Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 13, No. 2, 2025, pp. 538-553
DOI:10.22034/cmde.2024.60642.2597

Optimal control of fractional differential equations with interval uncertainty

Tahereh Shokouhi[1], **Mehdi Allahdadi**[1,*], **and Samaneh Soradi-Zeid**[2]

[1]Mathematics Faculty, University of Sistan and Baluchestan, Zahedan, Iran.

[2]Faculty of Industry and Mining (Khash), University of Sistan and Baluchestan, Zahedan, Iran.

Abstract

The purpose of this paper is to obtain numerical solutions of fractional interval optimal control problems. To do so, first, we obtain a system of fractional interval differential equations through necessary conditions for the optimality of these problems, via the interval calculus of variations in the presence of interval constraint arithmetic. Relying on the trapezoidal rule, we obtain a numerical approximation for the interval Caputo fractional derivative. This approach causes the obtained conditions to be converted to a set of algebraic equations which can be solved using an iterative method such as the interval Gaussian elimination method and interval Newton method. Finally, we solve some examples of fractional interval optimal control problems in order to evaluate the performance of the suggested method and compare the past and present achievements in this manuscript.

Keywords. Interval optimal control problems, Interval fractional calculus, Fractional interval differential equation, Interval iterative method, Interval arithmetic.

2010 Mathematics Subject Classification. 49-XX, 65Kxx,34A08.

1. Introduction

A fractional optimal control problem (FOCP) arises when an optimal control problem is combined through a differential equation with a fractional derivative and a collection of boundary statuses. These problems have become increasingly popular due to their wide range of applications in physics and engineering [19, 31, 39]. It has been shown that using fractional derivatives leads to more accurate behavior of dynamic systems [28, 33]. Because of this, the researchers are excited to develop numerical methods for solutions of FOCPs [5, 28, 31].

Fuzzy models express real-world, uncertain phenomena. The problem of solving differential equations with uncertainty of the fuzzy type have been considered in several articles, including, Allahviranloo et al., [7], Agarwal et al., [3], Arshad and Lupulescu [9, 10], Agarwal et al., [1, 2], Salahshour et al., [34, 35] and, Mazandarani and Kamyad [27]. So, the use of fuzzy concepts in optimal control theory enables us to realistically illustrate processes such as physical models and chaos control [15, 40]. For fundamental theories, we refer the interested readers to [16, 30]. Furthermore, through uncertainty in the sense of fuzzy fractional dynamics, it is shown that many phenomena can be modeled by fractional fuzzy differential equations [4, 8, 25]. Obviously, any cut of fuzzy number $\tilde{a}$ can be written as an interval number. In order to achieve more accurate calculations and reduce errors, the interval model is preferred over the fuzzy model as it provides both upper and lower boundaries for uncertainties. This allows us for more certainty when determining upper and lower bounds. In this article, we have used single level interval arithmetic, which is a sub-branch of constrained interval arithmetic, defined by Chalco-Cano et al., in [12]. One of the advantages of single level constrained interval arithmetic is that it can be implemented efficiently using the IEEE 754 standard for floating point arithmetic, which provides special values for signed infinities, signed zeros, and exact/inexact flags. These features allow the interval operations to be performed without the need for extra tests or branches, which can improve the speed and accuracy of the computation.

Received: 18 February 2024 ; Accepted: 08 June 2024.
$*$ Corresponding author. Email: m_ allahdadi@math.usb.ac.ir .

In recent years, interval differential equations have been gaining interest from the scientific community. For a better understanding of fundamental theories, we recommend readers to see [23, 38, 41–45], and the references mentioned in them. Authors in [36, 37] have used a monotone iterative approach for fractional differential equations in the presence of delay parameters and interval uncertainty. A Riemann-Liouville fractional integral was introduced for Abel's interval equation [24]. Finding optimal control for the fractional interval optimal control problems (FIOCPs) which involves a system of fractional interval differential equations (FIDEs) is still an attractive problem. Allahdadi et al., [6] presented an accurate method for solving fractional interval linear quadratic regulator optimal control problems based on the fractional interval Pontryagin's minimum principle (PMP) and using the Laplace transformation method. In this study, we use the trapezoidal rule to obtain a numerical approximation for the interval Caputo fractional derivative. This approach allows us to convert the obtained PMP conditions into a set of algebraic equations that can be solved using an iterative method such as the interval Gaussian elimination method or interval Newton method.

We introduce a new numerical scheme to efficiently solve a specific type of interval optimal control problem that involves fractional differential equations. Our paper also presents another approach to interval arithmetic, known as constrained interval arithmetic. Our motivation for this work is to offer an alternative framework for solving these types of problems. One can refer to [21, 22] for detailed information about the properties and calculations related to it. Here, the main objective is to propose a generalization of PMP conditions in beside to fractional operators and interval uncertainty to derive the essential conditions for FIOCPs. Actually, we will create an optimal interval solution, which includes the lower and upper bounds of the solution to account for the uncertainty. This is crucial in finding the best solution for the FIOCPs and will provide a point-valued control for the user at each period.

Overall, this article is organized as follows. Section 2 is allocated to the preliminary definitions of interval arithmetic and fractional calculus. We exploit the necessary conditions for the optimality of FIOCPs that have been presented by FIDEs in section 3. Then, we transform these conditions into a system of algebraic equations through a fractional derivative approach. We try to derive the numerical solutions of this system using the approximation methods for solving algebraic equations with interval uncertainty. In section 4, numerical examples demonstrate the suggested method's performance. The study's conclusions are summarized in section 5.

2. Preliminaries

Here, we express the basic concepts which are required in other sections. These contexts are derived from references [11, 12, 14, 21–23, 26, 29].

Definition 2.1. [21] Let $\mathcal{K} = \{[\underline{v}, \overline{v}] \mid \underline{v}, \overline{v} \in \mathbb{R}, \quad \underline{v} \leq \overline{v}\}$, $\mathbf{v} = [\underline{v}, \overline{v}], \mathbf{z} = [\underline{z}, \overline{z}] \in \mathcal{K}$, and $\vartheta \in \mathbb{R}$. Addition and scalar multiplication of interval numbers are specified respectively as follows:

$$\mathbf{v} + \mathbf{z} = [\underline{v}, \overline{v}] + [\underline{z}, \overline{z}] = [\underline{v} + \underline{z}, \overline{v} + \overline{z}],$$

and

$$\vartheta \mathbf{v} = \vartheta[\underline{v}, \overline{v}] = \begin{cases} [\vartheta\underline{v}, \vartheta\overline{v}], & \vartheta > 0, \\ [0, 0], & \vartheta = 0, \\ [\vartheta\overline{v}, \vartheta\underline{v}], & \vartheta < 0. \end{cases}$$

Also, the interval multiplication is given by

$$[\underline{v}, \overline{v}].[\underline{z}, \overline{z}] = \Big[min\{\overline{v}.\overline{z}, \underline{v}.\overline{z}, \underline{v}.\underline{z}, \overline{v}.\underline{z}\}, max\{\overline{v}.\overline{z}, \underline{v}.\overline{z}, \underline{v}.\underline{z}, \overline{v}.\underline{z}\}\Big].$$

Furthermore, if we exclude division by an interval $\mathbf{z} = [\underline{z}, \overline{z}]$ containing 0, then the inverse of this interval is defined as follows:

$$\frac{1}{\mathbf{z}} = [\frac{1}{\overline{z}}, \frac{1}{\underline{z}}],$$

and, it immediately follows that:

$$\frac{[\underline{v}, \overline{v}]}{[\underline{z}, \overline{z}]} = [\underline{v}, \overline{v}].\frac{1}{[\underline{z}, \overline{z}]}.$$

Through these operations, $\mathcal{K}$ is not a vector space, because, $\mathbf{z}-\mathbf{z}$ and $\mathbf{z}\div\mathbf{z}$ are never 0 and 1, respectively, unless $\mathbf{z}$ is a real number with zero width. For example, suppose $\mathbf{z}=[1,2]$. If we divide $\mathbf{z}$ by itself, and also we subtract it from itself, we will have:

$$\frac{[1,2]}{[1,2]}=[\frac{1}{2},2]\neq 1, \quad [1,2]-[1,2]\neq 0.$$

This unwanted extra interval width is called the dependency. As previously mentioned, standard interval arithmetic has shortcomings which is related to the algebraic construction of the interval variables. To address this, we introduce constrained interval arithmetic, which covers this challenge. Furthermore, it aims to reduce the overestimation of functions with duplicated variables. We have used single level interval arithmetic, which is a sub-branch of constrained interval arithmetic. Based on the concept of constrained interval arithmetic, Chalco-Cano et al., [12] defined a new concept of arithmetic, called single level constrained interval arithmetic which computes the range of a function using intervals with floating point endpoints. It is based on two principles: soundness and optimality. Soundness means that the interval result must contain all the possible values of the real function. Optimality means that the interval result must be as narrow as possible, given the limitations of the floating point representation.

Definition 2.2. [12] Let $\mathbf{v}=[\underline{v},\overline{v}]$ be an interval. Then, a continuous function $\mathbf{v}_\lambda:[0,1]\to\mathbb{R}$, which is defined as:

$$\mathbf{v}_\lambda=(1-\lambda)\overline{v}+\lambda\underline{v}=\overline{v}-W(\mathbf{v})\lambda, \qquad \lambda\in[0,1],$$

is called the constraint function associated with $\mathbf{v}$, in which $W(\mathbf{v})=\overline{v}-\underline{v}$.

In this definition, the parameters $\underline{v}$, and $\overline{v}$ are well-known numbers. In constrained interval arithmetic, the maximum and minimum values of $\mathbf{v}$ can be obtained following the lowest and highest values of $\mathbf{v}_\lambda$:

$$\overline{v}=\max_{\lambda\in[0,1]}\{\mathbf{v}_\lambda\}, \quad \underline{v}=\min_{\lambda\in[0,1]}\{\mathbf{v}_\lambda\}.$$

Single level constrained interval arithmetic yields:

$$\begin{aligned}\mathbf{q}&=[\underline{q},\overline{q}]=\mathbf{v}\circ\mathbf{z}\\&=\{q|q=v\circ z,\forall v\in\mathbf{v}_\lambda,z\in\mathbf{z}_\lambda,q\in\mathbf{q}_\lambda,0\leq\lambda\leq 1\}\\&=\{q|\mathbf{q}_\lambda=(\lambda\underline{v}+(1-\lambda)\overline{v})\circ(\lambda\underline{z}+(1-\lambda)\overline{z}),0\leq\lambda\leq 1\},\end{aligned}$$

where

$$\overline{q}=\max_{\lambda\in[0,1]}\{\mathbf{q}_\lambda\}, \quad \circ\in\{+,-,\times,\div\}, \quad \underline{q}=\min_{\lambda\in[0,1]}\{\mathbf{q}_\lambda\}.$$

Definition 2.3. [26] The interval distance $\mathbb{D}$ which is denoted from $\mathcal{K}\times\mathcal{K}$ into $\mathbf{R}^+\cup\{0\}$, is defined as follows:

$$\mathbb{D}(\mathbf{z},\mathbf{v})=\max\{|\ \underline{z}-\underline{v}\ |,|\ \overline{z}-\overline{v}\ |\}.$$

This statement declares that the metric space $(\mathcal{K},\mathbb{D})$ is both separable and complete.

Definition 2.4. [26] Equipping $\mathbf{z}\in\mathcal{K}$ together with $\|\mathbf{z}\|=\max\{|\ \overline{z}\ |,|\ \underline{z}\ |\}$ makes $(\mathcal{K},+,.,\|.\|)$ be a normed quasi-linear space.

2.1. **Interval fractional operators and their properties.** This subsection embodies a survey of fractional calculus in mathematical interval analysis [14, 23].

Definition 2.5. The Riemann-Liouville fractional integrals of $\mathbf{S}\in L[t_0,t_f]$ are appointed as:

$$\begin{aligned}{}_{t_0}\mathfrak{J}_t^{\mu}\mathbf{S}(t)&=\frac{1}{\Gamma(\mu)}\int_{t_0}^{t}(t-\tau)^{\mu-1}\mathbf{S}(\tau)d\tau,\\{}_{t}\mathfrak{J}_{t_f}^{\mu}\mathbf{S}(t)&=\frac{1}{\Gamma(\mu)}\int_{t}^{t_f}(\tau-t)^{\mu-1}\mathbf{S}(\tau)d\tau.\end{aligned}$$

Since $\mathbf{S}(t) = [\underline{S}(t), \overline{S}(t)]$, then, it can be deduce that:

$$ {}_{t_0}\mathfrak{I}_t^{\mu}\mathbf{S}(t) = [{}_{t_0}\mathfrak{I}_t^{\mu}\underline{S}(t), {}_{t_0}\mathfrak{I}_t^{\mu}\overline{S}(t)], $$

where the lower and upper bounds can be expressed using Definition 2.2 as follows:

$$ {}_{t_0}\mathfrak{I}_t^{\mu}\overline{S}(t) = \max_{\lambda\in[0,1]}\{\frac{1}{\Gamma(\mu)}\int_{t_0}^{t}(t-\tau)^{\mu-1}\mathbf{S}_{\lambda}(\tau)d\tau\}, $$

and

$$ {}_{t_0}\mathfrak{I}_t^{\mu}\underline{S}(t) = \min_{\lambda\in[0,1]}\{\frac{1}{\Gamma(\mu)}\int_{t_0}^{t}(t-\tau)^{\mu-1}\mathbf{S}_{\lambda}(\tau)d\tau\}. $$

Definition 2.6. [14] Let $\mathbf{S}$ be a continuous first-order differentiable function and $0 < \mu < 1$. If $\mathbf{S}$ belongs to $L[t_0, t_f]$, the Caputo fractional derivatives of $\mathbf{S}(t)$ are determined as:

$$ {}_{t_0}^{C}D_t^{\mu}\mathbf{S}(t) = {}_{t_0}\mathfrak{I}_t^{1-\mu}\{\frac{d}{dt}\mathbf{S}(t)\} = \frac{1}{\Gamma(1-\mu)}\int_{t_0}^{t}(t-\tau)^{-\mu}\mathbf{S}'(\tau)d\tau, \tag{2.1} $$

$$ {}_{t}^{C}D_{t_f}^{\mu}\mathbf{S}(t) = {}_{t}\mathfrak{I}_{t_f}^{1-\mu}\{-\frac{d}{dt}\mathbf{S}(t)\} = \frac{-1}{\Gamma(1-\mu)}\int_{t}^{t_f}(\tau-t)^{-\mu}\mathbf{S}'(\tau)d\tau. $$

Now, using the constrained interval arithmetic, the Caputo fractional derivative of $\mathbf{S}(t)$ can be written as:

$$ {}_{t_0}^{C}D_t^{\mu}\mathbf{S}(t) = \Big[\min_{\lambda\in[0,1]}\{{}_{t_0}^{C}D_t^{\mu}\mathbf{S}_{\lambda}(t)\}, \max_{\lambda\in[0,1]}\{{}_{t_0}^{C}D_t^{\mu}\mathbf{S}_{\lambda}(t)\}\Big]. $$

2.2. **Numerical approximation of interval Caputo fractional derivative.** In this subsection, we want to develop a generalization of a numerical approach that has been derived by Diethelm [14] for computing the interval Caputo fractional derivative. Fractional derivatives are hereditary functions possessing a total memory of past states. Now, we want to compute the numerical approximation of the interval Caputo fractional order derivative. To achieve this goal, let $\{t_j = t_0 + jh : j = 0, 1, ..., N\}$ be a uniform grid on interval $[t_0, t_f]$, in which $h = \frac{t_f - t_0}{N}$. By approximating the integral in Equations (2.1) with a product trapezoidal method, thereby restricting $0 < \mu < 1$, we have:

$$ {}_{t_0}^{C}D_t^{\mu}\mathbf{S}_N(h) = \frac{h^{-\mu}}{\Gamma(2-\mu)}\sum_{j=0}^{N}a_{j,N}\Big(\mathbf{S}_{N-j}(h) - \sum_{k=0}^{\lceil\mu\rceil}\frac{(N-j)^k h^k}{k!}S^{(k)}(t_0)\Big), \tag{2.2} $$

$$ {}_{t_f}^{C}D_t^{\mu}\mathbf{S}_N(h) = \frac{-h^{-\mu}}{\Gamma(2-\mu)}\sum_{j=0}^{N}a_{j,N}\Big(\mathbf{S}_{N-j}(h) - \sum_{k=0}^{\lceil\mu\rceil}\frac{(N-j)^k h^k}{k!}\mathbf{S}^{(k)}(t_f)\Big), \tag{2.3} $$

in which $\mathbf{S}_j(h) = \mathbf{S}(t_j)$, and using the quadrature weights (derived from a product trapezoidal rule) leads:

$$ a_{j,N} = \begin{cases} (N-1)^{1-\mu} + N^{-\mu}(1-\mu) - N^{1-\mu}, & if\ j = N, \\ (j+1)^{1-\mu} + (j-1)^{1-\mu} - 2j^{1-\mu}, & if\ 0 < j < N, \\ 1, & if\ j = 0. \end{cases} $$

Furthermore, ${}_{t_0}^{C}D_t^{\mu}\mathbf{S}(t) = {}_{t_0}^{C}D_t^{\mu}\mathbf{S}_N(h) + O(h^{2-\mu})$. A similar result is obtained for the right interval Caputo fractional derivative.

3. Method of solution for FIOCPs

Our method offers an efficient way to obtain the optimal solution for the following FIOCP:

$$ minimize_{\mathbf{u}(.)}\ \mathbf{I}(\mathbf{x}(t), \mathbf{u}(t), t) = \mathbf{h}(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f}\mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t)dt, \tag{3.1} $$

subject to the dynamic constraints

$$ {}_{t_0}^{C}D_t{}^{\mu}\mathbf{x}(t) = \mathbf{a}(\mathbf{x}(t), \mathbf{u}(t), t), \tag{3.2} $$

and the initial condition

$$\mathbf{x}(t_0) = \mathbf{x}_0, \tag{3.3}$$

where $\mathbf{g}$ and $\mathbf{a} \in \mathcal{K}$ are integrands and continuous functions, respectively. The interval-valued state and control variables are defined by $\mathbf{x}(t)$ and $\mathbf{u}(t)$, $t \in [t_0, t_f]$, respectively. In addition, $\mathbf{h}(t)$ is a constant interval-valued function and ${}_{t_0}^{C}D_t^{\mu}$ is the Caputo fractional derivative of order $0 \le \mu \le 1$.

Our objective involves finding the interval-valued function $u(t)$ that minimizes the performance index (3.1), while also satisfying the constraints (3.2)-(3.3). We can achieve this, by converting the FIOCP (3.1)-(3.3) into a system of FIDEs. Then, we apply a numerical scheme to solve the obtained FIDEs, similar to the previous section. It can find useful information for FIOCP (3.1)-(3.3).

Definition 3.1. If $\mathbf{I}(\mathbf{x}, t)$ is an interval-valued functional, then the increment of $\mathbf{I}(\mathbf{x}, t)$ can be expressed as follows:

$$\Delta\mathbf{I}(\mathbf{x}, \delta\mathbf{x}, t) = \mathbf{I}(\mathbf{x} + \delta\mathbf{x}, t) - \mathbf{I}(\mathbf{x}, t).$$

It has been established that $\delta\mathbf{x}$ represents the variation of $\mathbf{x}$.

Definition 3.2. The increment of $\mathbf{I}(\mathbf{x}, t)$ is defined by:

$$\Delta\mathbf{I}(\mathbf{x}, \delta\mathbf{x}, t) = \delta\mathbf{I}(\mathbf{x}, \delta\mathbf{x}, t) + \mathbf{v}(\mathbf{x}, \delta\mathbf{x}, t). \left\|\delta\mathbf{x}\right\|.$$

Furthermore, $\mathbf{I}(\mathbf{x}, t)$ is differentiable at $\mathbf{x}$, if for any given positive value of ε, $\mathbb{D}\left(\mathbf{I}(\mathbf{x}, \delta\mathbf{x}, t), 0\right) < \varepsilon$ as $\|\delta\mathbf{x}(t)\| \to 0$.

Definition 3.3. Let $\mathbf{x}^*(t) = [\underline{x}(t), \overline{x}(t)] \in \mathcal{K}$ be a relative minimizer for $\mathbf{I}(\mathbf{x}, t) : [t_0, t_f] \to \mathcal{K}$. Then:

$$\Delta\mathbf{I}(\mathbf{x}, \delta\mathbf{x}, t) \ge 0,$$

which means $\forall \mathbf{x}(t) \in \mathcal{K}$, $\mathbf{I}(\mathbf{x}^*, t) \le \mathbf{I}(\mathbf{x}, t)$.

Theorem 3.4. *If $\boldsymbol{x}^*(t) \in \mathcal{K}$ is a relative minimizer of $\boldsymbol{I}(\boldsymbol{x}, t)$, then we have:*

$$\delta\boldsymbol{I}(\boldsymbol{x}^*, \delta\boldsymbol{x}^*, t) = [\delta\underline{I}(\boldsymbol{x}^*, \delta\boldsymbol{x}^*, t), \delta\overline{I}(\boldsymbol{x}^*, \delta\boldsymbol{x}^*, t)] = 0,$$

in which $\delta\boldsymbol{x}^*(t) \in \mathcal{K}$.

Proof. According to the assumption and using Definition 3.3, we get:

$$\Delta\mathbf{I}(\mathbf{x}^*, \delta\mathbf{x}^*, t) \ge 0.$$

Now, based on Definition 3.2, it can be concluded:

$$\Delta\mathbf{I}(\mathbf{x}^*, \delta\mathbf{x}^*, t) = [\delta\underline{I}(\mathbf{x}^*, \delta\mathbf{x}^*, t) + \mathbf{v}_1(\mathbf{x}^*, \delta\mathbf{x}^*, t). \left\|\delta\mathbf{x}^*(t)\right\|,$$
$$\delta\overline{I}(\mathbf{x}^*, \delta\mathbf{x}^*, t) + \mathbf{v}_2(x^*, \delta\mathbf{x}^*, t). \|\delta\mathbf{x}^*(t)\|],$$

in which, as $\|\delta\mathbf{x}^*(t)\| \to 0$, $\mathbf{v}_1 \to 0$ and $\mathbf{v}_2 \to 0$ and then we have:

$$\delta\mathbf{I}(\mathbf{x}^*, \delta\mathbf{x}^*, t) \ge 0. \tag{3.4}$$

Consequently:

$$\delta\overline{I} = \max_{\lambda \in [0,1]} \{\delta\mathbf{I}(\mathbf{x}^*_\lambda, \delta\mathbf{x}^*_\lambda, t)\} \ge 0. \tag{3.5}$$

Let $\delta\mathbf{x} = -k^2\delta\mathbf{y}$ and $k \in \mathbb{R}$. Based on the relation (3.4), we obtain:

$$\delta\mathbf{I}(\mathbf{x}^*, \delta x^*, t) = \delta\mathbf{I}(\mathbf{x}^*, -k^2\delta\mathbf{y}, t) \ge 0.$$

In view of scalar multiplication in Definition 2.1, it is clear that:

$$\delta\mathbf{I}(\mathbf{x}^*, -k^2\delta\ \mathbf{y}, t) = -k^2\delta\mathbf{I}(\mathbf{x}^*, \delta\mathbf{y}, t),$$

or equivalently:

$$[\delta\underline{I}(\mathbf{x}^*, -k^2\delta\mathbf{y}, t), \delta\overline{I}(\mathbf{x}^*, -k^2\delta\mathbf{y}, t)] = -k^2[\delta\underline{I}(\mathbf{x}^*, \delta\mathbf{y}, t), \delta\overline{I}(\mathbf{x}^*, \delta\mathbf{y}, t)].$$

Afterwards, we achieve:

$$\begin{aligned}\delta\underline{I}(\mathbf{x}^*,-k^2\delta\mathbf{y},t)&=-k^2\delta\overline{I}(\mathbf{x}^*,\delta\mathbf{y},t),\\ \delta\overline{I}(\mathbf{x}^*,-k^2\delta\mathbf{y},t)&=-k^2\delta\underline{I}(\mathbf{x}^*,\delta\mathbf{y},t).\end{aligned}\tag{3.6}$$

Using Equation (3.4) and the second term of Equation (3.6), results:

$$\delta\overline{I}(\mathbf{x}^*,\delta\mathbf{x}^*,t)\leq 0.$$

Based on Equation (3.5) and the relationship described above, we are directed to the following relation:

$$\delta\overline{I}(\mathbf{x}^*,\delta\mathbf{x}^*,t)=\max_{\lambda\in[0,1]}\{\delta\mathbf{I}(\mathbf{x}^*_\lambda,\delta\mathbf{x}^*_\lambda,t)\}=0.$$

Similarly, we obtain:

$$\delta\underline{I}(\mathbf{x}^*,\delta\mathbf{x}^*,t)=\min_{\lambda\in[0,1]}\{\delta\mathbf{I}(\mathbf{x}^*_\lambda,\delta\mathbf{x}^*_\lambda,t)\}=0.$$

Furthermore:

$$\delta\mathbf{I}(\mathbf{x}^*,\delta\mathbf{x}^*,t)=[\delta\underline{I},\delta\overline{I}]=0.$$

□

Now, we want to use the PMP to derive the necessary conditions for the optimality of FIOCP (3.1)-(3.3).

Theorem 3.5. *Let $\boldsymbol{x}^*$ and $\boldsymbol{u}^*$ be the admissible state and control variables with interval values for FIOCP (3.1)-(3.3), respectively. Then, $\boldsymbol{u}^*$ gives the interval optimal control for the interval-valued functional $\boldsymbol{I}$, if it satisfies the following boundary value problem:*

$$\begin{aligned}&{}_{t_0}^{C}D_t^{\mu}\boldsymbol{x}^*(t)=\frac{\partial\boldsymbol{H}}{\partial\boldsymbol{p}}(\boldsymbol{x}^*(t),\boldsymbol{p}^{\ *}(t),\boldsymbol{u}^*(t),t),\\ &{}_{t}^{C}D_{t_f}^{\mu}\boldsymbol{p}^*(t)=-\frac{\partial\boldsymbol{H}}{\partial\boldsymbol{x}}(\boldsymbol{x}^*(t),\boldsymbol{p}^*(t),\boldsymbol{u}^*(t),t),\\ &\frac{\partial\boldsymbol{H}}{\partial\boldsymbol{u}}(\boldsymbol{x}^*(t),\boldsymbol{p}^*(t),\boldsymbol{u}^*(t),t)=0,\\ &\boldsymbol{x}(t_0)=\boldsymbol{x}_0,\qquad\Big(\frac{\partial\boldsymbol{h}}{\partial\boldsymbol{x}^*_\lambda}-{}_{t}I_{t_f}^{1-\mu}\boldsymbol{p}^*_\lambda(t)\Big)\Big|_{t=t_f}=0,\end{aligned}\tag{3.7}$$

in which the interval-valued Lagrange multiplier is expressed by $\mathbf{p}$ and, $\boldsymbol{H}$ is the following Hamiltonian function:

$$\boldsymbol{H}(\boldsymbol{x}(t),\boldsymbol{p}(t),\boldsymbol{u}(t),t)=\boldsymbol{g}(\boldsymbol{x}(t),\boldsymbol{u}(t),t)+\boldsymbol{p}(t)\boldsymbol{a}(\boldsymbol{x}(t),\boldsymbol{u}(t),t).\tag{3.8}$$

Proof. At the beginning, we define:

$$\mathbf{I}(\mathbf{x},\delta\mathbf{x})=\int_{t_0}^{t_f}\Big[\mathbf{g}(\mathbf{x}(t),\mathbf{u}(t),t)+\mathbf{p}(t)\Big(\mathbf{a}(\mathbf{x}(t),\mathbf{u}(t),t)-{}_{t_0}^{C}D_t^{\mu}\mathbf{x}(t)\Big)\Big]dt+\mathbf{h}(\mathbf{x}(t_f),t_f).\tag{3.9}$$

In view of relation (3.8), we can express the generalized criterion $\underline{I}$ by modifying Equation (3.9) as follows:

$$\underline{I}(\mathbf{u},t)=\min_{\lambda\in[0,1]}\Big\{\int_{t_0}^{t_f}\Big[\mathbf{H}(\mathbf{x}_\lambda(t),\mathbf{p}_\lambda(t),\mathbf{u}_\lambda(t),t)-\mathbf{p}_\lambda(t){}_{t_0}^{C}D_t^{\mu}\mathbf{x}_\lambda(t)\Big]dt+\mathbf{h}(\mathbf{x}_\lambda(t_f),t_f)\Big\}.\tag{3.10}$$

The variation of $\underline{I}$ is determined as follows:

$$\begin{aligned}\delta\underline{I}(\mathbf{u}^*,\delta\mathbf{u}^*,t)=\min_{\lambda\in[0,1]}\Big\{\int_{t_0}^{t_f}&\Big[\frac{\partial\mathbf{H}}{\partial\mathbf{x}^*_\lambda}(\mathbf{x}^*_\lambda(t),\mathbf{p}^*_\lambda(t),\mathbf{u}^*_\lambda(t),t)\ \delta\mathbf{x}^*_\lambda(t)\\ &+\frac{\partial\mathbf{H}}{\partial\mathbf{u}^*_\lambda}(\mathbf{x}^*_\lambda(t),\mathbf{p}^*_\lambda(t),\mathbf{u}^*_\lambda(t),t)\ \delta\mathbf{u}^*_\lambda(t)+\frac{\partial\mathbf{H}}{\partial\mathbf{p}^*_\lambda}(\mathbf{x}^*_\lambda(t),\mathbf{p}^*_\lambda(t),\mathbf{u}^*_\lambda(t),t)\ \delta\mathbf{p}^*_\lambda(t)\\ &-{}_{t_0}^{C}D_t^{\mu}\mathbf{x}^{\ *}_{\ \lambda}(t)\ \delta\mathbf{p}^*_\lambda(t)-\mathbf{p}^*_\lambda(t)\ {}_{t_0}^{C}D_t^{\mu}\delta\mathbf{x}^*_\lambda(t)\Big]dt+\frac{\partial\mathbf{h}}{\partial\mathbf{x}^*_\lambda}\ \delta\mathbf{x}^*_\lambda(t_f)\Big\}.\end{aligned}\tag{3.11}$$

The use of integration by parts [32], leads:

$$\min_{\lambda\in[0,1]}\Big\{\int_{t_0}^{t_f}\mathbf{p}_\lambda^*(t)\,{}_{t_0}^{C}D_t^{\mu}\delta\mathbf{x}_\lambda^*(t)dt\Big\}=\min_{\lambda\in[0,1]}\Big\{\Big|{}_tI_{t_f}^{1-\mu}\mathbf{p}_\lambda^*(t)\delta\mathbf{x}_\lambda^*(t)\Big|_{t=t_f}-\int_{t_0}^{t_f}{}_t^CD_{t_f}^{\mu}\mathbf{p}_\lambda^*(t)\delta\mathbf{x}\,{}_\lambda^*(t)dt\Big\}. \tag{3.12}$$

When (3.12) is substituted into (3.11), a new formula can be derived as follows:

$$\begin{aligned}\delta\underline{I}(\mathbf{u}^*,\delta\mathbf{u}^*,t)=\min_{\lambda\in[0,1]}\Big\{\int_{t_0}^{t_f}&\Big(\frac{\partial\mathbf{H}}{\partial\mathbf{x}_\lambda^*}(\mathbf{x}_\lambda^*(t),\mathbf{p}_\lambda^*(t),\mathbf{u}_\lambda^*(t),t)+{}_t^CD_{t_f}^{\mu}\mathbf{p}_\lambda^*(t)\Big)\delta\mathbf{x}_\lambda^*(t)\\&+\frac{\partial\mathbf{H}}{\partial\mathbf{u}_\lambda^*}(\mathbf{x}_\lambda^*(t),\mathbf{p}_\lambda^*(t),\mathbf{u}_\lambda^*(t),t)\delta\mathbf{u}_\lambda^*(t)+\Big(\frac{\partial\mathbf{H}}{\partial\mathbf{p}_\lambda^*}(\mathbf{x}_\lambda^*(t),\mathbf{p}_\lambda^*(t),\mathbf{u}_\lambda^*(t),t)-{}_{t_0}^CD_t^{\mu}\mathbf{x}_\lambda^*(t)\Big)\delta\mathbf{p}_\lambda^*(t)dt\\&+\Big(\frac{\partial\mathbf{h}}{\partial\mathbf{x}_\lambda^*}-{}_tI_{t_f}^{1-\mu}\mathbf{p}_\lambda^*(t)\Big)|_{t=t_f}\delta\mathbf{x}_\lambda^*(t_f)\Big\}=0.\end{aligned}$$

According to Theorem 3.4 and Definition 3.2, the following analysis can be done:

$$\mathbb{D}\Big(\min_{\lambda\in[0,1]}\Big\{\frac{\partial\mathbf{H}}{\partial\mathbf{u}_\lambda^*}(\mathbf{x}_\lambda^*(t),\mathbf{p}_\lambda^*(t),\mathbf{u}_\lambda^*(t),t\Big\},0\Big)\le\varepsilon,\qquad\|\delta\mathbf{u}_\lambda^*(t)\|\to0,$$

$$\mathbb{D}\Big(\min_{\lambda\in[0,1]}\Big\{\frac{\partial\mathbf{H}}{\partial\mathbf{p}_\lambda^*}(\mathbf{x}_\lambda^*(t),\mathbf{p}_\lambda^*(t),\mathbf{u}_\lambda^*(t),t)-{}_{t_0}^CD_t^{\mu}\mathbf{x}_\lambda^*(t)\Big\},0\Big)\le\varepsilon,\qquad\|\delta\mathbf{p}_\lambda^*(t)\|\to0,$$

$$\mathbb{D}\Big(\min_{\lambda\in[0,1]}\Big\{\frac{\partial\mathbf{H}}{\partial\mathbf{x}_\lambda^*}(\mathbf{x}_\lambda^*(t),\mathbf{p}_\lambda^*(t),\mathbf{u}_\lambda^*(t),t)+{}_t^CD_{t_f}^{\mu}\mathbf{p}_\lambda^*(t)\Big\},0\Big)\le\varepsilon,\qquad\|\delta\mathbf{x}_\lambda^*(t)\|\to0.$$

Since $\mathbf{x}(t_f)$ is uncertain, δt_f and $\delta\mathbf{x}_\lambda^*(t_f)$ are optional. So, we have:

$$\min_{\lambda\in[0,1]}\Big\{\Big(\frac{\partial\mathbf{h}}{\partial\mathbf{x}_\lambda^*}-{}_tI_{t_f}^{1-\mu}\mathbf{p}_\lambda^*(t)\Big)\Big|_{t=t_f}\Big\}=0.$$

Consequently:

$$\begin{aligned}{}_{t_0}^CD_t^{\mu}\underline{x}^*(t)&=\min_{\lambda\in[0,1]}\Big\{\frac{\partial\mathbf{H}}{\partial\mathbf{p}_\lambda^*}(\mathbf{x}_\lambda^*(t),\mathbf{p}_\lambda^*(t),\mathbf{u}_\lambda^*(t),t)\Big\},\\{}_{t_0}^CD_t^{\mu}\underline{p}^*(t)&=\min_{\lambda\in[0,1]}\Big\{\frac{\partial\mathbf{H}}{\partial\mathbf{x}_\lambda^*}(\mathbf{x}_\lambda^*(t),\mathbf{p}_\lambda^*(t),\mathbf{u}_\lambda^*(t),t)\Big\},\\\underline{u}^*(t)&=\min_{\lambda\in[0,1]}\Big\{\frac{\partial\mathbf{H}}{\partial\mathbf{u}_\lambda^*}(\mathbf{x}_\lambda^*(t),\mathbf{p}_\lambda^*(t),\mathbf{u}_\lambda^*(t),t)\Big\},\\\underline{x}(t_0)&=min_{\lambda\in[0,1]}\Big\{\mathbf{x}_{0\lambda}\Big\}.\end{aligned}$$

We find that $\overline{I}$ vary similarly. Then, we have:

$$\begin{aligned}{}_{t_0}^CD_t^{\mu}\overline{x}^*(t)&=\max_{\lambda\in[0,1]}\Big\{\frac{\partial\mathbf{H}}{\partial\mathbf{p}_\lambda^*}(\mathbf{x}_\lambda^*(t),\mathbf{p}_\lambda^*(t),\mathbf{u}_\lambda^*(t),t)\Big\},\\{}_{t_0}^CD_t^{\mu}\overline{p}^*(t)&=\max_{\lambda\in[0,1]}\Big\{\frac{\partial\mathbf{H}}{\partial\mathbf{x}_\lambda^*}(\mathbf{x}_\lambda^*(t),\mathbf{p}_\lambda^*(t),\mathbf{u}_\lambda^*(t),t)\Big\},\\\overline{u}^*(t)&=\max_{\lambda\in[0,1]}\Big\{\frac{\partial\mathbf{H}}{\partial\mathbf{u}_\lambda^*}(\mathbf{x}_\lambda^*(t),\mathbf{p}_\lambda^*(t),\mathbf{u}_\lambda^*(t),t)\Big\},\\\overline{x}(t_0)&=\max_{\lambda\in[0,1]}\Big\{\mathbf{x}_{0\lambda}\Big\},\quad\max_{\lambda\in[0,1]}\Big\{\Big(\frac{\partial\mathbf{h}}{\partial\mathbf{x}_\lambda^*}-{}_tI_{t_f}^{1-\mu}\mathbf{p}_\lambda^*(t)\Big)\Big|_{t=t_f}\Big\}=0.\end{aligned}$$

The proof is complete based on the interval solutions obtained through constrained interval arithmetic. □

When the FIOCP is used with a quadratic performance index, we have a special case as follows:

$$minimize_{\mathbf{u}(.)} \ \mathbf{I}(\mathbf{x}(t), \mathbf{u}(t), t) = \int_{t_0}^{t_f} \frac{1}{2}\Big[\mathbf{u}^T(t)\,\mathbf{R}\mathbf{u}(t) + \mathbf{x}^T(t)\,\mathbf{Q}\mathbf{x}(t)\Big]dt, \tag{3.13}$$

$${}_{t_0}^{C}D_t^{\mu}\mathbf{x} = \mathbf{B}\mathbf{u}(t) + \ \mathbf{A}\mathbf{x}(t), \tag{3.14}$$

$$\mathbf{x}(t_0) = \mathbf{x}_0. \tag{3.15}$$

Here, it is assumed that $\mathbf{A}$ and $\mathbf{B}$ are fixed interval matrices, $\mathbf{Q} \geq 0$ is a fixed interval symmetric positive semi-definite matrix and, $\mathbf{R} > 0$ is a fixed interval symmetric positive definite matrix. Using the results of Theorem 3.5, we have the following system of FIDEs:

$$\begin{aligned}
&{}_{t}^{C}D_{t_f}^{\mu}\mathbf{p}^*(t) = \mathbf{A}\mathbf{p}^*(t) + \mathbf{Q}\mathbf{x}^*(t), \\
&{}_{t_0}^{C}D_t^{\mu}\mathbf{x}^*(t) = \mathbf{A}\mathbf{x}^*(t) + \mathbf{B}\mathbf{u}^*(t), \\
&\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{p}^*(t)\mathbf{B}, \\
&\mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{p}(t_f) = 0.
\end{aligned} \tag{3.16}$$

According to the numerical calculation of the Caputo fractional derivative, we divide the entire time domain $[t_0, t_f]$ into N equal parts. So, Equations (3.16) can be transformed into a nonlinear system of algebraic equations which can be solved using the direct interval methods [17]. As soon as $\mathbf{p}^*$ is known, the optimal control can be achieved from $\mathbf{u}^*(t) = -\mathbf{R}^{-1}\mathbf{p}^*(t)\mathbf{B}$.

4. Illustrative Example

To show the utility and effectiveness of the proposed method for obtaining the optimal controller of fractional interval differential equations, two examples have been provided. The simulated results have been carried out using the interval toolbox INTLAB v10 which is a downloadable software package and works through MATLAB.

Example 4.1. Consider the following FIOCP:

$$minimize_{\mathbf{u}(.)} \ \mathbf{I}(\mathbf{x}(t), \mathbf{u}(t), t) = \int_0^1 \mathbf{u}^2(t)dt, \tag{4.1}$$

with dynamic system:

$${}_{0}^{C}D_t^{\mu}\mathbf{x}(t) = \mathbf{u}(t) - [b, 3-2b]\mathbf{x}(t), \quad 0 \leq b \leq 1, \tag{4.2}$$

and initial condition:

$$\mathbf{x}(0) = 1. \tag{4.3}$$

To minimize the performance index (4.1), our objective is to find the interval control $\mathbf{u}(t)$ that holds at the FIDE (4.2) with the initial condition (4.3). We construct the interval Hamiltonian function as follows:

$$\mathbf{H}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{p}, t) = \mathbf{u}^2(t) - \mathbf{r}\mathbf{p}(t)\mathbf{x}(t) + \mathbf{p}(t)\mathbf{u}(t),$$

in which $\mathbf{r} = [b, 3-2b]$. So, the necessary optimality conditions based on Theorem 3.5 are obtained as follows:

$${}_{0}^{C}D_t^{\mu}\mathbf{x}(t) = \mathbf{u}(t) - \mathbf{r}\mathbf{x}(t), \tag{4.4}$$

$${}_{t}^{C}D_1^{\mu}\mathbf{p} = \mathbf{r}\mathbf{p}(t), \tag{4.5}$$

$$2\mathbf{u}(t) + \mathbf{p}(t) = 0, \tag{4.6}$$

and

$$\mathbf{x}(0) = 1, \ \ \mathbf{p}(1) = 0. \tag{4.7}$$

According to Equation (4.6), we can come to an equivalent conclusion as follows:

$$\mathbf{u}(t) = -\frac{1}{2}\mathbf{p}(t). \tag{4.8}$$

By replacing Equation (4.8) in (4.5) and using the numerical approximation of Caputo-type derivatives described in Equations (2.2) and (2.3), Equations (4.4) and (4.5) can be rewritten as follows:

$$\frac{h^{-\mu}}{\Gamma(2-\mu)} \sum_{i=0}^{j} a_{i,j}\Big(\mathbf{x}_{j-i}(h) - \mathbf{x}_0\Big) = -\frac{1}{2}\mathbf{p}_j(h) - \mathbf{r}\mathbf{x}_j(h), \qquad j = 1, ..., N,$$

$$\frac{h^{-\mu}}{\Gamma(2-\mu)} \sum_{i=j}^{N} a_{i-j,j}\Big(\mathbf{p}_N - \mathbf{p}_i(h)\Big) = \mathbf{r}\mathbf{p}_j(h), \qquad j = 0, 1, ..., N-1.$$

Now, to solve these equations with respect to the boundary conditions (4.7), we use the interval Gaussian elimination method [17], with the finite number of iterations (N=10). In Table 1, we obtain the results for $\mu = 1$ and $b = 0.5$ in contrast to the reported numerical solutions in [16] and [30]. From the perspective of this table, it is obvious that the obtained results for optimal control in [16] are completely incorrect and did not provide an interval number (the lower bound is larger than the upper bound!). Figure 1 displays the state and control variables for different values of μ and b. According to Table 2, we noticed that the numerical solutions become smaller when the value of μ increases, therefore, the initial condition is established. Figure 2 shows the matching between the upper and lower bounds for $b = \mu = 1$. All results show that the optimal control leads to zero, which is the expected optimal solution for the problem (4.1)-(4.3). These behaviors indicate that this generalization of the PMP approach for fractional interval models has the potential to be used satisfactorily by researchers.

TABLE 1. Comparing results for Example 4.1 with $b = 0.5$ and $\mu = 1$.

	fuzzy PMP [16]		fuzzy PMP [30]		our approach	
t	**x**	**u**	**x**	**u**	**x**	**u**
0.0	[1.0000, 1.0000]	[−0.0756, −0.5830]	[1.0000, 1.0000]	[−0.5830, −0.0756]	[1.0000, 1.0000]	[−0.4001, −0.1000]
0.1	[0.8122, 0.8930]	[−0.0923, −0.6128]	[0.8122, 0.8930]	[−0.6128, −0.0923]	[0.7833, 0.9024]	[−0.3273, −0.0818]
0.2	[0.6550, 0.7882]	[−0.1113, −0.6432]	[0.6550, 0.7882]	[−0.6432, −0.1113]	[0.6142, 0.8091]	[−0.2667, −0.0666]
0.3	[0.5251, 0.6855]	[−0.1360, −0.6762]	[0.5251, 0.6855]	[−0.6762, −0.1360]	[0.4749, 0.7196]	[−0.2154, −0.0538]
0.4	[0.4162, 0.5844]	[−0.1661, −0.7108]	[0.4162, 0.5844]	[−0.7108, −0.1661]	[0.3555, 0.6334]	[−0.1715, −0.0428]
0.5	[0.3240, 0.4848]	[−0.2029, −0.7473]	[0.3240, 0.4848]	[−0.7473, −0.2029]	[0.2500, 0.5500]	[−0.1334, −0.0333]
0.6	[0.2449, 0.3864]	[−0.2478, −0.7856]	[0.2449, 0.3864]	[−0.7856, −0.2478]	[0.1545, 0.4693]	[−0.1001, −0.0250]
0.7	[0.1755, 0.2889]	[−0.3026, −0.8259]	[0.1755, 0.2889]	[−0.8259, −0.3026]	[0.0666, 0.3908]	[−0.0706, −0.0176]
0.8	[0.1133, 0.1922]	[−0.3696, −0.8682]	[0.1133, 0.1922]	[−0.8682, −0.3696]	[−0.0154, 0.3143]	[−0.0445, −0.0111]
0.9	[0.0555, 0.0960]	[−0.4515, −0.9127]	[0.0555, 0.0960]	[−0.9127, −0.4515]	[−0.0929, 0.2397]	[−0.0211, −0.0052]
1.0	[0.0000, 0.0000]	[−0.5514, −0.9595]	[0.0000, 0.0000]	[−0.9595, −0.5514]	[−0.1667, 0.1667]	[−0.0000, 0.0000]

Example 4.2. Find the optimal solution of the following interval optimal control problem:

$$minimize_{\mathbf{u}(.)} \quad \mathbf{I}(\mathbf{x}(t), \mathbf{u}\,(t)\,, t) = \frac{1}{2}\int_0^1 (\mathbf{x}^2(t) + \mathbf{u}^2)dt, \tag{4.9}$$

with the state dynamic:

$${}_0^C D_t^{\mu}\mathbf{x}(t) = \mathbf{u}(t) + t\mathbf{x}(t), \tag{4.10}$$

and initial condition:

$$\mathbf{x}(0) = [b, 2-b], \quad 0 \le b \le 1. \tag{4.11}$$

First, using Theorem 3.5, we obtain the necessary conditions for the optimization of FIOCP (4.9)-(4.11) as follows:

$${}_t^C D_1^{\mu}\mathbf{p}(t) = -\Big(t\mathbf{p}(t) + \mathbf{x}(t)\Big), \tag{4.12}$$

TABLE 2. Comparing results for Example 4.1 with $\mu = 0.8, 0.9, 1$ and $b = 0.5$.

μ	0.8		0.9		1	
t	**x**	**u**	**x**	**u**	**x**	**u**
0.0	$[1.0000, 1.0000]$	$[-0.4001, -0.1000]$	$[1.0000, 1.0000]$	$[-0.4001, -0.1000]$	$[1.0000, 1.0000]$	$[-0.4001, -0.1000]$
0.1	$[0.7122, 0.8784]$	$[-0.3143, -0.0785]$	$[0.7474, 0.8915]$	$[-0.3215, -0.0803]$	$[0.7833, 0.9024]$	$[-0.3273, -0.0818]$
0.2	$[0.5464, 0.7839]$	$[-0.2554, -0.0638]$	$[0.5804, 0.7979]$	$[-0.2616, -0.0653]$	$[0.6142, 0.8091]$	$[-0.2667, -0.0666]$
0.3	$[0.4183, 0.6976]$	$[-0.2074, -0.0518]$	$[0.4478, 0.7101]$	$[-0.2119, -0.0529]$	$[0.4749, 0.7196]$	$[-0.2154, -0.0538]$
0.4	$[0.3105, 0.6162]$	$[-0.1666, -0.0416]$	$[0.3352, 0.6264]$	$[-0.1694, -0.0423]$	$[0.3555, 0.6334]$	$[-0.1715, -0.0428]$
0.5	$[0.2153, 0.5385]$	$[-0.1310, -0.0327]$	$[0.2356, 0.5460]$	$[-0.1325, -0.0331]$	$[0.2500, 0.5500]$	$[-0.1334, -0.0333]$
0.6	$[0.1287, 0.4636]$	$[-0.0994, -0.0248]$	$[0.1453, 0.4682]$	$[-0.1000, -0.0249]$	$[0.1545, 0.4693]$	$[-0.1001, -0.0250]$
0.7	$[0.0484, 0.3909]$	$[-0.0710, -0.0177]$	$[0.0617, 0.3927]$	$[-0.0711, -0.0177]$	$[0.0666, 0.3908]$	$[-0.0706, -0.0176]$
0.8	$[-0.0273, 0.3201]$	$[-0.0453, -0.0113]$	$[-0.0168, 0.3192]$	$[-0.0450, -0.0112]$	$[-0.0154, 0.3143]$	$[-0.0445, -0.0111]$
0.9	$[-0.0994, 0.2510]$	$[-0.0217, -0.0054]$	$[-0.0914, 0.2474]$	$[-0.0215, -0.0053]$	$[-0.0929, 0.2397]$	$[-0.0211, -0.0052]$
1.0	$[-0.1686, 0.1832]$	$[0.0000, 0.0000]$	$[-0.1627, 0.1771]$	$[0.0000, 0.0000]$	$[-0.1667, 0.1667]$	$[0.0000, 0.0000]$

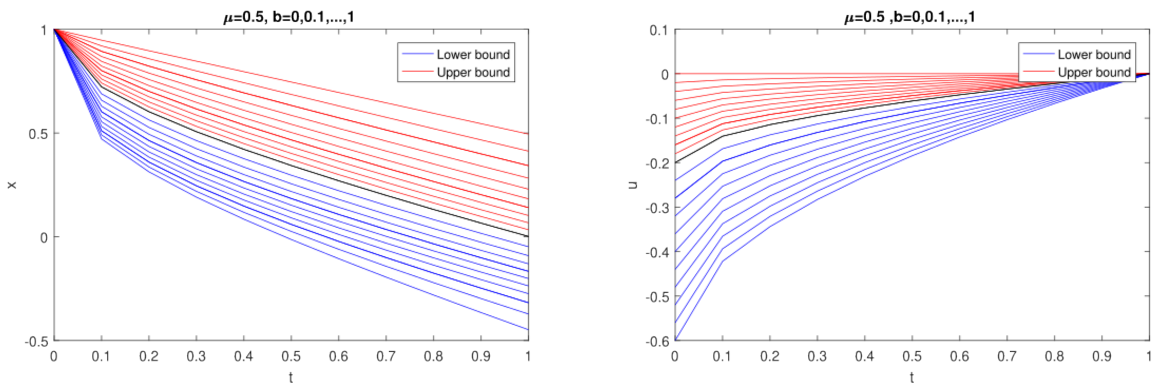

FIGURE 1. The state and control functions for Example 4.1 with $\mu = 0.5$ and $b \in [0, 1]$.

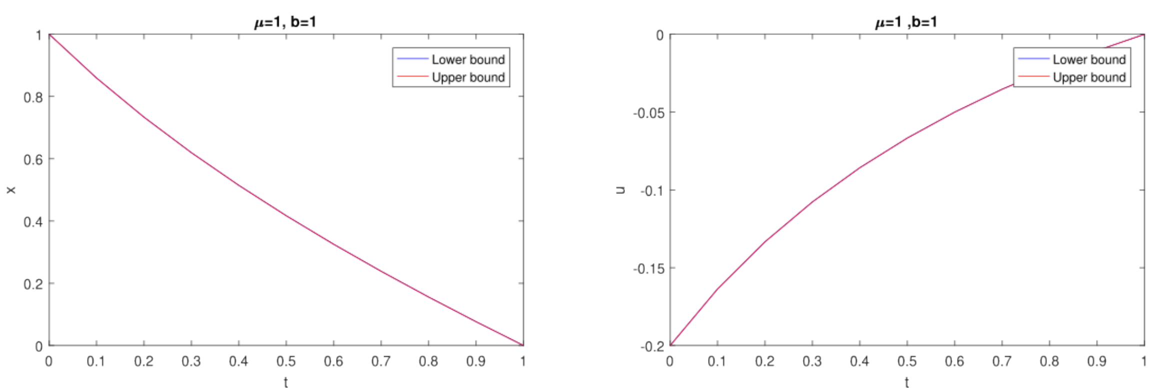

FIGURE 2. The state and control functions for Example 4.1 with $b = \mu = 1$.

$$ {}_{0}^{C}D_t^{\mu}\mathbf{x}(t) = t\,\mathbf{x}(t) + \mathbf{u}(t), \tag{4.13}$$

$$\mathbf{u}(t) + \mathbf{p}(t) = 0, \tag{4.14}$$

in which $\mathbf{p}(1) = 0$. Equation (4.14) provides $\mathbf{u}(t) = -\mathbf{p}(t)$. Due to the numerical approximation of Caputo-type derivatives described in Equations (2.2) and (2.3), we can converted Equations (4.13) and (4.12) to the following algebraic equations:

$$\frac{h^{-\mu}}{\Gamma(2-\mu)} \sum_{i=0}^{j} a_{i,j}\Big(\mathbf{x}_{j-i}(h) - \mathbf{x}_0\Big) = t_j\, \mathbf{x}_j(h) - \mathbf{p}_j(h), \qquad j = 1, ..., N,$$

$$\frac{h^{-\mu}}{\Gamma(2-\mu)} \sum_{i=j}^{N} a_{i-j,j}\Big(\mathbf{p}_N - \mathbf{p}_i(h)\Big) = -t_j\, \mathbf{p}_j(h) - \mathbf{x}_j(h), \qquad i = 0, 1, ..., N-1.$$

To obtain the optimal numerical solutions of these equations, we use the interval Newton method [18] with the relevant boundary conditions and the finite number of iterations (N=10). Although this problem previously addressed in [29], the authors solely presented the ideas of robust and non-robust solutions through the application of fuzzy PMP methodology. The evaluated state and control functions for different values of b and μ are presented in Table 3, Table 4, Figure 3 and Figure 4. These results indicate that the obtained approximate solutions for FIOCP (4.9)-(4.11) perform well for $\mu \in (0, 1]$ and $b \in [0, 1]$. Our approach is validated and effective, as confirmed by the figures depicted through the proposed method.

Table 3. Comparing results for Example 4.2 with $b = 0, 0.5, 1$ and $\mu = 0.77$.

b	0		0.5		1	
t	$\mathbf{x}$	$\mathbf{u}$	$\mathbf{x}$	$\mathbf{u}$	$\mathbf{x}$	$\mathbf{u}$
0.0	$[0.0000, 2.0000]$	$[-0.5000, -0.5000]$	$[0.5000, 1.5000]$	$[-0.5000, -0.5000]$	1.0000	−0.5000
0.1	$[0.0000, 1.5296]$	$[-0.4501, -0.3073]$	$[0.3858, 1.1577]$	$[-0.4177, -0.3436]$	0.7647	−0.3897
0.2	$[0.0000, 1.2100]$	$[-0.4001, -0.2233]$	$[0.3054, 0.9165]$	$[-0.3534, -0.2618]$	0.6049	−0.3165
0.3	$[0.0000, 0.9366]$	$[-0.3500, -0.1681]$	$[0.2363, 0.7091]$	$[-0.2966, -0.2033]$	0.4682	−0.2572
0.4	$[0.0000, 0.6841]$	$[-0.3000, -0.1276]$	$[0.1724, 0.5173]$	$[-0.2450, -0.1579]$	0.3420	−0.2069
0.5	$[0.0000, 0.4370]$	$[-0.2501, -0.0960]$	$[0.1100, 0.3301]$	$[-0.1974, -0.1208]$	0.2185	−0.1629
0.6	$[0.0000, 0.1821]$	$[-0.2001, -0.0703]$	$[0.0457, 0.1373]$	$[-0.1530, -0.0897]$	0.0910	−0.1238
0.7	$[-0.0970, 0.0000]$	$[-0.1501, -0.0487]$	$[-0.0729, -0.0242]$	$[-0.1115, -0.0629]$	−0.0484	−0.0886
0.8	$[-0.4278, 0.0000]$	$[-0.1001, -0.0302]$	$[-0.3205, -0.1068]$	$[-0.0723, -0.0394]$	−0.2139	−0.0565
0.9	$[-0.8854, 0.0000]$	$[-0.0501, -0.0142]$	$[-0.6603, -0.2200]$	$[-0.0353, -0.0186]$	−0.4426	−0.0271
1.0	$[-2.0000, 0.0000]$	$[-0.0001, 0.0000]$	$[-0.3639, -0.2746]$	$[-0.0001, 0.0000]$	−1.0000	0.0000

Example 4.3. In this example, we consider the fractional interval model of continuously stirred tank reactor in the form below:

$$\begin{aligned} &{}_0^C D_t^{\mu} \mathbf{x}_1(t) = -\mathbf{k}_1 \mathbf{x}_1(t) - \mathbf{k}_3 \mathbf{x}_1^2(t) + (C_{AF} - \mathbf{x}_1(t))\mathbf{u}(t), \\ &{}_0^C D_t^{\mu} \mathbf{x}_2(t) = \mathbf{k}_1 \mathbf{x}_1(t) - \mathbf{k}_2 \mathbf{x}_2(t) + \mathbf{x}_2(t)\mathbf{u}(t), \\ &\mathbf{x}_1(0) = 0.189, \quad \mathbf{x}_2(0) = 0.0936, \end{aligned} \tag{4.15}$$

where $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are the concentration of substances, C_{AF} signify the concentration of $\mathbf{x}_1(t)$ and $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ are reaction rate constants. These parameters take the values $C_{AF} = 10$, $\mathbf{k}_1 = [50 - b, 50 + b], \mathbf{k}_2 = [100 - b, 100 + b]$, $\mathbf{k}_3 = [10 - b, 10 + b]$ from the experimental analysis in [13, 20]. In order to solve the optimal tracking control problem mentioned above, here we consider the performance index as follows:

$$\mathbf{J}(\mathbf{u}) = \int_0^1 (\mathbf{x}_1^2(t) + \mathbf{x}_2^2(t) + \mathbf{u}^2(t))dt. \tag{4.16}$$

TABLE 4. Comparing results for Example 4.2 with $\mu = 0.8, 0.9, 1$ and $b = 0.5$.

μ	0.8		0.9		1	
t	**x**	**u**	**x**	**u**	**x**	**u**
0.0	$[0.5000, 1.5000]$	$[-0.5000, -0.5000]$	$[0.5000, 1.5000]$	$[-0.5000, -0.5000]$	$[0.5000, 1.5000]$	$[-0.5000, -0.5000]$
0.1	$[0.3858, 1.1577]$	$[-0.4195, -0.3485]$	$[0.3959, 1.1880]$	$[-0.4246, -0.3630]$	$[0.4040, 1.2123]$	$[-0.4286, -0.3749]$
0.2	$[0.3054, 0.9165]$	$[-0.3551, -0.2654]$	$[0.3144, 0.9434]$	$[-0.3598, -0.2764]$	$[0.3218, 0.9656]$	$[-0.3637, -0.2857]$
0.3	$[0.2363, 0.7091]$	$[-0.2979, -0.2057]$	$[0.2429, 0.7288]$	$[-0.3015, -0.2129]$	$[0.2482, 0.7448]$	$[-0.3044, -0.2187]$
0.4	$[0.1724, 0.5173]$	$[-0.2458, -0.1593]$	$[0.1765, 0.5298]$	$[-0.2483, -0.1635]$	$[0.1797, 0.5392]$	$[-0.2501, -0.1666]$
0.5	$[0.1100, 0.3301]$	$[-0.1979, -0.1216]$	$[0.1120, 0.3363]$	$[-0.1993, -0.1237]$	$[0.1133, 0.3400]$	$[-0.2001, -0.1250]$
0.6	$[0.0457, 0.1373]$	$[-0.1533, -0.0900]$	$[0.0463, 0.1390]$	$[-0.1539, -0.0908]$	$[0.0463, 0.1392]$	$[-0.1539, -0.0909]$
0.7	$[-0.0729, -0.0242]$	$[-0.1116, -0.0630]$	$[-0.0729, -0.0242]$	$[-0.1116, -0.0630]$	$[-0.0722, -0.0240]$	$[-0.1112, -0.0625]$
0.8	$[-0.3205, -0.1068]$	$[-0.0723, -0.0394]$	$[-0.3163, -0.1054]$	$[-0.0720, -0.0391]$	$[-0.3046, -0.1015]$	$[-0.0715, -0.0384]$
0.9	$[-0.6603, -0.2200]$	$[-0.0352, -0.0186]$	$[-0.6361, -0.2120]$	$[-0.0350, -0.0183]$	$[-0.5730, -0.1909]$	$[-0.0345, -0.0178]$
1.0	$[-1.5100, -0.5029]$	$[-0.0001, 0.0000]$	$[-1.5498, -0.5082]$	$[-0.0001, 0.0000]$	$[-0.9800, -0.3899]$	$[-0.0001, 0.0000]$

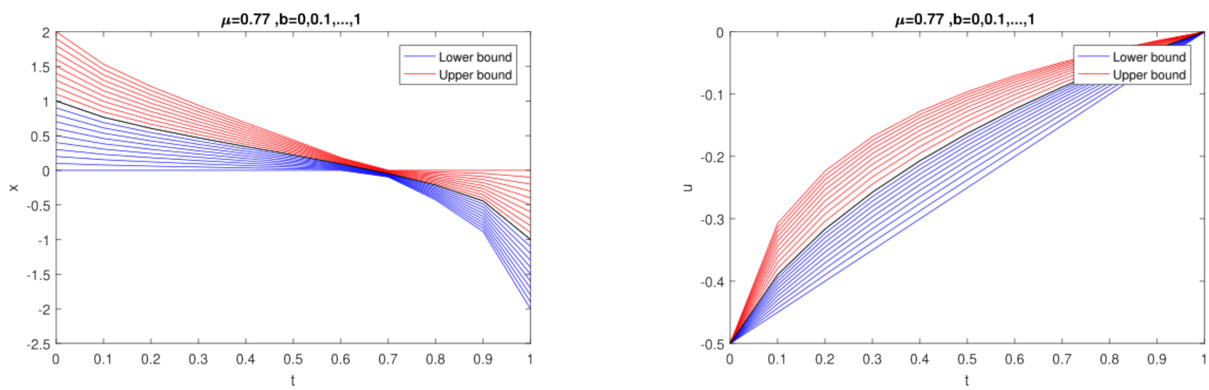

FIGURE 3. The state and control variables for Example 4.2 with $b \in [0, 1]$ and $\mu = 0.77$.

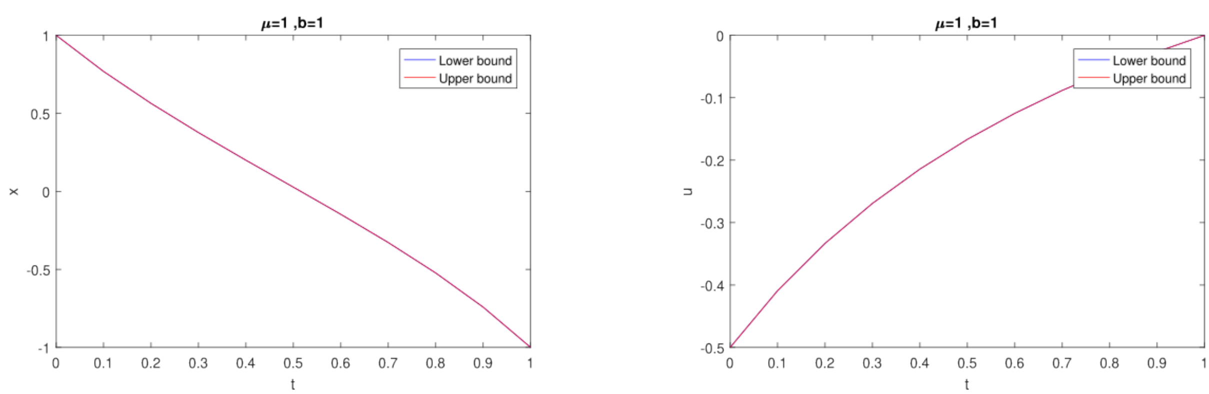

FIGURE 4. The state and control variables for Example 4.2 with $\mu = 1$ and $b = 1$.

The objective is to find the control input $\mathbf{u}(t)$ and the corresponding state trajectory $\mathbf{x}(t) = (\mathbf{x}_1(t), \mathbf{x}_2(t))$ for system (4.15), which minimizes the performance index (4.16). The approximate solution for this problem in the case of $b = 0, \mu = 1$ has been provided in [13, 20], in which the above-mentioned problem is reduced to a classical optimal control problem. Here, we apply the proposed method developed in Section 3 to solve this problem. So, from Theorem

3.5, we derive the following conditions:

$$
\begin{aligned}
&{}_t^C D_1^{\mu}\mathbf{x}_1(t) = -\mathbf{k}_1\mathbf{x}_1(t) - \mathbf{k}_3\mathbf{x}_1^2(t) + (C_{AF} - \mathbf{x}_1(t))\mathbf{u}(t),\\
&{}_t^C D_1^{\mu}\mathbf{x}_2(t) = \mathbf{k}_1\mathbf{x}_1(t) - \mathbf{k}_2\mathbf{x}_2(t) + \mathbf{x}_2(t)\mathbf{u}(t),\\
&{}_t^C D_1^{\mu}\mathbf{p}_1(t) = -\mathbf{p}_1(t)(\mathbf{k}_1 - 2\mathbf{k}_3 - \mathbf{u}(t)),\\
&{}_t^C D_1^{\mu}\mathbf{p}_2(t) = 2\mathbf{x}_2(t) - \mathbf{p}_2(t)(\mathbf{k}_2 - \mathbf{u}(t)),\\
&2\mathbf{u}(t) + \mathbf{p}_1(t)(C_{AF} - \mathbf{x}_1(t)) + \mathbf{p}_2(t)\mathbf{x}_2(t) = 0.
\end{aligned} \tag{4.17}
$$

Due to the numerical approximation of fractional derivative, Equations (4.17) convert to the following algebraic system:

$$
\begin{aligned}
&\frac{h^{-\mu}}{\Gamma(2-\mu)}\sum_{i=0}^{j} a_{i,j}\Big(\mathbf{x}_{1,j-i}(h) - \mathbf{x}_1(0)\Big) = -\mathbf{k}_1\mathbf{x}_{1,j}(h) - \mathbf{k}_3\mathbf{x}_{1,j}^2(t) + (C_{AF} - \mathbf{x}_{1,j}(h))\mathbf{u}(h), \qquad j = 1,...,N,\\
&\frac{h^{-\mu}}{\Gamma(2-\mu)}\sum_{i=0}^{j} a_{i,j}\Big(\mathbf{x}_{2,j-i}(h) - \mathbf{x}_2(0)\Big) = \mathbf{k}_1\mathbf{x}_{1,j}(h) - \mathbf{k}_2\mathbf{x}_{2,j}(h) + \mathbf{x}_{2,j}(h)\mathbf{u}(h), \qquad j = 1,...,N,\\
&\frac{h^{-\mu}}{\Gamma(2-\mu)}\sum_{i=j}^{N} a_{i-j,j}\Big(\mathbf{p}_{1,N} - \mathbf{p}_{1,i}(h)\Big) = -\mathbf{p}_{1,j}(h)(\mathbf{k}_1 - 2\mathbf{k}_3 - \mathbf{u}(h)), \qquad i = 0,1,...,N-1.\\
&\frac{h^{-\mu}}{\Gamma(2-\mu)}\sum_{i=j}^{N} a_{i-j,j}\Big(\mathbf{p}_{2,N} - \mathbf{p}_{2,i}(h)\Big) = 2\mathbf{x}_{2,j}(h) - \mathbf{p}_{2,j}(t)(\mathbf{k}_2 - \mathbf{u}(h)), \qquad i = 0,1,...,N-1.
\end{aligned}
$$

Simulation results including the cost value are summarized in Table 5. Some comparative results are included between our numerical findings (in the conventional fractional interval sense) and those of the recent methods [13] for $b = 0, \mu = 1, 0.93.0, 9$ in this table. The reported results in this table confirm the validity and accuracy of our numerical achievements in comparison with the available results. Figure 5 displays the state and control solutions of the classical optimal control problem with $b = 0, \mu = 1$. From the perspective view of this figure, the lower and upper bounds are coincident for $b = 0, \mu = 1$. Also, Figure 6 plots these solutions for $\mu = 0.93$ and different choices of b. As can be seen, the numerical solution goes to the classic integer solution when $\mu \to 1$ and $b \to 0$.

TABLE 5. The cost value **J** for Example 4.3 with particular choices of derivative operator and $b = 0$.

μ	The proposed method	Method in [13]
0.9	0.004461	0.010770
0.93	0.000034	0.000428
1	0.000000	0.000080

5. Conclusion

An efficient method based on the synergy between the fractional interval calculus of variations and constrained interval arithmetic was presented to gain the optimal control for fractional interval dynamical systems. The main aim of this approach is based on the PMP conditions to convert the FIOCP into a system of FIDEs. The Numerical solution of the introduced system utilizes a high accuracy for the state and control parameters. Simulations with different values of μ and b show the effectiveness of the proposed method and support the theoretical findings. Furthermore, it was found that the results obtained by this approach are better in contrast to other literature. The proposed approximation is easy to implement and capable of solving FIOCPs. The results of this paper present an encouraging view for future discussion. Indeed, for the recently developed of fractional equations, the theoretical justification on the merit of this method will enhance the engineers, physicists, and other scientists to use these results for their practical objectives.

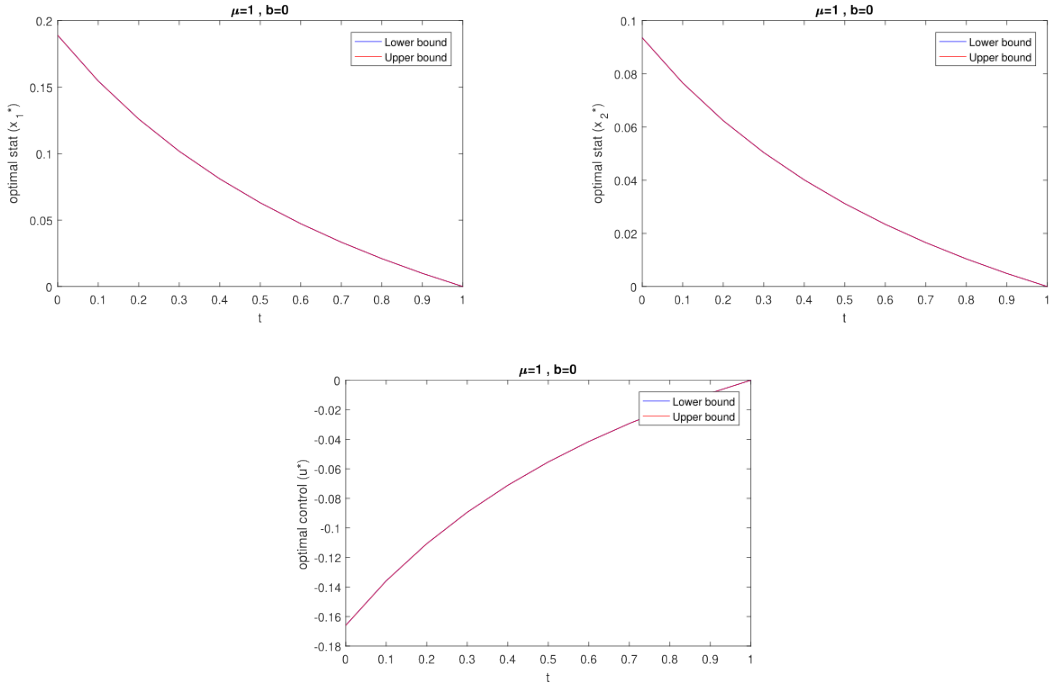

FIGURE 5. The state and control variables for Example 4.3 with $\mu = 1$ and $b = 0$.

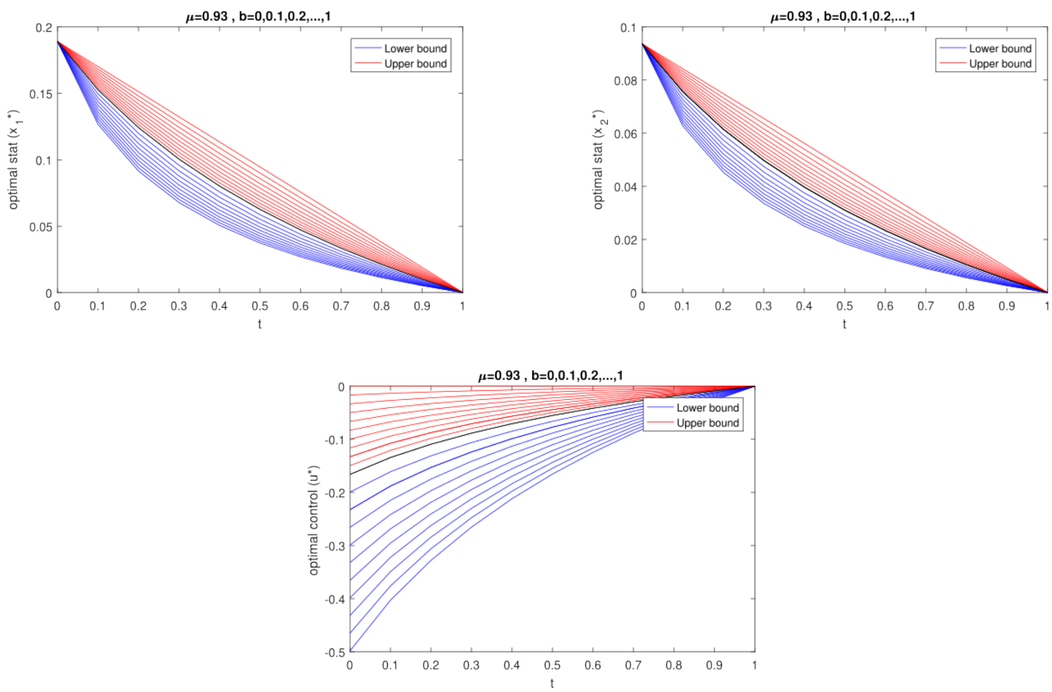

FIGURE 6. The state and control variables for Example 4.3 with $b \in [0, 1]$ and $\mu = 0.93$ which the black curve corresponds to $b = 0$.

References

[1] R. P. Agarwal, S. Arshad, D. ÓRegan, and V. Lupulescu, *A Schauder fixed point theorem in semilinear spaces and applications*, Fixed point Theorey Appl., *2013* (2013), 1-13.

[2] R. P. Agarwal, S. Arshad, D. ÓRegan, and V. Lupulescu, *Fuzzy fractional integral equations under compactness type condition*, Fract. Calc. Appl. Anal., *15* (2012), 572-590.

[3] R. P. Agarwal, V. Lakshmikantham, and J. J. Nieto, *On the concept of solution for fractional differential equations with uncertainty*, Nonlinear Anal., *72*(6) (2010), 2859-2862.

[4] Z. Alijani, D. Baleanu, B. Shiri, and G. C. Wu, *Spline collocation methods for systems of fuzzy fractional differential equations*, Chaos, Solitons Fractals, *131* (2020), 109510.

[5] M. Alipour and S. Soradi Zeid, *Optimal control of Volterra integro-differential equations based on interpolation polynomials and collocation method*, Computational Methods for Differential Equations, *11* (2023), 52-64.

[6] M. Allahdadi, S. Soradi-Zeid, and T. Shokouhi, *Accurate solutions of interval linear quadratic regulator optimal control problems with fractional-order derivative*, Soft Comput., (2023), 1-18.

[7] T. Allahviranloo, S. Salahshour, and S. Abbasbandy, *Explicit solutions of fractional differential equations with uncertainty*, Soft Comput., *16* (2012), 297-302.

[8] T. V. An, N. D. Phu, and N. Van Hoa, *A survey on non-instantaneous impulsive fuzzy differential equations involving the generalized Caputo fractional derivative in the short memory case*, Fuzzy Sets Syst., *443* (2022), 160-197.

[9] S. Arshad and V. Lupulescu, *Fractional differential equation with the fuzzy initial condition*, Electron. J. Differ. Equations, *2011* (2011), 1-8.

[10] S. Arshad and V. Lupulescu, *On the fractional differential equations with uncertainty*, Nonlinear Anal., Theory Methods Appl., *74* (2011), 3685-3693.

[11] F. B. Bergamaschi and R. H. N. Santiago, *A Study on Constrained Interval Arithmetic*. In Explainable AI and Other Applications of Fuzzy Techniques: Proceedings of the 2021 Annual Conference of the North American Fuzzy Information Processing Society, NAFIPS 2021, Springer International Publishing, (2022), 26-36.

[12] Y. Chalco-Cano, W. A. Lodwick, and B. Bede, *Single level constraint interval arithmetic,* Fuzzy Sets Syst., *257* (2014), 146-168.

[13] R. Dehghan and M. Kianpour,, *Using positive semi-definite optimization to solve the problem of optimal control of control systems of isothermal continuous stirred tank reactors*, J. Oper. Res. Appl., *52* (2017), 1-13.

[14] K. Diethelm, N. J. Ford, A. D. Freed, and Y. Luchko, *Algorithms for the fractional calculus: a selection of numerical methods*, Comput. Methods Appl. Mech. Eng., *194* (2005), 743-773.

[15] N. Eqra, R. Vatankhah, and M. Eghtesad, *A novel adaptive multi-critic based separated-states neuro-fuzzy controller: Architecture and application to chaos control*, ISA. transactions, *111* (2021), 57-70.

[16] B. Farhadinia, *Pontryagin's minimum principle for fuzzy optimal control problems*, Iran. J. Fuzzy Syst., *11* (2014), 27-43.

[17] J. Garloff, *Interval Gaussian elimination with pivot tightening*, SIAM J. Matrix Anal. Appl., *30*(4) (2009), 1761-1772.

[18] E. R. Hansen and R. I. Greenberg, *An interval Newton method*, SIAM J. Matrix Anal. Appl., *12*(2-3) (1983), 89-98.

[19] M. K. Kar, A. K. Singh, S. Kumar, and B. Rout, *Application of Fractional-Order PID Controller to Improve Stability of a Single-Machine Infinite-Bus System*, J. Instit. Engin., (India): Series B, *105*(1) (2024), 77-92.

[20] S. Kuntanapreeda and P. M. Marusak, *Nonlinear extended output feedback control for CSTRs with van de Vusse reaction*, Computers & Chemical Engineering, *41* (2012), 10-23.

[21] W. A. Lodwick, *Constrained interval arithmetic*. Denver: University of Colorado at Denver, Center for Comput. Math., 1999.

[22] W. A. Lodwick, *Interval and fuzzy analysis: A unified approach*, Adv. Imaging Electron. Phys., *148* (2007), 76192.

[23] V. Lupulescu, *Fractional calculus for interval-valued functions*, Fuzzy Sets Syst., *265* (2014), 63-85.

[24] V. Lupulescu and N. Van Hoa, *Interval Abel integral equation*, Soft Comput., *21*(10) (2017), 2777-2784.

[25] S. Maitama and W. Zhao, *Homotopy analysis Shehu transform method for solving fuzzy differential equations of fractional and integer order derivatives*, Comput. Appl. Math., *40* (2021), 1-30.
[26] S. Markov, *Calculus for interval functions of a real variable*, Comput., *22*(4) (1979), 325-337.
[27] M. Mazandarani and A. V. Kamyad, *Modified fractional Euler method for solving fuzzy fractional initial value problem*, Commun. Nonlinear Sci. Numer. Simul., *18*(1) (2013), 12-21.
[28] S. Mohammadi and S. R. Hejazi, *Presentation of the model and optimal control of non-linear fractional-order chaotic system of glucose-insulin*, Comput. Methods Biomech. Biomed. Eng., *27* (7) (2024), 836-848.
[29] A. Mohammed, Z. T. Gong, and M. Osman, *A Numerical Technique for Solving Fuzzy Fractional Optimal Control Problems*, J. Comput. Anal. Appl., *3* (2021), 413-430.
[30] A. M. Mustafa, Z. T. Gong, and M. Osman, *The solution of fuzzy variational problem and fuzzy optimal control problem under granular differentiability concept*, Int. J. Comput. Math., *98*(8) (2021), 1495-520.
[31] M. D. Patil, K. Vadirajacharya, and S. W. Khubalkar, *Design and tuning of digital fractional-order PID controller for permanent magnet DC motor*, IETE. J. Research, *69*(7) (2023), 4349-4359.
[32] S. Pooseh, R. Almeida, and D. F. Torres, *Fractional order optimal control problems with free terminal time*, J. Ind. Manage. Optim., *10*(2)(2014), 363-381.
[33] S. Sabermahani and Y. Ordokhani, *Solving distributed-order fractional optimal control problems via the Fibonacci wavelet method*, J. Vib. Control, *30*(1-2) (2024), 418-432.
[34] S. Salahshour, T. Allahviranloo, and S. Abbasbandy, *Solving fuzzy fractional differential equations by fuzzy Laplace transforms*, Commun. Nonlinear Sci. Numer. Simul., *17*(3) (2012), 1372-1381.
[35] S. Salahshour, T. Allahviranloo, S. Abbasbandy, and D. Baleanu, *Existence and uniqueness results for fractional differential equations with uncertainty*, Adv. Differ. Equations, *2012* (2012), 1-12.
[36] N. Van Hoa, *Existence results for extremal solutions of interval fractional functional integro-differential equations*, Fuzzy Sets Syst., *347* (2018), 29-53.
[37] H. Vu, V. Lupulescu, and N. Van Hoa, *Existence of extremal solutions to interval-valued delay fractional differential equations via monotone iterative technique*, J. Intell. Fuzzy Syst., *34*(4) (2018), 2177-2195.
[38] H. Wang, R. Rodrguez-Lpez, and A. Khastan, *On the stopping time problem of interval-valued differential equations under generalized Hukuhara differentiability*, Inf. Sci., *579* (2021), 776-795.
[39] T. Wang, A. San-Millan, and S. S. Aphale, *Quantifying the performance enhancement facilitated by fractional-order implementation of classical control strategies for nanopositioning*, ISA. transactions, 2024.
[40] X. Xia and T. Li, *A fuzzy control model based on BP neural network arithmetic for optimal control of smart city facilities*, Pers. Ubiquitous Comput., *23* (2019), 453-463.
[41] C. Yang, W. Lu, and Y. Xia, *Positioning accuracy analysis of industrial robots based on non-probabilistic time-dependent reliability*, IEEE. Trans. Reliab., 2023.
[42] C. Yang, W. Lu, and Y. Xia, *Reliability-constrained optimal attitude-vibration control for rigid-flexible coupling satellite using interval dimension-wise analysis*, Reliab. Eng. Syst. Saf., *237* (2023), 109382.
[43] C. Yang and Y. Xia, *Interval Pareto front-based multi-objective robust optimization for sensor placement in structural modal identification,* Reliab. Eng. Syst. Saf., *242* (2024), 109703.
[44] C. Yang and Y. Xia, *Interval uncertainty-oriented optimal control method for spacecraft attitude control*, IEEE. Trancs. Aerosp. Electron. Syst., 2023.
[45] T. Zhou, Z. Yuan, and T. Du, *On the fractional integral inclusions having exponential kernels for interval-valued convex functions*, Math. Sci., *17*(2) (2023), 107-120.