



On unique solutions of integral equations by progressive contractions

Osman Tunç^{1,*}, John R. Graef², and Cemil Tunç³

¹Department of Computer Programming, Baskale Vocational School, Van Yuzuncu Yil University, 65080, Campus, Van, Turkey.

²Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA.

³Department of Mathematics, Faculty of Sciences, Van Yuzuncu Yil University, 65080, Van, Turkey.

Abstract

The authors consider Hammerstein type integral equations for the purpose of obtaining new results on the uniqueness of solutions on an infinite interval. The approach used in the proofs is based on the technique called progressive contractions due to T. A. Burton. Here the authors apply Burton's method to a general Hammerstein type integral equation that also yields existence of solutions. In most of the existing literature investigators prove uniqueness of solutions of integral equations by applying some type of fixed point theorem which can be tedious and challenging, often patching together solutions on short intervals after making complicated translations. In this article, using the progressive contractions throughout three simple short steps, each of the three steps is an elementary contraction mapping on a short interval, we improve the technique due to T. A. Burton for a general Hammerstein type integral equation and obtain the uniqueness of solutions on an infinite interval. These are advantages of the used method to prove the uniqueness of solutions.

Keywords. Existence, Uniqueness, Hammerstein integral equation, Fixed point, Progressive contractions.

2010 Mathematics Subject Classification. 45D05, 45G10, 47H09, 47H10.

1. INTRODUCTION

In 2016, Burton [4] gave conditions guaranteeing the global existence and uniqueness of solutions to scalar integro-differential equations (IDEs) of the form

$$x'(t) = g(t, x(t)) + \int_0^t A(t-s)f(s, x(s))ds.$$

In 2017, Burton and Purnaras [8] delved into the following scalar integral equation (IE) with a variable delay of the form

$$x(t) = L(t) + g(t, x(t)) + \int_0^t A(t-s)[f(s, x(s)) + f(s, x(s-r(s)))]ds.$$

In the paper of Burton and Purnaras [[8], Lemma 2.1], Lemma 2.1 is the key to the extension to delay equations, i.e., Lemma 2.1 extends progressive contractions to delay equations. Thereby, taking into account Lemma 2.1, then using the method of progressive contractions, in [8] sufficient conditions were obtained with regard to global unique solutions of this delay IE.

Received: 13 November 2023; Accepted: 26 February 2024.
* Corresponding author. Email: osmantunc89@gmail.com .

Later, in 2019, Burton [6] presented a very simple and straightforward way to obtain a unique solution to the scalar nonlinear integral equation IE

$$x(t) = g(t, x(t)) + \int_0^t A(t-s)f(s, x(s))ds. \quad (1.1)$$

on the infinite interval $[0, \infty)$. To accomplish this, he introduced the method of progressive contractions (also see [5]). As a consequence, he avoided the process of establishing existence on a possibly short interval, translating the equation to a later starting time, and then connecting a solution on the new interval to the previous one.

In 2020, Ilea and Otrocol [18] applied Burton's technique to the IEs

$$x(t) = \int_0^t K(t, s, x(s))ds \quad (1.2)$$

and

$$x(t) = g(t, x(t)) + \int_0^t f(t, s, x(s))ds \quad (1.3)$$

to obtain existence of solutions in a Banach space. In particular, they proved that equation (1.2) has a unique solution in $C([0, b], B)$ and the IE (1.3) has a unique solution in $C([0, b] \times B, B)$, where $b \in \mathbb{R}$, $b > 0$, and B is a Banach space.

As can be seen from the references given in this paper, there have been a variety of integral equations studied in the literature using fixed point theorems and other techniques with an eye on obtaining different qualitative properties of the solutions. For example, for the application of fixed point theorems such as Schauder's fixed point theorem and Banach's contraction theorem to IEs, see the monographs of Abbas *et al.* [1], Burton [3], and Miller [18], as well as the papers [23–25]. For problems involving fractional derivatives, again see [1] as well as the papers [2, 19, 20, 25]. For questions on well-posedness and regularity of solutions, see [2, 17]; for stability and attractivity of solutions see [11, 19, 20, 22, 27]; and for oscillatory behavior, see [13–16]. For existence and uniqueness of solutions of integral and integro-differential equations, see [8, 28]. For Ulam stabilities of integral equations, see [29, 30].

Motivated by the works of Burton ([4], [6]), Ilea and Otrocol [18], and some of the others mentioned above, here we consider the general Hammerstein type integral equation

$$x(t) = p(t) + g(t, x(t)) + h(t, x(t)) \int_0^t A(t-s)f(t, s, x(s))ds, \quad (1.4)$$

where $\mathbb{R}^+ = [0, \infty)$, $p \in C(\mathbb{R}^+, \mathbb{R})$, $g, h \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$, $A \in C((0, \infty), \mathbb{R})$, and $f \in C(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$. There does not appear to be any work on uniqueness of solutions of equations of the type (1.4) on intervals $[0, E]$ with $E > 0$, or on $[0, \infty)$, in which the progressive contraction method of Burton [6] is used to prove the main results. The outcomes of this study advance and improve those of Burton [4–6] and Ilea and Otrocol [18].

In addition to presenting new results, we hope that this paper will help popularize the method of *progressive contractions*. The remaining sections of this study are as follows. Section 2 contains our main results, namely, Theorems 1 and 2, and the last section discusses our findings.

2. THE MAIN RESULTS

As we turn to our main results on the uniqueness of solutions, we refer to the type of proof we use as *progressive contractions* as named by Burton [6]. Basic information on the complete metric spaces used here can be found in [3] and [12]. For our results we need the following conditions to hold.

(H₁) There exist positive constants h_0 , g_0 , and f_0 such that, for all $s, t \in \mathbb{R}^+$ and all $x, y \in \mathbb{R}$,

$$|h(t, x)| \leq h_0,$$

$$|g(t, x) - g(t, y)| \leq g_0|x - y|, \text{ with } g_0 \in (0, 1),$$



and

$$|f(t, s, x) - f(t, s, y)| \leq f_0|x - y|.$$

(H₂) For any $\phi \in C(\mathbb{R}^+, \mathbb{R})$,

$\int_0^t A(t-s)\phi(s)ds$ is continuous,

$\int_0^t |A(s)|ds$ is continuous and converges to zero as $t \rightarrow 0^+$,

and for T small enough,

$$\int_0^T |A(s)| ds < \frac{(1 - g_0)}{2f_0g_0}.$$

Remark 1. Let $T, T_1, T_2 \in \mathbb{R}$ be positive constants such that for $T_2 - T_1 \leq T$ and $T_1 \leq t \leq T_2$, a change of variable shows that

$$\int_{T_1}^t |A(t-s)| ds < \frac{(1 - g_0)}{2f_0h_0}. \tag{2.1}$$

Theorem 1. Assume that conditions (H₁) and (H₂) hold. Then, for every $E > 0$, the IE (1.4) admits is a unique solution on $[0, E]$.

Proof. We divide the interval $[0, E]$ into n equal parts in such a way that the equal length of each part is denoted by S , where $S < T$, and the terminal points of each subinterval is denoted by T_i for $i = 1, 2, \dots, n$. That is, the subintervals are $[0, T_1], [T_1, T_2], \dots, [T_{n-1}, T_n] = [T_{n-1}, E]$.

We first let $(\mathcal{M}_1, \|\cdot\|)$ be the complete metric space of continuous functions $\phi \in C([0, T_1], \mathbb{R})$ together with the supremum norm. We define a mapping $P_1 : \mathcal{M}_1 \rightarrow \mathcal{M}_1$ by

$$(P_1\phi)(t) = p(t) + g(t, \phi(t)) + h(t, \phi(t)) \int_0^t A(t-s)f(t, s, \phi(s))ds. \tag{2.2}$$

for $\phi \in \mathcal{M}_1$. Then, for $\phi, \psi \in \mathcal{M}_1$ and $t \in [0, T_1]$,

$$\begin{aligned} \|(P_1\phi)(t) - (P_1\psi)(t)\| &\leq g_0|\phi(t) - \psi(t)| \\ &\quad + h_0 \int_0^t |A(t-s)| |f(t, s, \phi(s)) - f(t, s, \psi(s))| ds \\ &\leq g_0|\phi(t) - \psi(t)| + f_0h_0 \int_0^t |A(t-s)| |\phi(s) - \psi(s)| ds \\ &\leq g_0\|\phi(t) - \psi(t)\| + \frac{1 - g_0}{2}\|\phi(t) - \psi(t)\| \\ &= \left(\frac{1 + g_0}{2}\right) \|\phi(t) - \psi(t)\|. \end{aligned}$$

Hence, P_1 is a contraction, and so P_1 has a unique fixed point $\xi_1(t)$ on the interval $[0, T_1]$, that is,

$$(P_1\xi_1)(t) = \xi_1(t) = p(t) + g(t, \xi_1(t)) + h(t, \xi_1(t)) \int_0^t A(t-s)f(t, s, \xi_1(s))ds \tag{2.3}$$

for $t \in [0, T_1]$.



Now let $(\mathcal{M}_2, \|\cdot\|)$ be the complete metric space of continuous functions $\phi \in C([0, T_2], \mathbb{R})$ that satisfied

$$\phi(t) = \xi_1(t) \text{ on } [0, T_1]$$

together with the supremum norm. Define a mapping $P_2 : \mathcal{M}_2 \rightarrow C([0, T_2], \mathbb{R})$ by

$$(P_2\phi)(t) = p(t) + g(t, \phi(t)) + h(t, \phi(t)) \int_0^t A(t-s)f(t, s, \phi(s))ds$$

for $\phi \in \mathcal{M}_2$. From (2.2), we see that for $t \in [0, T_1]$,

$$(P_2\phi)(t) = (P_2\xi_1)(t) = \xi_1.$$

For $\phi, \psi \in \mathcal{M}_2$ and $t \in [0, T_2]$, we have

$$\begin{aligned} \|(P_2\phi)(t) - (P_2\psi)(t)\| &\leq g_0|\phi(t) - \psi(t)| \\ &\quad + h_0 \int_0^t |A(t-s)| |f(t, s, \phi(s)) - f(t, s, \psi(s))| ds \\ &\leq g_0|\phi(t) - \psi(t)| + f_0 h_0 \int_0^t |A(t-s)| |\phi(s) - \psi(s)| ds \\ &\leq g_0\|\phi(t) - \psi(t)\| + \frac{1-g_0}{2}\|\phi(t) - \psi(t)\| \\ &= \left(\frac{1+g_0}{2}\right)\|\phi(t) - \psi(t)\|. \end{aligned}$$

Therefore, $P_2 : \mathcal{M}_2 \rightarrow \mathcal{M}_2$ and P_2 is a contraction, so P_2 has a unique fixed point on the interval $[0, T_2]$, that is,

$$(P_2\xi_2)(t) = \xi_2(t) = p(t) + g(t, \xi_2(t)) + h(t, \xi_2(t)) \int_0^t A(t-s)f(t, s, \xi_2(s))ds \quad (2.4)$$

for $t \in [0, T_2]$ and $\xi_2(t) = \xi_1(t)$ for $t \in [0, T_1]$.

Proceeding by induction, we arrive at the complete metric space $(\mathcal{M}_n, \|\cdot\|)$ of continuous functions $\phi \in C([0, T_n], \mathbb{R})$ that satisfy

$$\phi(t) = \xi_{n-1}(t) \text{ on } [0, T_{n-1}]$$

together with the supremum norm. We define the mapping $P_n : \mathcal{M}_n \rightarrow C([0, T_n], \mathbb{R})$ by

$$(P_n\phi)(t) = p(t) + g(t, \phi(t)) + h(t, \phi(t)) \int_0^t A(t-s)f(t, s, \phi(s))ds$$

for $\phi \in \mathcal{M}_n$. In a completely analogous fashion we show that $P_n : \mathcal{M}_n \rightarrow \mathcal{M}_n$ and P_n is a contraction whose fixed point $\xi_n(t)$ agrees with $\xi_{n-1}(t)$ on the interval $[0, T_{n-1}]$. This proves the theorem. \square

Now that we have the existence and uniqueness of a solution to our problem on any finite interval, it is reasonable to ask if this approach can be extended to obtain the same result on the whole real line. The affirmative answer is given in the next theorem.

Theorem 2. *Under conditions (H_1) and (H_2) , Equation (1.4) has a unique solution on $[0, \infty)$.*

Proof. Using the approach taken in the proof of Theorem 1, we can construct a unique solution $\xi_n(t)$ on every interval of the form $[0, n]$, for every $n \in \mathbb{N}_+$, and which agrees with the unique solution $\xi_{n-1}(t)$ on the interval $[0, n-1]$. We can then extend solutions on $[0, n]$ to the interval $[0, \infty)$ by defining $\xi_n(t)$ for $t > n$ by $\xi_n(n)$. This results in a sequence $\{\xi_n(t)\}$ of uniformly continuous functions that converges uniformly on the compact intervals $[0, n]$ to a solution of the problem. \square



Remark 2. We should mention that in Burton and Purnaras [8], the authors considered a scalar integral equation with a variable delay. However, in this paper, we considered the general Hammerstein type IE (1.4). Indeed, the delay IE of Burton and Purnaras [8] and the Hammerstein type IE (1.4) without delay are different. Next, Burton and Purnaras [[8], Lemma 2.1] applied the method of progressive contractions depending upon the conditions of Lemma 2.1. However, in this paper, we do not need to use Lemma 2.1. We carried out the application of the method of progressive contractions due to T. A. Burton to the general Hammerstein type IE (1.4). This is an essential novelty of this paper.

3. CONCLUSIONS AND DISCUSSION

In this work, a general nonlinear Hammerstein type IE was studied and new sufficient conditions for the existence of a unique solution of the equation on an infinite interval were obtained. The outcomes of this study provide new contributions to existence theory for integral equations.

As pointed out by Burton [6], the constant f_0 “is a function of E and may tend to ∞ as $E \rightarrow \infty$.” As a simple example to see this, suppose that $f(t, s, x) = x^2$. Then, $|f(t, s, x) - f(t, s, y)| = |(x - y)(x + y)| \leq 2E|x - y|$.

We did not find any results in the literature on the existence of a unique solution to IE (1.4) on finite or infinite intervals. If $p(t) = 0$, $h(t, x(t)) = 1$, and $f(t, s, x(s)) = f(s, x(s))$, the Hammerstein IE (1.4) reduces to equation (1.1) as considered by Burton [6]. On the other hand, if $p(t) = 0$, $h(t, x(t)) = 1$, and $f(t, s, x(s)) = K(t, s, x(s))$, then (1.4) becomes (1.2) of Ilea and Otrocol [18]. Similarly, for $p(t) = 0$, and $h(t, x(t)) = 1$, (1.4) reduces to IE (1.3) considered by Ilea and Otrocol [18].

Questions for future study include using different norms, such as the Bielecki or Chebyshev norms used in [18]. Other possible directions for additional study are to consider nonlinear fractional versions of equation (1.4) as well as vector systems. Results for general integro-differential equations that include the type considered by Burton in [4] would also be of interest, as would equations involving delay arguments or neutral terms.

As for the advantages of the technique called progressive contractions used in the proofs of this article, the uniqueness of solutions of nonlinear Hammerstein type IEs can be obtained via the progressive contractions throughout three simple short steps. Hence, this technique may be applied Hammerstein type IEs with and without delay in higher dimensions.

CREDIT AUTHORSHIP CONTRIBUTION STATEMENT

John R. Graef, Osman Tunç and Cemil Tunç: Conceptualization, Data curation, Formal analysis, Methodology, Project administration, Validation, Visualization. Osman Tunç: Writing – original draft, Validation, Visualization

DECLARATION OF COMPETING INTEREST

No potential conflict of interest was reported by the authors.

DATA AVAILABILITY

No data was used for the research described in the article.

ACKNOWLEDGMENT

All authors have read and agreed to the published version of the manuscript.

REFERENCES

- [1] S. Abbas, M. Benchohra, J. R. Graef, and J. Henderson, *Implicit Fractional Differential and Integral Equations, Existence and Stability*, De Gruyter Series in Nonlinear Analysis and Applications, De Gruyter, Berlin, 26 (2018).
- [2] Z. Arab and C. Tunç, *Well-posedness and regularity of some stochastic time-fractional integral equations in Hilbert space*, J. Taibah Univ. Sci., 16 (2022), 788–798. doi.org/10.1080/16583655.2022.2119587
- [3] T. A. Burton, *Stability and Periodic Solutions of Ordinary and Functional-differential Equations*, Math. in Sci. Engin., Academic Press, Orlando, 178 (1985).



- [4] T. A. Burton, *Existence and uniqueness results by progressive contractions for integro-differential equations*, *Nonlinear Dyn. Syst. Theory*, 16 (2016), 366–371.
- [5] T. A. Burton, *Integral equations, transformations, and a Krasnosel'skii-Schaefer type fixed point theorem*, *Electron. J. Qual. Theory Differ. Equ.*, (66) (2016), 1–13. doi: 10.14232/ejqtde.2016.1.66
- [6] T. A. Burton, *A note on existence and uniqueness for integral equations with sum of two operators: progressive contractions*, *Fixed Point Theory*, 20 (2019), 107–111.
- [7] T. A. Burton, *An existence theorem for a fractional differential equation using progressive contractions*, *J. Fract. Calc. Appl.*, 8(1) (2017), 168–172.
- [8] T. A. Burton and I. K. Purnaras, *Global existence and uniqueness of solutions of integral equations with delay: progressive contractions*. *Electron. J. Qual. Theory Differ. Equ.*, (49) (2017), 6.
- [9] H. V. S. Chauhan, B. S. Singh, C. Tunç, and O. Tunç, *On the existence of solutions of non-linear 2D Volterra integral equations in a Banach space*, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 116(101) (2022), 11.
- [10] A. Deep, Deepmala, and C. Tunç, *On the existence of solutions of some non-linear functional integral equations in Banach algebra with applications*, *Arab J. Basic Appl. Sci.*, 27 (2020), 279–286. doi.org/10.1080/25765299.2020.1796199
- [11] B. C. Dhage, A. V. Deshmukh, and J. R. Graef, *On asymptotic behavior of a nonlinear functional integral equation*, *Comm. Appl. Nonlinear Anal.*, 15 (2008), 55–67.
- [12] L. E. Èl'sgol'ts, *Introduction to the Theory of Differential Equations with Deviating Arguments*, Translated from the Russian by R. J. McLaughlin, Holden-Day, San Francisco, 1966.
- [13] J. R. Graef, S. R. Grace, and E. Tunç, *On the oscillation of certain integral equations*, *Publ. Math. Debrecen*, 90 (2017), 195–204.
- [14] J. R. Graef, S. R. Grace, and E. Tunç, *Asymptotic behavior of solutions of certain integro-differential equations*, *PanAmer. Math. J.*, 29 (2019), 45–60.
- [15] J. R. Graef, S. R. Grace, and E. Tunç, *On the asymptotic behavior of solutions of certain integro-differential equations*, *J. Appl. Anal. Comput.*, 9 (2019), 1305–1318.
- [16] J. R. Graef, S. R. Grace, S. Panigrahi, and E. Tunç, *On the oscillatory behavior of Volterra integral equations on time-scales*, *PanAmer. Math. J.*, 23 (2013), 35–41.
- [17] J. K. Hale, *Theory of Functional Differential Equations*, Second edition, *Appl. Math. Sci.*, Vol. 3, Springer-Verlag, New York, 1977.
- [18] V. Ilea and D. Otrocol, *On the Burton method of progressive contractions for Volterra integral equations*, *Fixed Point Theory*, 21 (2020), 585–593.
- [19] H. Khan, C. Tunç, and A. Khan, *Stability results and existence theorems for nonlinear delay-fractional differential equations with Φ_p -operator*, *J. Appl. Anal. Comput.*, 10 (2020), 584–597.
- [20] H. Khan, C. Tunç, W. Chen, and A. Khan, *Existence theorems and Hyers-Ulam stability for a class of Hybrid fractional differential equations with p -Laplacian operator*, *J. Appl. Anal. Comput.*, 8 (2018), 1211–1226.
- [21] R. K. Miller, *Nonlinear Volterra Integral Equations*, *Math. Lecture Note Series*, Benjamin, Menlo Park, 1971.
- [22] K. Shah and C. Tunç, *Existence theory and stability analysis to a system of boundary value problem*, *J. Taibah Univ. Sci.*, 11 (2017), 1330–1342.
- [23] M. Shoaib, M. Sarwar, and C. Tunç, *Set-valued fixed point results with application to Fredholm-type integral inclusion*, *J. Taibah Univ. Sci.*, 14 (2020), 1077–1088. doi.org/10.1080/16583655.2020.1803554
- [24] D. R. Smart, *Fixed Point Theorems*, *Cambridge Tracts in Math.*, No. 66, Cambridge Press, London-New York, 1974.
- [25] M. B. Zada, M. Sarwar, and C. Tunç, *Fixed point theorems in b -metric spaces and their applications to non-linear fractional differential and integral equations*, *J. Fixed Point Theory Appl.*, 20(25) (2018), 19. doi.org/10.1007/s11784-018-0510-0
- [26] C. Tunç, F. S. Alshammari, and F. T. Akyıldız, *Existence and uniqueness of solutions of Hammerstein type functional integral equations*. *Symmetry*, 15(12) (2023), 2205.



- [27] C. Tunç and O. Tunç, *On behaviours of functional Volterra integro-differential equations with multiple time lags*. Journal of Taibah University for Science, *12*(2) (2018), 173-179.
- [28] O. Tunç, C. Tunç, and J.-C. Yao, *Global Existence and Uniqueness of Solutions of Integral Equations with Multiple Variable Delays and Integro Differential Equations: Progressive Contractions*. Mathematics, *12* (2024), 171.
- [29] O. Tunç, C. Tunç, and J. -C. Yao, *New results on Ulam stabilities of nonlinear integral equations*. Mathematics., (2024) (in press).
- [30] O. Tunç, C. Tunç, *On Ulam stabilities of delay Hammerstein integral equation*. Symmetry, *15* (2023), 1736.

Uncorrected Proof

