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Integrated pests management and food security: A mathematical analysis

Abayomi Ayotunde Ayoade[1] **and Srinivasarao Thota**[2,*]
[1]Department of Mathematics, University of Lagos, Lagos, Nigeria.
[2]Department of Mathematics, Amrita School of Physical Sciences, Amrita Vishwa Vidyapeetham, Amaravati, Andhra Pradesh–522503, India.

Abstract

The basic necessities of life are food, shelter and clothing. Food is more necessary because the existence of life depends on food. In order to foster global food security, integrated pest management (IPM), an environmentally-friendly program, was designed to maintain the density of pest population in the equilibrium level below the economic damage. For years, mathematics has been an ample tool to solve and analyze various real-life problems in science, engineering, industry and so on but the use of mathematics to quantify ecological phenomena is relatively new. While efforts have been made to study various methods of pest control, the extent to which pests' enemies as well as natural treatment can reduce crop damage is new in the literature. Based on this, deterministic mathematical models are designed to investigate the prey-predator dynamics on a hypothetical crop field in the absence or presence of natural treatment. The existence and uniqueness of solutions of the models are examined using Derrick and Grossman's theorem. The equilibria of the models are derived and the stability analysed following stability principle of differential equations and Bellman and Cooke's theorem. The theoretical results of the models are justified by a means of numerical simulations based on a set of reasonable hypothetical parameter values. Results from the simulations reveal that the presence of pests' enemies on a farm without application of natural treatment may not avert massive crop destruction. It is also revealed that the application of natural treatment may not be enough to keep the density of the pest population below the threshold of economic damage unless the rate of application of natural treatment exceeds the growth rate of the pest.

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1. Introduction

Food security is defined as "a condition that occurs when every individual, at all times, has physical, social and economic access to sufficient, safe and nutritious foods that meet their dietary needs and food preferences for a balanced and healthy life" [15]. The definition encompasses four dimensions:

(1) availability or ample supply of food;
(2) affordability or food accessibility;
(3) food safety and quality of food utilization;
(4) supply stability without shortages or seasonal fluctuations [18].

Food security is threatened by many factors worldwide nowadays. Food damage due to plant diseases and pests are the main risks to food security, especially in developing countries [16]. Because pests cannot be controlled indiscriminately due to some inherent factors like human health, financial cost and environmental impact, there have been a number of integrated methods, such as Integrated Pest Management (IPM) to combat the menace of pests infestation and ensure food security.

The idea of IPM, an environmentally-friendly technique for handling pests, has been in use for a very long time (since the 1950s) [6, 26]. IPM is a science-based and sustainable technique that adopts chemical, biological, physical

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∗ Corresponding author. Email: t_srinivasarao@av.amrita.edu.

and cultural tools to minimize threats from pests in such a way that reduces general economic, environmental and health risks [45]. The theory of pest control has been dropped for pest management for years because a stable technique to checking the population of pests below the levels of economic damage is more reasonable and obtainable than eradicating or eliminating pests for economic and environmental reasons. IPM is therefore a method to manage pest population in a socially acceptable, economically viable, and environmentally secure manner.

IPM can be achieved through host plant resistance, cultural control, biological control, behavioral control, physical or mechanical control, microbial control or chemical control [9]. Host plant resistance, a major option in IPM, involves the application of pest-tolerant and pest-resistant cultivars that can tolerate or resist pest infestation and at the same time, reduce crop's vulnerability to pest damage. Cultural control involves implementing suitable agronomic methods that reduce or avoid pest damage or infestation such as selecting sterile seeds or planting materials, adjusting planting date, sanitation practices, crop rotation, intercropping, row spacing, fertilizer application, regulating irrigation practices just to mention few [11, 20, 27, 29, 34]. Biological control, on the other hand, is the use of pests' natural enemies such as lacewings and parasitic wasps to reduce their population below the threshold of economic damage. Pests' natural enemies can be induced either commercially or through conservation to achieve the purpose. Behavioral control involves the study of pests behaviours to manage their population through traps, baits and mating disruption [35]. Mechanical or physical control is similar to behavioural control and it involves the use of various mechanical or physical approaches to pest reduction including hand-picking, weed control, solarisation or steam sterilization, traps for rodents, and physical or visual birds deterrents to mention but a few [10, 17]. Microbial control involves the use of entomopathogenic microsporidia, bacteria, nematodes, fungi, or viruses, and fermentation byproducts of some microbes against plant parasitic nematodes, arthropod pests, and plant pathogens [28]. Lastly, chemical control involves the use of both organic and synthetic chemicals to combat pest infestation. The use of organic chemicals to reduce pest population is termed natural treatment. Since the introduction of IPM theory in 1950s, a good number of mathematical models has appeared addressing different aspects of IPM-based applications based on prey-predator mechanism [2, 7, 13, 14, 21–23, 25, 30–33, 36–41, 46, 47]. Aside ecological phenomena, deterministic mathematical models have been explored to study epidemiological problems in relation to the impact of vaccination in controlling infectious diseases [19]. It has also been used to study human behaviours with respect to drinking habits [24]. For the ecological phenomena regarding food security, several models have been developed, a good number of these models focus different methods of pest control [5, 22, 30, 32, 33, 39, 42–44, 47].

In [23], the authors investigated the impact of a hybrid approach with the release of natural treatment and the application of chemical pesticides on the population of pests whereas in [2], the scholars studied the effect of pest control and farmers' practices in the rice ecosystem on environmental sustainability. In [25], the interaction between guava borers and natural treatment was investigated. It is worth mentioning that some of the previous studies considered the use of chemicals for pest control which is toxic not only to the environment but to humans. It is therefore necessary to propose a new method of pest control devoid of the use of harmful chemicals. Besides, few studies have been dedicated to the analysis of situations where the fates of crops are subjected to the interplay of pests and natural enemies when the pest population is not influenced and when it is influenced by the natural treatment.

The motivation of the present study therefore is to consider the extent to which pests' enemies as well as natural treatment can reduce crop damage using locusts (pests) and wasps (natural enemies) as a frame. The study seeks to evaluate two scenarios by developing and analyzing two models. One model is designed to examine the effect on crops when both pests and their natural enemies are allowed to operate under natural setting without interference in the ecosystem while the other model is designed to assess the effect on crops when natural treatment is used to further influence the population of pests and alter the agroecosystem. Locusts are considered as pests while wasps are considered as natural enemies. The application of garlic spray is considered as the natural treatment. As far as we know, the dynamic behavior of the prey-predator phenomenon in terms of locusts, wasps and garlic spray has not been considered. This aroused our interest towards the investigation of the phenomenon to foster food security. The major contributions of this paper to the discipline of Mathematical Ecology are as follows.

- Development of a new prey-predator model incorporating locusts as preys, wasps as predators, and garlic spray as natural treatment.
- The application of Bellman and Cooke's theorem to establish equilibria stability of a prey-predator model.

- Establishing that the presence of pest's natural enemies on a farm might not avert massive crop destruction.
- Establishing that the application of natural treatment must exceed certain critical level before the density of pest population could be kept below the threshold of economic damage.

The rest of the article is arranged as follows. In section 2, the impact of pests on crops when the natural treatment is not applied is analyzed while in section 3, the impact of pests on crops when there is natural treatment is investigated. In section 4, numerical simulations are conducted to justify the theoretical results in sections 2 and 3. Finally, the conclusion of the work comes in section 5.

2. Model 1: Pests Impact on Crops in the Absence of Natural Treatment

Model 1 is built on the ground that every pest has a natural enemy and when the natural treatment is not applied to influence the population of pests, the density of the pest population can still be affected by the natural enemies. For instance, locusts are major pests to crops in sub-Saharan Africa and the natural enemy of locusts are wasps. A mathematical model is therefore developed to examine the extent to which wasps can go in reducing the density of locust population below the equilibrium of economic damage. To capture the dynamics of prey-predator on the hypothetical crop field, the locust u and the wasp v are assumed to co-exist on the crop field. The crop field is the agroecosystem for the two organisms. The locusts are pests that feed on the hypothetical crops while the wasps are the natural enemies of the pests that feed on the locusts. If the interaction between the locusts and the wasps on the crop field is allowed to take place under natural setting, that is, if neither the population locusts nor the population wasps are influenced, then the interaction between the two creatures on the crop field can be quantified in terms of the following first-order nonlinear ordinary differential equations

$$\frac{du}{dt} = \alpha u - \gamma u^2 - \epsilon uv, \tag{2.1}$$

$$\frac{dv}{dt} = -\beta v + \omega uv. \tag{2.2}$$

As stated earlier, u and v are considered the populations of locusts and wasps respectively. One of the basic assumptions of prey-predator models is that the population of preys is always higher than the population of predators and the model described by a system of Equations (2.1) and (2.2) is built around this general assumption. In Equation (2.1), if the population of locusts grows at rate α then αu is the term at which the locusts grow in the Malthusian way. Since the locusts compete for space, food and other things for survival, γ is considered the locusts carrying capacity rate of reduction due to internal conflict among the locusts. Also, since the locusts and wasps interaction is only for survival then ϵ is considered the rate of effect of wasps on the locusts. Notice that the terms γu^2 and ϵuv in Equation (2.1) are negative. It is as a result of competition between locust and locust (for γu^2) and locust and wasp (for ϵuv) which results in a reduction in the population of locusts. In other words, the term γu^2 is negative because of the internal conflict between the locusts while the term ϵuv is negative because the competition between the locusts and the wasps is to reduce the population of the locusts.

In Equation (2.2), β is the rate at which the wasps grow while ω is the rate of effect of locusts on the wasps. βv is negative because wasp's population decreases exponentially in the absence or scarceness of locusts while ωuv is positive because the competition between the locust and wasp is to increase the population of the wasp. Therefore, the model is a duality of first-order nonlinear differential equations. Equation (2.1) describes the changes in the population of the locusts with time while Equation (2.2) describes the changes in the population of the wasp with time. The changing and intensity of the dynamics of competition between the two creatures when there is no interference on the crop field can be driven by Equations (2.1) and (2.2).

2.1. **Existence and Uniqueness of Solutions.**

The validity and usability of a mathematical model depend on whether the model has a solution and whether the solution is unique. Refer to [1] for the definitions of the existence and uniqueness of solutions and Lipschitz criteria. Hence, the system of equations representing the dynamics of the locusts and wasps on the crop field when there is no application of natural treatment shall be subjected to the integrity test by verifying the existence and uniqueness of solutions for the model following Lipschitz criteria. Let

$$f_1 = \alpha u - \gamma u^2 - \epsilon uv, \tag{2.3}$$

$$f_2 = -\beta v + \omega uv. \tag{2.4}$$

Theorem 2.1. *[12] Let D^1 denotes the region*

$$|t - t_0| \leq a,\ \|x - x_0\| \leq b,\ x = (x_1, x_2, ..., x_n),\ x_0 = (x_{10}, x_{20}, ..., x_{n0}),$$

and suppose that $f(t, x)$ satisfies the Lipschitz condition

$$\|(t, x_1) - f(t, x_2)\| \leq k\|x_1 - x_2\|.$$

Whenever the pairs (t, x_1) and (t, x_2) belong to $D^{'}$, where k is a positive constant, then there is a constant $\delta > 0$ such that there exists a unique continuous vector solution $x(t)$ of the system in the interval $t - t_0 \leq \delta$. It is important to note that the condition is satisfied by the requirement that $\frac{\partial f_i}{\partial x_j}, i = 1, 2, ...,$ be continuous and bounded in $D^{'}$.

We now return to the system of Equations (2.1) and (2.2) and we are interested in the region $0 \leq \alpha \leq R$, we look for a bounded solution in this region whose partial derivatives satisfy $\delta \leq \alpha \leq 0$, where α and δ are positive constants.

Theorem 2.2. *Let D' denote the region $0 \leq \alpha \leq R$. Then the system of Equations (2.1) and (2.2) has a unique solution if it is established that $\frac{\partial f_i}{\partial x_j}, i = 1, 2$ are continuous and bounded in D'.*

From Equation (2.1), we obtain the partial derivatives below

$$\left|\frac{\partial f_1}{\partial u}\right| = |\alpha - 2\gamma u - \epsilon v| < \infty; \ \left|\frac{\partial f_1}{\partial v}\right| = |-\epsilon u| < \infty.$$

The above partial derivatives exist, are continuous and bounded. Similarly, for Equation (2.2), we show that

$$\left|\frac{\partial f_2}{\partial u}\right| = |\omega v| < \infty; \ \left|\frac{\partial f_2}{\partial v}\right| = |-\beta + \omega u| < \infty.$$

Since all the partial derivatives exist and are finite (bounded and defined), then the system of Equations (2.1) and (2.2) exists and has a unique solution in $\mathbb{R}^2$.

2.2. **Equilibria and Stability of Model 1.** When the locusts and wasps are allowed to interact under natural setting without the application of natural treatment, the dynamics are at equilibrium when the rate of change of each variable with respect to time in the system of Equations (2.1) and (2.2) is reduced to zero. i.e.

$$\alpha u - \gamma u^2 - \epsilon uv = 0, \tag{2.5}$$

$$-\beta v + \omega uv = 0. \tag{2.6}$$

Solving Equation (2.6),

$$v = 0 \text{ or } u = \frac{\beta}{\omega}.$$

In Equation (2.5),

$$u(\alpha - \gamma u - \epsilon v) = 0 \Rightarrow u = 0 \text{ or } \alpha - \gamma u - \epsilon v = 0.$$

Each of $v = 0$ and $u = \frac{\beta}{\omega}$ shall be substituted into $\alpha - \gamma u - \epsilon v = 0$ to determine the equilibria points. When $v = 0$ then

$$\alpha - \gamma u - \epsilon v = 0 \Rightarrow \alpha - \gamma u = 0 \Rightarrow u = \frac{\alpha}{\gamma}.$$

Therefore, we have

$$(u, v) = \left(\frac{\alpha}{\gamma}, 0\right). \tag{2.7}$$

Also, when

$$u = \frac{\beta}{\omega} \text{ then } \alpha - \gamma u - \epsilon v = 0 \Rightarrow \alpha - \gamma\left(\frac{\beta}{\omega}\right) - \epsilon v = 0 \Rightarrow v = \frac{\alpha\omega - \beta\gamma}{\epsilon\omega}.$$

We have

$$(u, v) = \left(\frac{\beta}{\omega}, \frac{\alpha\omega - \beta\gamma}{\epsilon\omega}\right). \tag{2.8}$$

Hence, the equilibria points for the system of Equations (2.1) and (2.2) are $(0,0)$, $\left(\frac{\alpha}{\gamma}, 0\right)$, and $\left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma}{\epsilon\omega}\right)$.

Having derived equilibrium points of the model, the stability of the model at each equilibrium point could be studied by computing the Jacobian matrix of the model and evaluating it at each equilibrium point. The stability of a model following Jacobian approach depends on the signs of the roots of the characteristic equation of the Jacobian matrix. The equilibrium of the model is stable if all the roots of the characteristic equation of the Jacobian matrix are negative otherwise the equilibrium of the model is unstable. The Jacobian matrix of the system of Equations (2.1) and (2.2) is therefore derived as

$$J(u, v) = \begin{pmatrix} \alpha - 2\gamma u - \epsilon v & -\epsilon u \\ \omega v & -\beta + \omega u \end{pmatrix}. \tag{2.9}$$

At the first equilibrium point, (0,0), the Jacobian (2.9) reduces to

$$J(0, 0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix}. \tag{2.10}$$

The characteristic equation of Equation (2.10) is

$$-(\alpha - \lambda)(\beta + \lambda) = 0. \tag{2.11}$$

From which

$$\lambda = \alpha \text{ or } \lambda = -\beta.$$

Since one of the roots is positive, the equilibrium point of the model at the point (0,0) is unstable.
For the second equilibrium point, $\left(\frac{\alpha}{\gamma}, 0\right)$, we substitute the point $\left(\frac{\alpha}{\gamma}, 0\right)$, into Equation (2.9) to obtain

$$J\left(\frac{\alpha}{\gamma}, 0\right) = \begin{pmatrix} \alpha - 2\alpha & -\frac{\epsilon\alpha}{\gamma} \\ 0 & -\beta + \frac{\omega\alpha}{\gamma} \end{pmatrix}. \tag{2.12}$$

The characteristic equation of Equation (2.12) is

$$(-\alpha - \lambda)\left(-\beta + \frac{\omega\alpha}{\gamma} - \lambda\right) = 0. \tag{2.13}$$

From which

$$\lambda = -\alpha \text{ or } \lambda = -\beta + \frac{\omega\alpha}{\gamma}.$$

If $M_1 = -\beta + \frac{\omega\alpha}{\gamma}$ then the signs of the two roots indicate that the equilibrium point of the model at the point $\left(\frac{\alpha}{\gamma}, 0\right)$ is stable if and only if $M_1 < 0$.

For the third equilibrium point, $\left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma}{\epsilon\omega}\right)$, we substitute the point $\left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma}{\epsilon\omega}\right)$, into Equation (2.9) to obtain

$$J\left(\frac{\beta}{\omega}, \frac{\alpha\omega - \beta\gamma}{\epsilon\omega}\right) = \begin{pmatrix} \alpha - 2\frac{\gamma\beta}{\omega} - \frac{(\alpha\omega-\beta\gamma)}{\omega} & -\frac{\epsilon\beta}{\omega} \\ \frac{\alpha\omega-\beta\gamma}{\epsilon} & 0 \end{pmatrix}. \tag{2.14}$$

The characteristic equation of Equation (2.14) is

$$\lambda^2 - \left[\alpha - 2\frac{\gamma\beta}{\omega} - \frac{(\alpha\omega - \beta\gamma)}{\omega}\right]\lambda + \frac{\beta(\alpha\omega - \beta\gamma)}{\omega} = 0. \tag{2.15}$$

Given Equation (2.15), we employ the theorem outlined in [8] to investigate the stability nature of the model at point $\left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma}{\epsilon\omega}\right)$. Following [8],

$$H(z) = P(z, e^z),$$

where $P(z, w)$ is the polynomial with the principle term. Assuming $H(iy), y \in \mathbb{R}$ is separated into real and imaginary parts i.e.

$$H(iy) = F(y) + iG(y). \tag{2.16}$$

If all zeros of $H(z)$ have negative real parts then zeros of $F(y)$ and $G^{'}(y)$ are real, simple and alternate and

$$F(0)G^{'}(0) - F^{'}(0)G(0) > 0 \ \forall \ y \in \mathbb{R}. \tag{2.17}$$

To apply the theorem, let $\lambda = iq$ in Equation (2.15) then

$$H(iq) = q^2 + \left[\alpha - 2\frac{\gamma\beta}{\omega} - \frac{(\alpha\omega - \beta\gamma)}{\omega}\right] iq + \frac{\beta(\alpha\omega - \beta\gamma)}{\omega}. \tag{2.18}$$

Separating Equation (2.18) into real and imaginary parts and use the result to determine each term in inequality (2.17), we have

$$F(q) = q^2 + \frac{\beta(\alpha\omega - \beta\gamma)}{\omega}, \qquad F(0) = \frac{\beta(\alpha\omega - \beta\gamma)}{\omega},$$

$$G(q) = \left[\alpha - 2\frac{\gamma\beta}{\omega} - \frac{(\alpha\omega - \beta\gamma)}{\omega}\right] q, \qquad G(0) = 0,$$

$$F^{'}(q) = 2q, \qquad F^{'}(0) = 0,$$

$$G^{'}(q) = \left[\alpha - 2\frac{\gamma\beta}{\omega} - \frac{(\alpha\omega - \beta\gamma)}{\omega}\right], \qquad G^{'}(0) = \left[\alpha - 2\frac{\gamma\beta}{\omega} - \frac{(\alpha\omega - \beta\gamma)}{\omega}\right].$$

Now, applying [8], $F(0)G'(0) - F'(0)G(0) > 0$ then,

$$\left(\frac{\beta(\alpha\omega - \beta\gamma)}{\omega}\right)\left[\alpha - 2\frac{\gamma\beta}{\omega} - \frac{(\alpha\omega - \beta\gamma)}{\omega}\right] - 0.0 > 0.$$

Assuming

$$M_2 = \left(\frac{\beta(\alpha\omega - \beta\gamma)}{\omega}\right)\left[\alpha - 2\frac{\gamma\beta}{\omega} - \frac{(\alpha\omega - \beta\gamma)}{\omega}\right], \tag{2.19}$$

then the equilibrium point of the model at the point $\left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma}{\epsilon\omega}\right)$ is stable if $M_2 > 0$ otherwise it is unstable. The results of the analysis show that there is a tendency for model 1 to exhibit both stable and unstable equilibria. It exhibits stable equilibrium at point $(0, 0)$ and may also exhibit either stable or unstable equilibrium at the other two points. The ecological interpretation of the analysis is that the presence of wasps may neutralize the effect of locusts on crops and averts mass destruction only when wasps are able to eliminate locusts, otherwise locusts' existence would instigate a major crop loss.

3. Model 2: Pests Impact on Crops in the Presence of Natural Treatment

In model 1 particularly in Equation (2.1), it is observed that the pest population grows in a Malthusian way (i.e. αu) which is based on the general assumption that the growth rate of pests is always higher than the growth rate of natural enemies in any ecological setup. The unbounded growth in the population of pests in the absence human interventions may have adverse effects on crops especially if the population of natural enemies is not enough to counterbalance the population of pests. For this reason, there is a need to further control the population of pests for the benefit of crops. The use of chemicals to control pests population is toxic not only to the environment but also to human health as it may instigate food poisoning. So, a selective approach, natural treatment, based on the IPM programme is embarked upon to influence the natural setting in the ecosystem and to further control the pests population. Since locusts are considered as pests, the best natural treatment is the application of garlic spray. Locust infestation can be treated naturally if two cups of garlic and ten cups of water are blended, boiled and allowed to sit overnight so that the mixture of one part of the solution and some water is sprayed on the leaves of vulnerable plants. Assuming garlic spray is applied to control the locust population and suppose the effort reduces the locust population at rate ϕ then the rate of change of locust population with respect to time in Equation (2.1) reduces to

$$\frac{du}{dt} = \alpha u - \gamma u^2 - \epsilon uv - \phi u,$$

but the rate of change of wasp population with respect to time remains the same as in Equation (2.2). Therefore, the new model when the natural treatment is used to influence the population of pests is

$$\frac{du}{dt} = \alpha u - \gamma u^2 - \epsilon uv - \phi u, \tag{3.1}$$

$$\frac{dv}{dt} = -\beta v + \omega uv. \tag{3.2}$$

If the system of Equations (3.1) and (3.2) is assumed to have solutions and if the solutions are assumed to be unique as in Model 1 then the equilibria of the model can be derived and the stability of the equilibria can be studied. To derive the equilibria for the model, the following equations are solved

$$\alpha u - \gamma u^2 - \epsilon uv - \phi u = 0, \tag{3.3}$$

$$-\beta v + \omega uv = 0. \tag{3.4}$$

From Equation (3.4), $v = 0$ or $u = \frac{\beta}{\omega}$. Also, in Equation (3.3),

$$u(\alpha - \gamma u - \epsilon v - \phi) = 0 \Rightarrow u = 0 \text{ or } \alpha - \gamma u - \epsilon v - \phi = 0.$$

The equilibria of the model are derived when each of $v = 0$ and $u = \frac{\beta}{\omega}$ are used to solve $\alpha - \gamma u - \epsilon v - \phi = 0$. When $v = 0$ then

$$\alpha - \gamma u - \epsilon v - \phi = 0 \Rightarrow \alpha - \gamma u - \phi = 0 \Rightarrow u = \frac{\alpha - \phi}{\gamma}.$$

Hence

$$(u, v) = \left(\frac{\alpha - \phi}{\gamma}, 0\right). \tag{3.5}$$

Also, when $u = \frac{\beta}{\omega}$ then

$$\alpha - \gamma u - \epsilon v - \phi = 0 \Rightarrow \alpha - \gamma\left(\frac{\beta}{\omega}\right) - \epsilon v - \phi = 0 \Rightarrow v = \frac{\alpha\omega - \beta\gamma - \phi\omega}{\epsilon\omega},$$

and hence

$$(u, v) = \left(\frac{\beta}{\omega}, \frac{\alpha\omega - \beta\gamma - \phi\omega}{\epsilon\omega}\right) \tag{3.6}$$

The equilibria for the model are therefore

$$(0, 0), \left(\frac{\alpha - \phi}{\gamma}, 0\right), \text{ and } \left(\frac{\beta}{\omega}, \frac{\alpha\omega - \beta\gamma - \phi\omega}{\epsilon\omega}\right).$$

To study the stable nature of the model, the Jacobian of the model is derived and evaluated at each equilibrium point. The Jacobian of the system of Equations (3.3) and (3.4) is

$$J(u, v) = \begin{pmatrix} \alpha - 2\gamma u - \epsilon v - \phi & -\epsilon u \\ \omega v & -\beta + \omega u \end{pmatrix}. \tag{3.7}$$

Evaluated at the equilibrium point (0,0), Equation (3.7) becomes

$$J(0,0) = \begin{pmatrix} \alpha - \phi & 0 \\ 0 & -\beta \end{pmatrix}. \tag{3.8}$$

The Jacobian (3.8) has the characteristic equation

$$-(\beta + \lambda)(\alpha - \phi - \lambda) = 0, \tag{3.9}$$

such that $\lambda = -\beta$ or $\lambda = \alpha - \phi$. Based on the signs of the two roots and supposing $M_3 = \alpha - \phi$, the equilibrium point (0,0) is locally asymptotically stable only if $M_3 < 0$.

For the equilibrium point, $\left(\frac{\alpha-\phi}{\gamma}, 0\right)$, the point is used to evaluate the Jacobian (3.7)and the result is

$$J\left(\frac{\alpha - \phi}{\gamma}, 0\right) = \begin{pmatrix} \phi - \alpha & -\frac{\epsilon(\alpha-\phi)}{\gamma} \\ 0 & -\beta + \frac{\omega(\alpha-\phi)}{\gamma} \end{pmatrix}. \tag{3.10}$$

In Equation (3.10),

$$\left|J\left(\frac{\alpha - \phi}{\gamma}, 0\right) - \lambda I\right| = 0 \Rightarrow (\phi - \alpha - \lambda)\left(-\beta + \frac{\omega(\alpha - \phi)}{\gamma} - \lambda\right) = 0, \tag{3.11}$$

so that $\lambda = \phi - \alpha$ or $\lambda = -\beta + \frac{\omega(\alpha-\phi)}{\gamma}$. Assuming $M_4 = \phi - \alpha$ and $M_5 = -\beta + \frac{\omega(\alpha-\phi)}{\gamma}$ then the equilibrium of the model at the point $\left(\frac{\alpha-\phi}{\gamma}, 0\right)$ is locally asymptotically stable only if $M_4 < 0$ and $M_5 < 0$.

For the remaining equilibrium point, $\left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma-\phi\omega}{\epsilon\omega}\right)$, the Jacobian (3.7) is evaluated at the point $\left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma-\phi\omega}{\epsilon\omega}\right)$. The result of the evaluation is

$$J\left(\frac{\beta}{\omega}, \frac{\alpha\omega - \beta\gamma - \phi\omega}{\epsilon\omega}\right) = \begin{pmatrix} \alpha - 2\frac{\gamma\beta}{\omega} - \frac{(\alpha\omega-\beta\gamma-\phi\omega)}{\omega} - \phi & -\frac{\epsilon\beta}{\omega} \\ \frac{\alpha\omega-\beta\gamma-\phi\omega}{\epsilon} & 0 \end{pmatrix}. \tag{3.12}$$

In Equation (3.12), $\left|J\left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma-\phi\omega}{\epsilon\omega}\right) - \lambda I\right| = 0$ gives

$$\lambda^2 - \left[\alpha - 2\frac{\gamma\beta}{\omega} - \frac{(\alpha\omega - \beta\gamma - \phi\omega)}{\omega} - \phi\right]\lambda + \frac{\beta(\alpha\omega - \beta\gamma - \phi\omega)}{\omega} = 0. \tag{3.13}$$

Using Equation (3.13), the stability of the equilibrium at the point $\left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma-\phi\omega}{\epsilon\omega}\right)$ can be examined following [8] as in Model 1. Therefore

$$\begin{aligned} F(q) &= q^2 + \frac{\beta(\alpha\omega - \beta\gamma - \phi\omega)}{\omega}, & F(0) &= \frac{\beta(\alpha\omega - \beta\gamma - \phi\omega)}{\omega}, \\ G(q) &= \left[\alpha - 2\frac{\gamma\beta}{\omega} - \frac{(\alpha\omega - \beta\gamma - \phi\omega)}{\omega} - \phi\right] q, & G(0) &= 0, \\ F'(q) &= 2q, & F'(0) &= 0, \\ G'(q) &= \left[\alpha - 2\frac{\gamma\beta}{\omega} - \frac{(\alpha\omega - \beta\gamma - \phi\omega)}{\omega} - \phi\right], & G'(0) &= \left[\alpha - 2\frac{\gamma\beta}{\omega} - \frac{(\alpha\omega - \beta\gamma - \phi\omega)}{\omega} - \phi\right]. \end{aligned}$$

With [8], $F(0)G'(0) - F'(0)G(0) > 0$ we get

$$\left(\frac{\beta(\alpha\omega - \beta\gamma - \phi\omega)}{\omega}\right)\left[\alpha - 2\frac{\gamma\beta}{\omega} - \frac{(\alpha\omega - \beta\gamma - \phi\omega)}{\omega} - \phi\right] - 0.0 > 0.$$

Let

$$M_6 = \left(\frac{\beta(\alpha\omega - \beta\gamma - \phi\omega)}{\omega}\right)\left[\alpha - 2\frac{\gamma\beta}{\omega} - \frac{(\alpha\omega - \beta\gamma - \phi\omega)}{\omega} - \phi\right] \tag{3.14}$$

Table 1. Definitions and values for parameters of the model.

Parameters	Nomenclatures	Hypothetical Values
α	preys growth rate	0.1 day^{-1}
γ	preys reduction rate due to internal competition	0.01 day^{-1}
ϵ	rate of effect of predators on preys	0.01 day^{-1}
β	predators growth rate	0.05 day^{-1}
ω	rate of effect of preys on predators	0.02 day^{-1}
ϕ	rate of efficacy of natural treatment	0.01 day^{-1}

then the equilibrium point of model 2 at the point $\left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma-\phi\omega}{\epsilon\omega}\right)$ is stable if $M_6 > 0$ otherwise it is unstable.The results of the analysis indicate that the model may exhibit stable and unstable equilibria. However, the existence of a stable equilibria is a function of α and ϕ, the pest growth rate and the rate of efficacy of natural treatment respectively. The ecological meaning of the result is that locust infestation on a farm could only be controlled via garlic spray if the rate of application is enough to suppress the locust growth rate.

4. Simulation and Discussion

To justify the theoretical results in sections 2 and 3, numerical simulations are conducted using the computer-in-built Runge-Kutta package implemented in Maple to put numerical values on the analytical results so as to examine and visualize the effect of changes in the values of various parameters on the structure and dynamics of the models. Since the actual values of the parameters are not easy to come by, a set of logical values in Table 1 are assumed for the parameters to conduct the simulation following the same idea as in [3, 4, 13, 14, 33].

For ease of reference, in section 2, the equilibria for Model 1 are:

$$(0,0), \left(\frac{\alpha}{\gamma}, 0\right) \text{ and } \left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma}{\epsilon\omega}\right).$$

The model is unstable at the equilibrium point (0,0) as the roots of the characteristic equations are

$$\lambda = \alpha \text{ and } \lambda = -\beta.$$

However, the stability of the model equilibrium is inconclusive at the equilibrium points

$$\left(\frac{\alpha}{\gamma}, 0\right) \text{ and } \left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma}{\epsilon\omega}\right).$$

The equilibrium of the model is stable at the equilibrium point $\left(\frac{\alpha}{\gamma}, 0\right)$ only if $M_1 < 0$ (the value of M_1 is in section 2 below Equation (2.13). Also, the equilibrium of the model at the equilibrium point $\left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma}{\epsilon\omega}\right)$ is stable if and only if $M_2 > 0$ (the value of M_2 is also in section 2 i.e., Equation (2.19). The parameter values in Table 1 are used as the base to determine the initial values of M_1 and M_2. These values are then varied to examine the effect of changes in the values of the parameters on the stability nature of Model 1 at each equilibrium point $\left(\frac{\alpha}{\gamma}, 0\right)$ and $\left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma}{\epsilon\omega}\right)$. The results of the analysis are presented in Tables 2 and 3 to visualize the dynamics of Model 1.

The initial values of M_1 and M_2 in Tables 2 and 3 are computed by using hypothetical values in Table 1 as the base for each of the parameters. The values of the parameters are then varied from S/No.2 through S/No.5 to examine the effect of the variations in the values of the parameters on the structure of Model 1. It is observed in Tables 2 and 3 that the conditions $M_1 < 0$ and $M_2 > 0$ are not satisfied which implied that Model 1 remained unstable within the parameter space of consideration. The implication of instability of Model 1 is that even though the natural enemies are preying on the pests, the presence of wasps is not enough not only to keep the density of the locusts population below the threshold of economic damage but also to avert massive crop destruction. It is therefore revealed from the simulations in Tables 2 and 3 that while the presence of natural enemies of pests on a farm might be beneficial

TABLE 2. Numerical results for stability of Model 1 at the point $\left(\frac{\alpha}{\gamma}, 0\right)$.

S/No.	α	γ	ϵ	β	ω	M_1	Remark
1.	0.1	0.01	0.01	0.05	0.02	0.15000	Unstable
2.	0.3	0.03	0.03	0.15	0.06	0.45000	Unstable
3.	0.5	0.05	0.05	0.25	0.10	0.75000	Unstable
4.	0.7	0.07	0.07	0.35	0.14	1.05000	Unstable
5.	0.9	0.09	0.09	0.45	0.18	1.35000	Unstable

TABLE 3. Numerical results for stability of Model 1 at the point $\left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma}{\epsilon\omega}\right)$.

S/No.	α	γ	ϵ	β	ω	M_2	Remarks
1.	0.1	0.01	0.01	0.05	0.02	-0.00009	Unstable
2.	0.3	0.03	0.03	0.15	0.06	-0.00235	Unstable
3.	0.5	0.05	0.05	0.25	0.10	-0.01172	Unstable
4.	0.7	0.07	0.07	0.35	0.14	-0.03216	Unstable
5.	0.9	0.09	0.09	0.45	0.18	-0.06834	Unstable

TABLE 4. Numerical results for stability of Model 2 at the point $(0, 0)$.

S/No.	α	γ	ϵ	β	ω	ϕ	M_3	Remarks
1.	0.1	0.01	0.01	0.05	0.02	0.01	0.09000	Unstable
2.	0.3	0.03	0.03	0.15	0.06	0.03	0.27000	Unstable
3.	0.5	0.05	0.05	0.25	0.10	0.60	-0.10000	Stable
4.	0.7	0.07	0.07	0.35	0.14	0.80	-0.10000	Stable
5.	0.9	0.09	0.09	0.45	0.18	0.95	-0.05000	Stable

to crops, the crop destruction by pests might not exist below the equilibrium level of economic damage without the application of natural treatment.

Having conducted numerical simulations for the analytical results of Model 1, numerical simulations were also conducted for the analytical results of Model 2. In section 3, the equilibria for Model 2 when natural treatment was incorporated are:

$$(0,0), \left(\frac{\alpha-\phi}{\gamma}, 0\right) \text{ and } \left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma-\phi\omega}{\epsilon\omega}\right).$$

The model is stable if and only if $M_3 < 0$ for the equilibrium point (0,0), M_4 and M_5 are both less than zero for the equilibrium point $\left(\frac{\alpha-\phi}{\gamma}, 0\right)$ and $M_6 > 0$ for the equilibrium point $\left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma-\phi\omega}{\epsilon\omega}\right)$. Refer to section 3 for the analytical values of M_3, M_4, M_5 and M_6. M_3 is below Equation (3.9), M_4 and M_5 are below Equation (3.11) while M_6 is Equation (3.14). The numerical values of M_3, M_4, M_5 and M_6 are computed following the same procedure as in Tables 2 and 3. The results of the analyses are presented in Tables 4-6.

The results of the simulations for the theoretical outcomes of Model 2 revealed the necessary and sufficient conditions for the workability of the IPM program when the natural treatment is applied to influence the population of pests despite the presence of the pests natural enemies on the farm. In Tables 4 and 6 (S/No.1 & S/No.2), it is observed that the conditions $M_3 < 0$ and $M_6 > 0$ are not satisfied and the model remains unstable in the regions despite the

TABLE 5. Numerical results for stability of Model 2 at the point $\left(\frac{\alpha-\phi}{\gamma}, 0\right)$.

S/No.	α	γ	ϵ	β	ω	ϕ	M_4	M_5	Remarks
1.	0.1	0.01	0.01	0.05	0.02	0.01	-0.09000	0.13000	Unstable
2.	0.3	0.03	0.03	0.15	0.06	0.03	-0.27000	0.39000	Unstable
3.	0.5	0.05	0.05	0.25	0.10	0.60	0.10000	-0.45000	Unstable
4.	0.7	0.07	0.07	0.35	0.14	0.80	0.10000	-0.55000	Unstable
5.	0.9	0.09	0.09	0.45	0.18	0.95	0.05000	-0.55000	Unstable

TABLE 6. Numerical results for stability of Model 2 at the point $\left(\frac{\beta}{\omega}, \frac{\alpha\omega-\beta\gamma-\phi\omega}{\epsilon\omega}\right)$.

S/No.	α	γ	ϵ	β	ω	ϕ	M_6	Remarks
1.	0.1	0.01	0.01	0.05	0.02	0.01	-0.00008	Unstable
2.	0.3	0.03	0.03	0.15	0.06	0.03	-0.00219	Unstable
3.	0.5	0.05	0.05	0.25	0.10	0.60	0.00703	Stable
4.	0.7	0.07	0.07	0.35	0.14	0.80	0.01684	Stable
5.	0.9	0.09	0.09	0.45	0.18	0.95	0.002784	Stable

application of natural treatment ϕ. Notice that the rate of application of natural treatment ϕ is lower than the growth rate of pests α in the regions (i.e. S/No.1 & S/No.2 in Table 4 and Table 6). On the other hand, as the rate of application of natural treatment ϕ exceeds the growth rate of pests α, the conditions $M_3 < 0$ and $M_6 > 0$ are satisfied and the model becomes stable. This is observable from S/No.3 - S/No.5 in Tables 4 and 6. The implication of stability of Model 2 is that the application of natural treatment in addition to the presence of pests' natural enemies on the crop field reduces the density of pest population below the threshold of economic damage.

In Table 5, the conditions $M_4 < 0$ and $M_5 < 0$ are not satisfied at any points of parameters space of consideration which implied instability for Model 2 at the equilibrium point $\left(\frac{\alpha-\phi}{\gamma}, 0\right)$. Nevertheless, while $M_4 < 0$ when the rate of application of natural treatment ϕ is lower than the growth rate of pests α (see S/No.1 & S/No.2 in Table 5), $M_5 < 0$ if the reverse is the case. That is when the rate of application of natural treatment ϕ is higher than the growth rate of pests α (see S/No.3 - S/No.5 in Table 5). It is therefore evident that the outcomes in S/No.1 & S/No.2 in Table 5 are odd, whereas the outcomes from S/No.3 - S/No.5 in Table 5 are in agreement with the results in Tables 4 and 6. Based on that argument, it is reasonable to conclude that Model 2 remains stable as long as the rate of application of natural treatment ϕ is higher than the growth rate of pests α. It is therefore deduced from the simulations that pest control and the reduction of pest population below the equilibrium of economic damage is a function of the rate of application of natural treatment. The rate of application of natural treatment must exceed the growth rate of pest before natural treatment application can achieve positive results in pest management.

5. CONCLUSION

In this paper, mathematical models for the workability of the IPM program have been presented and taken into consideration the effects of absence and presence of natural treatment in keeping the density of pests population below the equilibrium level of economic damage. The work has been carried out by studying the models both theoretically and numerically. Through the stability theory of differential equations, the dynamics of the models proposed were analyzed theoretically. In addition, we derived the conditions $M_1 < 0$ and $M_2 > 0$, which are necessary for only natural enemies to suppress pest attack on the farm and facilitate bumper harvests. Again, the conditions $M_3 < 0$, $M_5 < 0$ and $M_6 > 0$ were derived, which are necessary for the workability of the application of natural treatment.

We then use different values for the parameters of the models to compute stable and unstable points for the two models in terms of M_1, M_2, M_3, M_4, M_5, and M_6 to quantify and demonstrate the dynamics of the models. Results showed that the model with natural treatment yielded better results in terms of abundant harvests than the model without natural treatment. Furthermore, we indicated that natural treatment has a greater impact on the dynamics of the model particularly when the rate of application of the method exceeded certain critical level. It was shown that the application of natural treatment becomes reasonable as the conditions $M_3 < 0$, $M_5 < 0$, and $M_6 > 0$ tend to be fulfilled when $\phi > 0$ whenever $\alpha \to 0$ while the application natural treatment was not needed as the conditions $M_1 < 0$ and $M_2 > 0$ tend to be satisfied if $\beta > 0$ whenever $\alpha \to 0$, a situation that makes the presence of natural enemies on the farm sufficient to enhance bumper harvest. From the study, it is discovered that while the application of natural treatment is indispensable in pest control, positive results could not be achieved in the IPM program if the rate of application of natural treatment (organic chemicals ϕ) did not exceed the pests growth rate α. It was therefore established that:

- The presence of pest's natural enemies on a farm might not avert massive crop destruction,
- The application of natural treatment must exceed a certain critical level before the density of the pest population can be kept below the economic damage threshold.

Throughout the work, both theoretical and numerical methods have been utilized to gain insight into ecological mechanisms and for simplicity, the interaction between the pests and their natural enemies has been limited to a single pest (locusts) and a single natural enemy (wasps) respectively. It is hoped that the interaction between multiple pests and natural enemies would be applied to similar models to further thorough investigations of the ecological phenomena.

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