1)}$.

Theorem 5.3. *(Existence Theorem) Let f is continuous and $|f| \leq M$ on the 3-dimensional rectangle*

$$R = \{|t| \leq a, |y(t) - y_0| \leq b, |y(qt) - y_0| \leq b, a > 0,\ b > 0\}.$$

Suppose f satisfies Lipschitz condition in second and third variable with Lipschitz constants L_1 and L_2 such that

$$|f(t, y_1(t), y_1(qt)) - f(t, y_2(t), y_2(qt))| \leq L_1|y_1(t) - y_2(t)| + L_1|y_1(qt) - y_2(qt)|.$$

Then the successive approximations (5.3) converges on the interval $I = [-\zeta, \zeta]$, where $\zeta = min\left\{a, (\frac{\Gamma(\alpha+1)b}{M})^{\frac{1}{\alpha}}\right\}$ to a solution ϕ of the IVP (5.1) on I.

5.1. **Series Solution of Fractional Order Pantograph Equation.** Consider the fractional order pantograph equation as :

$$D^\alpha y(t) = ay(t) + by(qt), \quad y(0) = 1, \tag{5.4}$$

where $0 < \alpha \leq 1$, $0 < q < 1$, $a, b \in R$.

The solution of (5.4) using successive approximation is

$$y(t) = \sum_{m=0}^{\infty} \frac{t^{\alpha m}}{\Gamma(\alpha m + 1)} \prod_{j=0}^{m-1} \left(a + bq^{\alpha j}\right). \tag{5.5}$$

Theorem 5.4. *If $0 < q < 1$, then the power series (5.5) is convergent for all finite values of t.*

Theorem 5.5. *If $0 < q < 1$, $a, b \geq 0$, then*

$$E_\alpha(at^\alpha) \leq y(t) = \sum_{m=0}^{\infty} \frac{t^{\alpha m}}{\Gamma(\alpha m + 1)} \prod_{j=0}^{m-1} \left(a + bq^{\alpha j}\right) \leq E_\alpha((a+b)t^\alpha), \quad 0 \leq t < \infty.$$

5.2. **Series Solution of Fractional Order Ambartsumian Equation.** Consider the fractional order Ambartsumian equation as:

$$D^\alpha y(t) = -y(t) + \frac{1}{q} y\left(\frac{t}{q}\right), \quad y(0) = 1 \tag{5.6}$$

where $q > 1$ and is constant for the given model.

The solution of (5.6) using successive approximation is

$$y(t) = \sum_{m=0}^{\infty} \frac{t^{\alpha m}}{\Gamma(\alpha m + 1)} \prod_{j=0}^{m-1} \left(\frac{1}{q^{1+\alpha j}} - 1\right). \tag{5.7}$$

Theorem 5.6. *If $q > 1$, then the power series (5.7) is convergent for all finite values of t.*

6. System of fractional order differential equations with proportional delay

Consider the initial value problem (IVP)

$$\begin{aligned} D^{\alpha_i} y_i(t) &= f_i(t, \bar{y}(t), \bar{y}(qt)), 0 < \alpha_i \leq 1, 0 < q < 1 \\ y_i(0) &= {}^i y_0, \quad 1 \leq i \leq n, \end{aligned} \tag{6.1}$$

where D^{α_i} denotes Caputo fractional derivative, $\bar{y}(t) = (y_1(t), y_2(t) \cdots, y_n(t))$, $\bar{y}(qt) = (y_1(qt), y_2(qt) \cdots, y_n(qt))$ and $f = (f_1, f_2 \cdots, f_n)$ is a continuous function defined on the $(2n+1)$ dimensional rectangle

$$R = \{|t| \leq a, |y_i(t) -^i y_0| \leq b_i, |y_i(qt) -^i y_0| \leq b_i, a > 0,\ b_i > 0, 1 \leq i \leq n\}.$$

Theorem 6.1. *A function $\bar{\phi}$ is a solution of the IVP (6.1) on an interval I if and only if it is a solution of the integral equation*

$$y_i(t) =^i y_0 + \int_0^t \frac{(t-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f(x, \bar{y}(x), \bar{y}(qx)) dx \quad on \quad I, \tag{6.2}$$

where $\bar{\phi}_m = \left({}^1\phi_m, {}^2\phi_m, \cdots, {}^n\phi_m\right)$

Theorem 6.2. *Let $||f|| = M$ on rectangle R. The successive approximation*

$$\begin{aligned} {}^i\phi_0(t) &= {}^i y_0, \quad i = 0, 1, 2, \cdots. \\ {}^i\phi_{k+1}(t) &= y_0 + \int_0^t \frac{(t-x)^{\alpha_i - 1}}{\Gamma(\alpha_i)} f(x, \bar{\phi}_k(x), \bar{\phi}_k(qx)) dx. \quad k = 0, 1, 2, \cdots. \end{aligned} \tag{6.3}$$

exist and continuous on the interval $I = [-\zeta, \zeta]$, where

$$\zeta = min \left\{ a, \left(\frac{\Gamma(\alpha_1 + 1) b_1}{M} \right)^{\frac{1}{\alpha_1}}, \cdots, \left(\frac{\Gamma(\alpha_n + 1) b_n}{M} \right)^{\frac{1}{\alpha_n}} \right\}.$$

If t is in interval I then $(t, \bar{y}_m(t), \bar{y}_m(qt))$ is in rectangle R and $||\bar{y}_m(t) - \bar{y}(0)|| \leq M \sum_{i=1}^m \frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)}$, $||\bar{y}_m(qt) - \bar{y}(0)|| \leq M \sum_{i=1}^m \frac{|t|^{\alpha_i}}{\Gamma(\alpha_i+1)}$ $\forall m$.

Theorem 6.3. *Let f be a continuous function defined on the rectangle*

$$R = \{|t| \leq a, |y_i(t) -^i y_0| \leq b_i, |y_i(qt) -^i y_0| \leq b_i, a > 0,\ b_i > 0, 1 \leq i \leq n\}.$$

Suppose f satisfies Lipschitz condition in second and third variable with Lipschitz constants L_1 and L_2 such that $|f(t, \bar{y}(t), \bar{y}(qt)) - f(t, \bar{y}(t), \bar{y}(qt))| \leq L_1 |\bar{y}_1(t) - \bar{y}_2(t)| + L_1 |\bar{y}_1(qt) - \bar{y}_2(qt)|$. Then the successive approximations (6.3) converges on the interval $I = [-\zeta, \zeta]$, where $\zeta = min \left\{ a, \left(\frac{\Gamma(\alpha_1+1) b_1}{M} \right)^{\frac{1}{\alpha_1}}, \cdots, \left(\frac{\Gamma(\alpha_n+1) b_n}{M} \right)^{\frac{1}{\alpha_n}} \right\}$ to a solution of the ϕ of the IVP (6.1) on I.

6.1. System of Fractional Order Pantograph Equation.

Consider the system of fractional order pantograph equation

$$D^\alpha y(t) = Ay(t) + By(qt), \quad y(0) = y_0, \quad 0 < \alpha \leq 1 \tag{6.4}$$

where $0 < q < 1$, $A = (a_{ij})_{n \times n}$, $B = (b_{ij})_{n \times n}$ and $y = [y_1, y_2, \cdots, y_n]^T$.
The solution of (4.4) using successive approximation is

$$y(t) = \left[\sum_{k=0}^{\infty} \prod_{j=1}^{k} (A + Bq^{-(k-j)\alpha}) \frac{t^{k\alpha}}{\Gamma(k\alpha + 1)} \right] \lambda. \tag{6.5}$$

Theorem 6.4. *For $0 < q < 1$, the power series (6.5) is convergent for $t \in R$.*

6.2. **System of Fractional Order Ambartsumian Equations.** In this section, we generalize the Ambartsumian Equation (2.9) to the system of fractional order Ambartsumian equations [25] as:

$$D^{\alpha}y(t) = -Iy(t) + By\left(\frac{t}{q}\right), \quad y(0) = \lambda, \quad 0 < \alpha \leq 1, \tag{6.6}$$

where D^{α} denotes Caputo fractional derivative, I is the identity matrix of order n, $1 < q$,

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \ \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} \text{ and } B = \begin{bmatrix} \frac{1}{q} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{q} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{q} \end{bmatrix}_{n\times n}.$$

Applying SAM to the initial value problem (6.6), we have

$$y(t) = y(0) - IJ^{\alpha}y(t) + BJ^{\alpha}y\left(\frac{t}{q}\right). \tag{6.7}$$

Suppose $\phi_k(t)$ be the kth approximate solution, where the initial approximate solution is taken as

$$\phi_0(t) = \lambda. \tag{6.8}$$

For $k \geq 1$, the recurrent formula as below:

$$\phi_k(t) = \lambda - IJ^{\alpha}\phi_{k-1}(t) + BJ^{\alpha}\phi_{k-1}\left(\frac{t}{q}\right). \tag{6.9}$$

From the recurrent formula, we have

$$\begin{aligned}
\phi_1(t) &= \lambda - IJ^{\alpha}\phi_0(t) + BJ^{\alpha}\phi_0\left(\frac{t}{q}\right) \\
&= \lambda - I\frac{\lambda t^{\alpha}}{\Gamma(\alpha+1)} + B\frac{\lambda t^{\alpha}}{\Gamma(\alpha+1)} \\
&= \left(I + (-I+B)\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)\lambda, \\
\phi_2(t) &= \lambda - IJ^{\alpha}\phi_1(t) + BJ^{\alpha}\phi_1\left(\frac{t}{q}\right) \\
&= \lambda - IJ^{\alpha}\left[\left(I + (-I+B)\frac{t^{\alpha}}{\Gamma(\alpha+1)}\right)\lambda\right] + BJ^{\alpha}\left(I + (-I+B)\frac{q^{-\alpha}t^{\alpha}}{\Gamma(\alpha+1)}\right)\lambda \\
&= \lambda - I\left[\frac{\lambda t^{\alpha}}{\Gamma(\alpha+1)} + (-I+B)\frac{\lambda t^{2\alpha}}{\Gamma(2\alpha+1)}\right] + B\left[\frac{\lambda t^{\alpha}}{\Gamma(\alpha+1)} + (-I+B)\frac{\lambda q^{-\alpha}t^{2\alpha}}{\Gamma(2\alpha+1)}\right] \\
&= \left[I + (-I+B)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (-I+Bq^{-\alpha})(-I+B)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right]\lambda,
\end{aligned}$$

$$\begin{aligned}
\phi_3(t) &= \left[I + (-I+B)\frac{t^{\alpha}}{\Gamma(\alpha+1)} + (-I+Bq^{-\alpha})(-I+B)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\right. \\
&\quad \left. + (-I+Bq^{-2\alpha})(-I+Bq^{-\alpha})(-I+B)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right]\lambda, \\
&\cdots, \\
\phi_k(t) &= \left[I + \sum_{m=1}^{k}\prod_{j=1}^{m}(-I+Bq^{-(m-j)\alpha})\frac{t^{m\alpha}}{\Gamma(m\alpha+1)}\right]\lambda
\end{aligned}$$

As $k \to \infty, \quad \phi_k(t) \to y(t)$

$$y(t) = \left[I + \sum_{k=1}^{\infty} \prod_{j=1}^{k} (-I + Bq^{-(k-j)\alpha}) \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \right] \lambda.$$

If we set $\prod_{j=1}^{k}(-I + Bq^{(k-j)\alpha}) = I$, for $k = 0$, then

$$y(t) = \left[\sum_{k=0}^{\infty} \prod_{j=1}^{k} (-I + Bq^{-(k-j)\alpha}) \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \right] \lambda. \tag{6.10}$$

Theorem 6.5. *For $q > 1$, the power series*

$$y(t) = \left[\sum_{k=0}^{\infty} \prod_{j=1}^{k} (-I + Bq^{-(k-j)\alpha}) \frac{t^{k\alpha}}{\Gamma(k\alpha+1)} \right] \lambda$$

is convergent for $t \in R$.

Proof. Result follows immediately by ratio test [4]. □

7. Illustrative Examples

Example 7.1. Consider the non-linear differential equations with proportional delay [2, 17, 26, 27]

$$\frac{dy(t)}{dt} = 1 - 2y^2\left(\frac{t}{2}\right), \quad y(0) = 0. \tag{7.1}$$

The corresponding integral equation is

$$y(t) = \int_0^t \left(1 - 2u^2\left(\frac{t}{2}\right)\right) dx. \tag{7.2}$$

By using successive approximation method (2.10), we obtain

$$\begin{aligned}
\phi_0(t) &= 0,\\
\phi_1(t) &= t,\\
\phi_2(t) &= t - \frac{t^3}{6},\\
\phi_3(t) &= t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{8064},\\
\phi_4(t) &= t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{61t^9}{23224320} - \frac{67t^{11}}{3406233600} + \frac{t^{13}}{12881756160} - \frac{t^{15}}{7990652436480},\\
\phi_5(t) &= t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{61t^9}{23224320} - \cdots - \frac{t^{31}}{106266419988615169375835859588218880},
\end{aligned}$$

and so on.

The exact solution of (7.1) is $y(t) = \sin t$.

The 5-term solutions of (7.1) using Adomian decomposition method (ADM) [17], variational iteration method (VIM) [26], homotopy analysis method (HAM) [2], optimal homotopy asymptotic method (OHAM) [27] are same and is given by

$$\begin{aligned}
y(t) &= t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040} + \frac{t^9}{362880} - \frac{t^{11}}{39916800} + \frac{t^{13}}{6227020800}\\
&\quad - \frac{t^{15}}{1307674368000} + \frac{t^{17}}{355687428096000}.
\end{aligned} \tag{7.3}$$

The 4-term OHAM solution [27] of (7.1) is

$$y(t) \quad = \quad t - 0.166665t^3 + 0.00832857t^5 - 0.000192105t^7. \tag{7.4}$$

We compare 5^{th} approximation solution (SAM) and 5-term solutions (ADM, VIM, HAM) with exact solution in Figure 1. and 4^{th} approximation solution (SAM) with 4-term solution (OHAM) in Figure 2. The absolute errors in computation are shown in Figure 3-4. It can be observed that SAM solution is better than the solution obtained by using other methods.

8. Figures

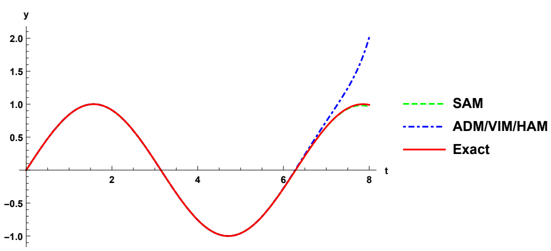

Figure 1. Comparison of SAM, ADM/VIM/HAM solutions with exact solution of Equation (7.1).

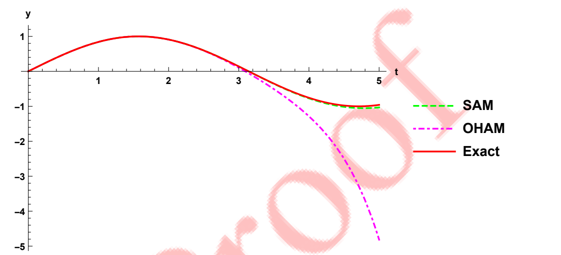

Figure 2. Comparison of SAM, OHAM solutions with exact solution of Equation (7.1).

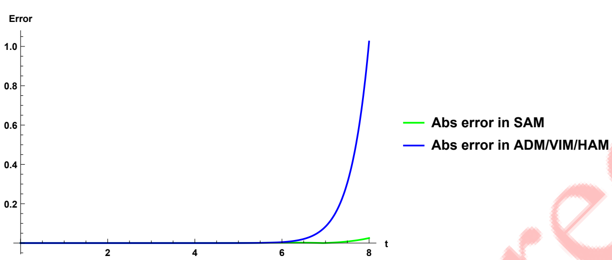

Figure 3. Comparison of absolute errors in SAM and ADM/VIM/HAM solutions.

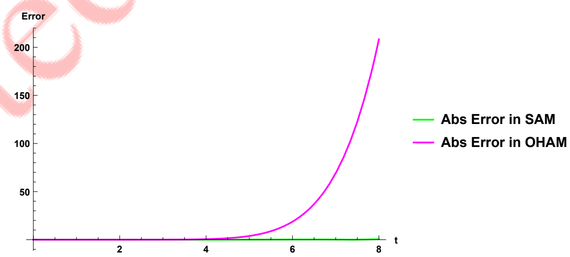

Figure 4. Comparison of absolute errors in SAM and OHAM solutions.

Remark 8.1. The ADM/VIM/HAM and OHAM solutions given in [2, 17, 26, 27] are considered only for the interval $[0, 1]$. Here we have successfully extended the solution using SAM in intervals $[0, 8]$.

9. Conclusions

Using SAM, we were able to solve non-linear DEs with proportional delay. We also obtained the stability, uniqueness, and existence results for several specific types of time-dependent DDEs. The convergence of the pantograph and Ambartsumian equations' series solution was examined. The analysis did in case of integer order for DEs with proportional delay and time-dependent DDEs have extended non integer order with Caputo fractional derivative. Finally, obtained results are supported with illustrative examples.

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