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On unique solutions of integral equations by progressive contractions

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Abstract

The authors consider Hammerstein-type integral equations for the purpose of obtaining new results on the uniqueness of solutions on an infinite interval. The approach used in the proofs is based on the technique called progressive contractions due to T. A. Burton. Here the authors apply the Burtons method to a general Hammerstein type integral equation that also yields the existence of solutions. In most of the existing literature, investigators prove uniqueness of solutions of integral equations by applying some type of fixed point theorem which can be tedious and challenging, often patching together solutions on short intervals after making complicated translations. In this article, using the progressive contractions throughout three simple short steps, each of the three steps is an elementary contraction mapping on a short interval, we improve the technique due to T. A. Burton for a general Hammerstein type integral equation and obtain the uniqueness of solutions on an infinite interval. These are advantages of the used method to prove the uniqueness of solutions.

Keywords. Existence, Uniqueness, Hammerstein integral equation, Fixed point, Progressive contractions.
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1. Introduction

In 2016, Burton [4] gave conditions guaranteeing the global existence and uniqueness of solutions to scalar integro-differential equations (IDEs) of the form

$$x'(t) = g(t, x(t)) + \int_0^t A(t-s) f(s, x(s)) ds.$$

In 2017, Burton and Purnaras [8] delved into the following scalar integral equation (IE) with a variable delay of the form

$$x(t) = L(t) + g(t, x(t)) + \int_0^t A(t-s) \left[f(s, x(s)) + f(s, x(s - r(s)))\right] ds.$$

In the paper of Burton and Purnaras [[8], Lemma 2.1], Lemma 2.1 is the key to the extension to delay equations, i.e., Lemma 2.1 extends progressive contractions to delay equations. Therefore, taking into account Lemma 2.1, then using the method of progressive contractions, in [8] sufficient conditions were obtained with regard to global unique solutions of this delay IE.

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Later, in 2019, Burton [6] presented a very simple and straightforward way to obtain a unique solution to the scalar nonlinear integral equation IE

$$x(t) = g(t, x(t)) + \int_0^t A(t-s) f(s, x(s)) ds, \tag{1.1}$$

on the infinite interval $[0, \infty)$. To accomplish this, he introduced the method of progressive contractions (also see [5]). As a consequence, he avoided the process of establishing existence on a possibly short interval, translating the equation to a later starting time, and then connecting a solution on the new interval to the previous one.

In 2020, Ilea and Otrocol [18] applied Burton's technique to the IEs

$$x(t) = \int_0^t K(t, s, x(s)) ds, \tag{1.2}$$

and

$$x(t) = g(t, x(t)) + \int_0^t f(t, s, x(s)) ds, \tag{1.3}$$

to obtain the existence of solutions in a Banach space. In particular, they proved that Equation (1.2) has a unique solution in $C([0, b], B)$ and the IE (1.3) has a unique solution in $C([0, b) \times B, B)$, where $b \in \mathbb{R}$, $b > 0$, and B is a Banach space.

As can be seen from the references given in this paper, there have been a variety of integral equations studied in the literature using fixed point theorems and other techniques with an eye on obtaining different qualitative properties of the solutions. For example, for the application of fixed point theorems such as Schauder's fixed point theorem and Banach's contraction theorem to IEs, see the monographs of Abbas *et al.* [1], Burton [3], and Miller [18], as well as the papers [23, 24, 30]. For problems involving fractional derivatives, again see [1] as well as the papers [2, 19, 20, 30]. For questions on well-posedness and regularity of solutions, see [2, 17]; for stability and attractivity of solutions see [11, 19, 20, 22, 25]; and for oscillatory behavior, see [13–16]. For the existence and uniqueness of solutions of integral and integro- differential equations, see [8, 28]. For Ulam stabilities of integral equations, see [27, 29].

Motivated by the works of Burton ([4], [6]), Ilea and Otrocol [18], and some of the others mentioned above, here we consider the general Hammerstein-type integral equation

$$x(t) = p(t) + g(t, x(t)) + h(t, x(t)) \int_0^t A(t-s) f(t, s, x(s)) ds, \tag{1.4}$$

where $\mathbb{R}^+ = [0, \infty)$, $p \in C(\mathbb{R}^+, \mathbb{R})$, g, $h \in C(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$, $A \in C((0, \infty), \mathbb{R})$, and $f \in C(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$. There does not appear to be any work on uniqueness of solutions of equations of the type (1.4) on intervals $[0, E]$ with $E > 0$, or on $[0, \infty)$, in which the progressive contraction method of Burton [6] is used to prove the main results. The outcomes of this study advance and improve those of Burton [4–6] and Ilea and Otrocol [18].

In addition to presenting new results, we hope that this paper will help popularize the method of *progressive contractions*. The remaining sections of this study are as follows. Section 2 contains our main results, namely, Theorems 2.2 and 2.3, and the last section discusses our findings.

2. The main results

As we turn to our main results on the uniqueness of solutions, we refer to the type of proof we use as *progressive contractions* as named by Burton [6]. Basic information on the complete metric spaces used here can be found in [3] and [12]. For our results, we need the following conditions to hold.

(H_1) There exist positive constants h_0, g_0, and f_0 such that, for all s, $t \in \mathbb{R}^+$ and all x, $y \in \mathbb{R}$,

$$|h(t, x)| \le h_0,$$

$$|g(t,x) - g(t,y)| \leq g_0|x-y|, \text{ with } g_0 \in (0,1),$$

and

$$|f(t,s,x) - f(t,s,y)| \leq f_0|x-y|.$$

(H_2) For any $\phi \in C(\mathbb{R}^+, \mathbb{R})$,

$\int_0^t A(t-s)\phi(s)ds$ is continuous,

$\int_0^t |A(s)|ds$ is continuous and converges to zero as $t \to 0^+$,

and for T small enough,

$\int_0^T |A(s)|\, ds < \frac{(1-g_0)}{2f_0 g_0}$.

Remark 2.1. Let $T,\ T_1,\ T_2 \in \mathbb{R}$ be positive constants such that for $T_2 - T_1 \leq T$ and $T_1 \leq t \leq T_2$, a change of variable shows that

$$\int_{T_1}^t |A(t-s)|\, ds < \frac{(1-g_0)}{2f_0 h_0}. \tag{2.1}$$

Theorem 2.2. *Assume that conditions* (H_1) *and* (H_2) *hold. Then, for every* $E > 0$*, the IE* (1.4) *admits is a unique solution on* $[0, E]$*.*

Proof. We divide the interval $[0, E]$ into n equal parts in such a way that the equal length of each part is denoted by S, where $S < T$, and the terminal points of each subinterval is denoted by T_i for $i = 1, 2, \ldots, n$. That is, the subintervals are $[0, T_1], [T_1, T_2], \ldots, [T_{n-1}, T_n] = [T_{n-1}, E]$.

We first let $(\mathcal{M}_1, \|\cdot\|)$ be the complete metric space of continuous functions $\phi \in C([0, T_1], \mathbb{R})$ together with the supremum norm. We define a mapping $P_1 : \mathcal{M}_1 \to \mathcal{M}_1$ by

$$(P_1\phi)(t) = p(t) + g(t, \phi(t)) + h(t, \phi(t)) \int_0^t A(t-s) f(t, s, \phi(s)) ds. \tag{2.2}$$

for $\phi \in \mathcal{M}_1$. Then, for $\phi,\ \psi \in \mathcal{M}_1$ and $t \in [0, T_1]$,

$$\begin{aligned}
\|(P_1\phi)(t) - (P_1\psi)(t)\| &\leq g_0|\phi(t) - \psi(t)| \\
&+ h_0 \int_0^t |A(t-s)|\, |f(t,s,\phi(s)) - f(t,s,\psi(s))| ds \\
&\leq g_0|\phi(t) - \psi(t)| + f_0 h_0 \int_0^t |A(t-s)||\phi(s) - \psi(s)| ds \\
&\leq g_0\|\phi(t) - \psi(t)\| + \frac{1-g_0}{2}\|\phi(t) - \psi(t)\| \\
&= \left(\frac{1+g_0}{2}\right)\|\phi(t) - \psi(t)\|.
\end{aligned}$$

Hence, P_1 is a contraction, and so P_1 has a unique fixed point $\xi_1(t)$ on the interval $[0, T_1]$, that is,

$$(P_1\xi_1)(t) = \xi_1(t) = p(t) + g(t, \xi_1(t)) + h(t, \xi_1(t)) \int_0^t A(t-s) f(t, s, \xi_1(s)) ds, \tag{2.3}$$

for $t \in [0, T_1]$.

Now let $(\mathcal{M}_2, \|\cdot\|)$ be the complete metric space of continuous functions $\phi \in C([0,T_2],\mathbb{R})$ that satisfied

$$\phi(t) = \xi_1(t), \text{ on } [0,T_1],$$

together with the supremum norm. Define a mapping $P_2 : \mathcal{M}_2 \to C([0,T_2],\mathbb{R})$ by

$$(P_2\phi)(t) = p(t) + g(t,\phi(t)) + h(t,\phi(t)) \int_0^t A(t-s)f(t,s,\phi(s))ds,$$

for $\phi \in \mathcal{M}_2$. From (2.2), we see that for $t \in [0,T_1]$,

$$(P_2\phi)(t) = (P_2\xi_1)(t) = \xi_1.$$

For ϕ, $\psi \in \mathcal{M}_2$ and $t \in [0,T_2]$, we have

$$\begin{aligned}
\|(P_2\phi)(t) - (P_2\psi)(t)\| &\leq g_0|\phi(t)-\psi(t)| \\
&+ h_0 \int_0^t |A(t-s)|\,|f(t,s,\phi(s)) - f(t,s,\psi(s))|ds \\
&\leq g_0|\phi(t)-\psi(t)| + f_0h_0 \int_0^t |A(t-s)||\phi(s)-\psi(s)|ds \\
&\leq g_0\|\phi(t)-\psi(t)\| + \frac{1-g_0}{2}\|\phi(t)-\psi(t)\| \\
&= \left(\frac{1+g_0}{2}\right)\|\phi(t)-\psi(t)\|.
\end{aligned}$$

Therefore, $P_2 : \mathcal{M}_2 \to \mathcal{M}_2$ and P_2 is a contraction, so P_2 has a unique fixed point on the interval $[0,T_2]$, that is,

$$(P_2\xi_2)(t) = \xi_2(t) = p(t) + g(t,\xi_2(t)) + h(t,\xi_2(t)) \int_0^t A(t-s)f(t,s,\xi_2(s))ds, \tag{2.4}$$

for $t \in [0,T_2]$ and $\xi_2(t) = \xi_1(t)$ for $t \in [0,T_1]$.

Proceeding by induction, we arrive at the complete metric space $(\mathcal{M}_n, \|\cdot\|)$ of continuous functions $\phi \in C([0,T_n],\mathbb{R})$ that satisfy

$$\phi(t) = \xi_{n-1}(t) \text{ on } [0,T_{n-1}]$$

together with the supremum norm. We define the mapping $P_n : \mathcal{M}_n \to C([0,T_n],\mathbb{R})$ by

$$(P_n\phi)(t) = p(t) + g(t,\phi(t)) + h(t,\phi(t)) \int_0^t A(t-s)f(t,s,\phi(s))ds$$

for $\phi \in \mathcal{M}_n$. In a completely analogous fashion we show that $P_n : \mathcal{M}_n \to \mathcal{M}_n$ and P_n is a contraction whose fixed point $\xi_n(t)$ agrees with $\xi_{n-1}(t)$ on the interval $[0,T_{n-1}]$. This proves the theorem. □

Now that we have the existence and uniqueness of a solution to our problem on any finite interval, it is reasonable to ask if this approach can be extended to obtain the same result on the whole real line. The affirmative answer in given in the next theorem.

Theorem 2.3. *Under conditions (H_1) and (H_2), Equation* (1.4) *has a unique solution on* $[0,\infty)$.

Proof. Using the approach taken in the proof of Theorem 2.2, we can construct a unique solution $\xi_n(t)$ on every interval of the form $[0,n]$, for every $n \in \mathbb{N}_+$, and which agrees with the unique solution $\xi_{n-1}(t)$ on the interval $[0,n-1]$. We can then extend solutions on $[0,n]$ to the interval $[0,\infty)$ by defining $\xi_n(t)$ for $t > n$ by $\xi_n(n)$. This results in a sequence $\{\xi_n(t)\}$ of uniformly continuous functions that converge uniformly on the compact intervals $[0,n]$ to a solution of the problem. □

Remark 2.4. We should mention that in Burton and Purnaras [8] , the authors considered a scalar integral equation with a variable delay. However, in this paper, we considered the general Hammerstein type IE (1.4). Indeed, the delay IE of Burton and Purnaras [8] and the Hammerstein type IE (1.4) without delay are different. Next, Burton and Purnaras [[8], Lemma 2.1] applied the method of progressive contractions depending upon the conditions of Lemma 2.1. However, in this paper, we do not need to use Lemma 2.1. We carried out the application of the method of progressive contractions due to T. A. Burton to the general Hammerstein type IE (1.4). This is an essential novelty of this paper.

3. Conclusions and Discussion

In this work, a general nonlinear Hammerstein type IE was studied, and new sufficient conditions for the existence of a unique solution of the equation on an infinite intervals were obtained. The outcomes of this study provide new contributions to existence theory for integral equations.

As pointed out by Burton [6], the constant f_0 "is a function of E and may tend to ∞ as $E \to \infty$." As a simple example to see this, suppose that $f(t,s,x) = x^2$. Then, $|f(t,s,x) - f(t,s,y)| = |(x-y)(x+y)| \le 2E|x-y|$.

We did not find any results in the literature on the existence of a unique solution to IE (1.4) on finite or infinite intervals. If $p(t) = 0$, $h(t,x(t)) = 1$, and $f(t,s,x(s)) = f(s,x(s))$, the Hammerstein IE (1.4) reduces to equation (1.1) as considered by Burton [6]. On the other hand, if $p(t) = 0$, $h(t,x(t)) = 1$, and $f(t,s,x(s)) = K(t,s,x(s))$, then (1.4) becomes (1.2) of Ilea and Otrocol [18]. Similarly, for $p(t) = 0$, and $h(t,x(t)) = 1$, (1.4) reduces to IE (1.3) considered by Ilea and Otrocol [18].

Questions for future study include using different norms, such as the Bielecki or Chebyshev norms used in [18]. Other possible directions for additional study are to consider nonlinear fractional versions of Equation (1.4) as well as vector systems. Results for general integro-differential equations that include the type considered by Burton in [4] would also be of interest, as would equations involving delay arguments or neutral terms.

As for the advantages of the technique called progressive contractions used in the proofs of this article, the uniqueness of solutions of nonlinear Hammerstein-type IEs can be obtained via the progressive contractions throughout three simple short steps. Hence, this technique may be applied Hammerstein-type IEs with and without delay in higher dimensions.

CRediT authorship contribution statement

John R. Graef, Osman Tunç and Cemil Tunç: Conceptualization, Data curation, Formal analysis, Methodology, Project administration, Validation, Visualization. Osman Tunç: Writing original draft, Validation, Visualization

Declaration of competing interest

No potential conflict of interest was reported by the authors.

Data availability

No data was used for the research described in the article.

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