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Application of general Lagrange scaling functions for evaluating the approximate solution time-fractional diffusion-wave equations

Sedigheh Sabermahani[1], **Yadollah Ordokhani**[1,*], **and Praveen Agarwal**[2,3,4]

[1]Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, Iran.

[2]Department of Mathematics, Anand International College of Engineering, Jaipur 303012, Rajesthan, India.

[3]Nonlinear Dynamics Research Center (NDRC), Ajman University, Ajman, UAE.

[4]International Center for Basic and Applied Sciences, Jaipur, 302029, India.

Abstract

This manuscript provides an efficient technique for solving time-fractional diffusion-wave equations using general Lagrange scaling functions (GLSFs). In GLSFs, by selecting various nodes of Lagrange polynomials, we get various kinds of orthogonal or non-orthogonal Lagrange scaling functions. The general Riemann-Liouville fractional integral operator (GRLFIO) of GLSFs is obtained generally. General Riemann-Liouville fractional integral operator of the general Lagrange scaling function is calculated exactly using the Hypergeometric functions. The operator extraction method is precisely calculated and this has a direct impact on the accuracy of our method. The operator and optimization method are implemented to convert the problem to a set of algebraic equations. Also, error analysis is discussed. To demonstrate the efficiency of the numerical scheme, some numerical examples are examined.

Keywords. Time-fractional diffusion-wave equation, General Riemann-Liouville pseudo-operational matrix, Optimization method, General Lagrange scaling function.

2010 Mathematics Subject Classification. 35R11, 26A33, 65M70.

1. Introduction

Application in mathematical modeling of anomalous diffusive systems, description of fractional random walk, unification of diffusion and wave propagation phenomenon [11, 16] is a part of wide usage anomalous diffusion equations.

One of the most important of these kinds of equations is the fractional diffusion-wave equation (FD-WEs). FD-WE indicates the duality behavior of wave and diffusion equations to a local disturbance.

The universal electromagnetic, acoustic, electrical network, signal processing, and mechanical responses may be modeled accurately utilizing time-FD-WEs (TFDWEs) [14].

Also, finding an analytical solution of fractional diffusion-wave equations is difficult, then developing numerical algorithms to solve them is of great importance. Several numerical techniques are proposed to solve these equations such as a combination method via the difference method and Galerkin spectral method [4], a spectral tau method [2], a meshless method [3], the Galerkin method using the second kind Chebyshev wavelets [24], local discontinuous Galerkin method [12], the second kind Chebyshev polynomial method [13], the Quasi-Boundary value method [28], the Galerkin finite element method [6], a pseudospectral Sinc method [26], fractional-order Bernoulli function method [27], the Crank-Nicolson method [8], and so on.

A set of schemes to quantitatively analyze academic publications is named bibliometric [23]. Also, an important role in creating effective science policies is empirical evaluations of scientific and technological research. Bibliometric analysis is effective on factors that raise the contribution of research in a subject area and guides researchers to produce effective investigations [1, 20].

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∗ Corresponding author. Email: ordokhani@alzahra.ac.ir .

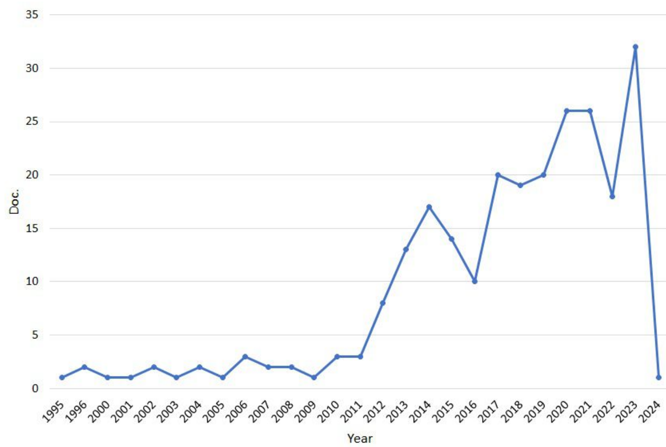

FIGURE 1. Annual number of publications per year in Scopus.

TABLE 1. The top 10 journals in the considered topic in Scopus.

Journals	**Number of publications**
Fractional Calculus and Applied Analysis	12
Numerical Algorithms	10
Journal of Computational and Applied Mathematics	10
Computers & Mathematics with Applications	10
Applied Numerical Mathematics	9
Applied Mathematics and Computation	9
Mathematics	9
Journal of Computational Physics	8
Applied Mathematics Letters	8
Journal of Scientific Computing	6
International Journal of Computer Mathematics	6

For appraising the importance and impact of academic studies on the topic of TFDWEs, here, we present a brief bibliometric analysis to display the state of publication in a valid database, Scopus. For this purpose, we consider the following keyword on 22nd October 2023. Brief considered Keyword: "fractional diffusion*wave equation*" OR "fractional*order diffusion*wave equation*" OR "time*fractional diffusion*wave equation*" OR "time-fractional-order diffusion*wave equation*" The search of the considered keyword at the title in Scopus concluded "249" publications. The annual number of publications in Scopus is shown in Figure 1. As for journals, "Fractional Calculus and Applied Analysis" is the most prolific journal in Scopus. The top 10 journals are tabulated in Table 1.

Most of the publications are written in English, however, one output is in Chinese and Russian. Of the 164 authors that are found on analyzing, the most prolific author is found to be "T. Wei" with 17 publications. Also, the value of "Authors per document" is 1.66, and the "Collaboration Index" is 1.92. The newest keywords that appear in Scopus outputs, are displayed in the visualization map (See Figure 2).

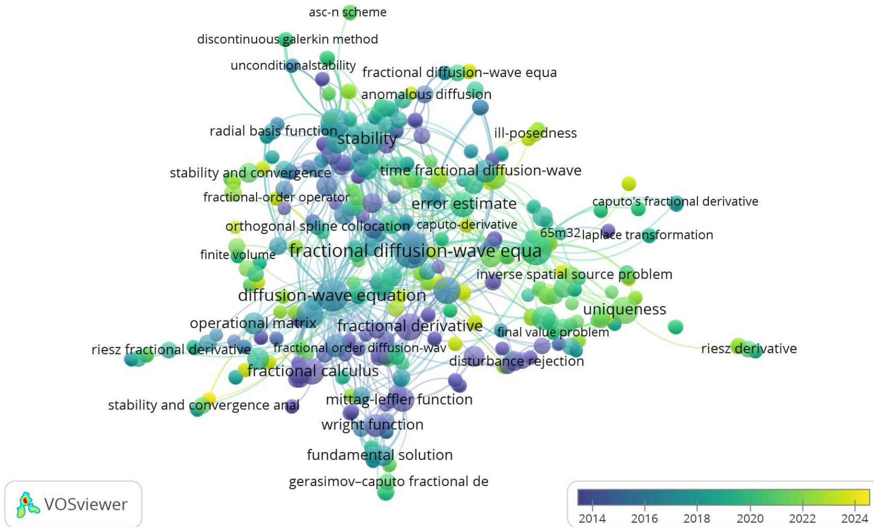

FIGURE 2. Visualization map of co-occurring keywords over time in the considered study in Scopus.

In this figure, the color of a keyword indicates the average year in which documents that include the term appeared. Top occurrence keywords include "fractional diffusion-wave equation", "stability", "diffusion-wave equation", "Caputo derivative", and "convergence". Finally, due to the present discussion, a novel approximation scheme is suggested for finding an approximate solution to TFDWEs. Currently, the following TFDWE is suggested with initial and boundary conditions [29]:

$$D_t^{\nu}\mathcal{U}(x,t) + D_t\mathcal{U}(x,t) = D_x^2\mathcal{U}(x,t) + \mathcal{Q}(x,t), \quad 0 \le x, t \le 1,\ 1 < \nu \le 2, \tag{1.1}$$

$$\begin{cases} \mathcal{U}(x,0) = \mu_0(x), \\ \mathcal{U}_t(x,0) = \mu_1(x), \quad 0 \le x \le 1, \end{cases} \tag{1.2}$$

and

$$\begin{cases} \mathcal{U}(0,t) = \xi_0(t), \\ \mathcal{U}(1,t) = \xi_1(t), \quad 0 \le t \le 1. \end{cases} \tag{1.3}$$

where, $\mu_0(x), \mu_1(x)$, $\xi_0(t)$ and $\xi_1(t)$ are given functions, and $\mathcal{U}(x,t)$ is an unknown function. Also, D_t^{ν} denotes the Caputo derivative of order $1 < \nu \le 2$ relative to variable t which is recalled in [21, 22].

On the other hand, for some decades, wavelets are powerful and efficient mathematical tools for designing the numerical method for solving some different kinds of fractional equations, such as Fibonacci [20], Boubaker [18], Mott [17], Touchard [15], the second kind Chebyshev [24] wavelets, etc.

Recently in [21, 22], we introduced fractional-order Lagrange polynomials and fractional-order general Lagrange scaling functions and applied these new functions to solve the fractional differential equations and the authors showed these functions are proper for the approximation of smooth and non-smooth functions.

The outline of the rest of the manuscript is organized as follows. Section 2 contains a summary of some definitions that are needed in this work and we present two-dimensional GLSFs. Section 3 is devoted to a new general Riemann-Liouville fractional integral operator for the GLSFs. The GRLFIO is derived generally. Section 4 develops an efficient numerical method for solving TFDWEs. Section 5 analyses the approximation using GLSFs. The numerical results are carried out in section 6. Finally, the conclusion is included in section 7.

2. Preliminaries

Here, some of the necessary definitions are recalled.

Definition 2.1. The RLFIO of order $\nu \geq 0$ of $f : [0, b] \to \mathcal{R}$ is defined as [21]

$$I^{\nu}u(t) = \begin{cases} \frac{1}{\Gamma(\nu)} \int_0^t u(s)(t-s)^{\nu-1}ds, & \nu > 0, t > 0, \\ u(t), & \nu = 0. \end{cases} \tag{2.1}$$

Also, we have the following property.

$$I^{\nu}t^k = \begin{cases} \frac{\Gamma(k+1)}{\Gamma(k+\nu+1)} t^{k+\nu}, & k > -1, \\ 0, & \textit{otherwise}. \end{cases} \tag{2.2}$$

Definition 2.2. Hypergeometric function ${}_2F_1(a, b, c; z)$ for $|z| < 1$ is defined as [19]

$${}_2F_1(a, b, c; z) = \sum_{i=0}^{\infty} \frac{(a)_i (b)_i}{(c)_i} \cdot \frac{z^i}{i!}, \tag{2.3}$$

in which

$$(q)_i = \begin{cases} 1, & \text{if } i = 0, \\ q(q+1)\cdots(q+i-1), & \text{if } i > 0, \end{cases}$$

and, a, b, c, z are real numbers. Also, we have

$${}_2F_1(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,\mathrm{d}t,$$

that is the integral form of the hypergeometric function [19].

Lemma 2.3. *Considering ${}_2F_1$ as the hypergeometric function, we get [19]*

$$I^{\gamma}(t^r \chi_c(t)) = \begin{cases} \frac{\Gamma(r+1)t^{r+\gamma}}{\Gamma(r+\gamma+1)} - \frac{t^{\gamma-1}c^{r+1}}{\Gamma(\gamma)(r+1)} {}_2F_1\left(1-\gamma, r+1, r+2; \frac{c}{t}\right), & t \geq c, \\ 0, & t < c, \end{cases} \tag{2.4}$$

subject to χ_c is the unit step function.

2.1. **GLSFs.** We consider nodal points $x_i,\ i = 0, 1, \ldots, n$. For any fixed non-negative integer number n, the Lagrange interpolating polynomials are defined in the following form [22]:

$$\mathcal{L}_i(x) := \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x - x_j)}{(x_i - x_j)}. \tag{2.5}$$

Moreover, there are no explicit formulas for the specified points x_i.
Authors [22] proposed another representation of these polynomials as follows:

$$\mathcal{L}_i(x) = \sum_{s=0}^{n} \zeta_{is} x^{n-s}, \quad i = 0, 1, \ldots, n, \tag{2.6}$$

where

$$\zeta_{i0} = \frac{1}{\prod_{\substack{j=0 \\ j \neq i}}^{n} (x_i - x_j)},$$

$$\zeta_{is} = \frac{(-1)^s}{\prod_{\substack{j=0\\ j\neq i}}^{n}(x_i - x_j)} \sum_{k_s=k_{s-1}+1}^{n} \cdots \sum_{k_1=0}^{n-s+1} \prod_{r=1}^{s} x_{k_r},$$

and $i \neq k_1 \neq \ldots \neq k_s,\ s = 1, 2, \ldots, n.$

Lemma 2.4. *The following feature is established for the Lagrange polynomials [21]*

$$\int_0^1 \mathcal{L}_m(x)\mathcal{L}_{m'}(x)dx = \sum_{s_1=0}^{n}\sum_{s_2=0}^{n} \frac{\zeta_{ms_1}\zeta_{m's_2}}{2n - s_1 - s_2 + 1}. \tag{2.7}$$

GLSFs are defined as the following formula [21]:

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{\mathcal{K}-1}{2}}\tilde{\mathcal{L}}_m(2^{\mathcal{K}-1}x - n + 1), & \frac{n-1}{2^{\mathcal{K}-1}} \leq x < \frac{n}{2^{\mathcal{K}-1}}, \\ 0, & otherwise, \end{cases} \tag{2.8}$$

and

$$\tilde{\mathcal{L}}_m(x) = \frac{1}{\sqrt{\omega_m}}\mathcal{L}_m(x), \tag{2.9}$$

which, $n = 1, 2, \ldots, 2^{\mathcal{K}-1}$, $m = 0, 1, \ldots, \mathcal{M} - 1$ and ω_m are determined using Eq. (2.7), when $m' = m$.

If we consider the nodal points $(x_m, m = 0, 1, \ldots, \mathcal{M}-1)$ as the roots of Legendre polynomials, we have interpolation scaling function [7].

The two-dimensional GLSFs (2D-GLSFs) are introduced on the region $[0, 1) \times [0, 1)$ as the following form

$$\begin{aligned} \psi_{n_1,m_1,n_2,m_2}(x,t) &= \psi_{n_1,m_1}(x)\psi_{n_2,m_2}(t) \\ &= \begin{cases} 2^{\frac{\mathcal{K}_1+\mathcal{K}_2-2}{2}}\tilde{\mathcal{L}}_{m_1}(2^{\mathcal{K}_1-1}x - n_1 + 1)\tilde{\mathcal{L}}_{m_2}(2^{\mathcal{K}_2-1}t - n_2 + 1), & x \in [\frac{n_1-1}{2^{\mathcal{K}_1-1}}, \frac{n_1}{2^{\mathcal{K}_1-1}}), \\ & t \in [\frac{n_2-1}{2^{\mathcal{K}_2-1}}, \frac{n_2}{2^{\mathcal{K}_2-1}}), \\ 0, & otherwise. \end{cases} \end{aligned} \tag{2.10}$$

2.1.1. *Approximation based on GLSFs.* The approximation for an arbitrary function $u(x)$ on $[0, 1)$ can be presented utilizing GLSFs as

$$u(x) \simeq \sum_{n=1}^{2^{\mathcal{K}-1}}\sum_{m=0}^{\mathcal{M}-1} \tilde{u}_{n,m}\psi_{n,m}(x) = \tilde{U}^T\Psi_{\mathcal{K}}^{\mathcal{M}}(x). \tag{2.11}$$

We obtain the coefficient vector $\tilde{U}$ in the following way

$$\tilde{U} = \mathcal{D}^{-1}\langle u(x), \Psi_{\mathcal{K}}^{\mathcal{M}}(x)\rangle, \quad \mathcal{D} = \langle \Psi_{\mathcal{K}}^{\mathcal{M}}(x), \Psi_{\mathcal{K}}^{\mathcal{M}}(x)\rangle, \tag{2.12}$$

in which

$$\tilde{U} = [\tilde{u}_{1,0}, \tilde{u}_{1,1}, \cdots, \tilde{u}_{1,\mathcal{M}-1}, \tilde{u}_{2,0}, \cdots, \tilde{u}_{2,\mathcal{M}-1}, \cdots, \tilde{u}_{2^{\mathcal{K}-1},\mathcal{M}-1}]^T, \tag{2.13}$$

$$\Psi_{\mathcal{K}}^{\mathcal{M}}(x) = [\psi_{1,0}, \psi_{1,1}, \cdots, \psi_{1,\mathcal{M}-1}, \psi_{2,0}, \cdots, \psi_{2,\mathcal{M}-1}, \cdots, \psi_{2^{\mathcal{K}-1},\mathcal{M}-1}]^T.$$

Let $u(x,t)$ is an arbitrary function in $L^2([0,1)\times[0,1))$. We can express this function as follows

$$u(x,t) \simeq \sum_{n_1=1}^{2^{\mathcal{K}_1-1}} \sum_{m_1=0}^{\mathcal{M}_1-1} \sum_{n_2=1}^{2^{\mathcal{K}_2-1}} \sum_{m_2=0}^{\mathcal{M}_2-1} u_{n_1,m_1,n_2,m_2}\psi_{n_1,m_1,n_2,m_2}(x,t) = \Psi_{\mathcal{K}_1}^{\mathcal{M}_1 T}(x) U \Psi_{\mathcal{K}_2}^{\mathcal{M}_2}(t), \tag{2.14}$$

we derive the coefficient matrix U as follows

$$U = \tilde{\mathcal{D}}^{-1}\langle\langle u(x,t), \Psi_{\mathcal{K}_1}^{\mathcal{M}_1}(x)\rangle\Psi_{\mathcal{K}_2}^{\mathcal{M}_2}(t)\rangle\widehat{\mathcal{D}}^{-1}, \tag{2.15}$$
$$\tilde{\mathcal{D}} = \langle\Psi_{\mathcal{K}_1}^{\mathcal{M}_1}(x), \Psi_{\mathcal{K}_1}^{\mathcal{M}_1}(x)\rangle, \quad \widehat{\mathcal{D}} = \langle\Psi_{\mathcal{K}_2}^{\mathcal{M}_2}(t), \Psi_{\mathcal{K}_2}^{\mathcal{M}_2}(t)\rangle.$$

3. General Riemann-Liouville fractional integral operator of GLSFs

In the current part, we present a GRLFIO of GLSFs regardless the nodal points for Lagrange polynomials. We extract formula for computing the GRLFIO for GLSFs in the following form

$$I^\nu \Psi_{\mathcal{K}}^{\mathcal{M}}(x) = \tilde{\Psi}_{\mathcal{K}}^{\mathcal{M}}(x,\nu). \tag{3.1}$$

We derive the components of $\tilde{\Psi}(x,\gamma)$. For this aim, according to Eqs. (2.6), (2.8), and binomial expansion, GLSFs can expanded in the following form

$$\psi_{n,m}(x) = \frac{2^{\frac{\mathcal{K}-1}{2}}}{\sqrt{w_m}}\chi_{\left(\frac{n-1}{2^{\mathcal{K}-1}}\right)}(x)\mathcal{L}_m(2^{\mathcal{K}-1}x-n+1) - \frac{2^{\frac{\mathcal{K}-1}{2}}}{\sqrt{w_m}}\chi_{\left(\frac{n}{2^{\mathcal{K}-1}}\right)}(x)\mathcal{L}_m(2^{\mathcal{K}-1}x-n+1). \tag{3.2}$$

By applying the Riemann-Liouville integral operator (I^ν) on both sides of Eq. (3.2), we get

$$I^\nu(\psi_{n,m}(x)) = I^\nu\left(\frac{2^{\frac{\mathcal{K}-1}{2}}}{\sqrt{w_m}}\chi_{\left(\frac{n-1}{2^{\mathcal{K}-1}}\right)}(x)\mathcal{L}_m(2^{\mathcal{K}-1}x-n+1)\right) - I^\nu\left(\frac{2^{\frac{\mathcal{K}-1}{2}}}{\sqrt{w_m}}\chi_{\left(\frac{n}{2^{\mathcal{K}-1}}\right)}(x)\mathcal{L}_m(2^{\mathcal{K}-1}x-n+1)\right). \tag{3.3}$$

Then, we get

$$\begin{aligned} \mathcal{L}_m(2^{\mathcal{K}-1}x-n+1) &= \sum_{s=0}^{M-1}\zeta_{ms}(2^{\mathcal{K}-1}x-n+1)^{M-1-s} \\ &= \sum_{s=0}^{M-1}\sum_{i=0}^{M-1-s}\zeta_{ms}(1-n)^{M-1-s-i}(2^{\mathcal{K}-1})^i x^i, \end{aligned} \tag{3.4}$$

so, Eq. (3.3) can be rewritten as

$$\begin{aligned} I^\nu(\psi_{n,m}(x)) = {} & \frac{2^{\frac{\mathcal{K}-1}{2}}}{\sqrt{w_m}}\sum_{s=0}^{M-1}\sum_{i=0}^{M-1-s}\zeta_{ms}(1-n)^{M-1-s-i}(2^{\mathcal{K}-1})^i \\ & \times\left(I^\gamma(\chi_{\left(\frac{n-1}{2^{\mathcal{K}-1}}\right)}(x)x^i) - I^\gamma(\chi_{\left(\frac{n}{2^{\mathcal{K}-1}}\right)}(x)x^i)\right). \end{aligned}$$

Thus, due to Lemma 2.4 and the above discussion, we achieve

$$I^\nu(\psi_{n,m}(x)) = \begin{cases} 0, & x \in \left[0, \left(\frac{n-1}{2^{\mathcal{K}-1}}\right)\right), \\ \theta(x), & t \in \left[\left(\frac{n-1}{2^{\mathcal{K}-1}}\right), \left(\frac{n}{2^{\mathcal{K}-1}}\right)\right), \\ \theta(x)-\tilde{\theta}(x), & x \in \left[\left(\frac{n}{2^{\mathcal{K}-1}}\right), 1\right), \end{cases} \tag{3.5}$$

where

$$\theta(t) = \frac{2^{\frac{\mathcal{K}-1}{2}}}{\sqrt{w_m}} \sum_{s=0}^{M-1} \sum_{i=0}^{M-1-s} \zeta_{ms}(1-n)^{M-1-s-i}(2^{\mathcal{K}-1})^i$$
$$\times \left[x^{i+\nu} \frac{\Gamma(i+1)}{\Gamma(i+\nu+1)} - \frac{x^{\nu-1}}{\Gamma(\nu)(i+1)} \left(\frac{n-1}{2^{\mathcal{K}-1}}\right)^{(i+1)} {}_2F_1\left(1-\nu, i+1, i+2; x\left(\frac{n-1}{2^{\mathcal{K}-1}}\right)\right)\right],$$

and

$$\tilde{\theta}(t) = \frac{2^{\frac{\mathcal{K}-1}{2}}}{\sqrt{w_m}} \sum_{s=0}^{M-1} \sum_{i=0}^{M-1-s} \zeta_{ms}(1-n)^{M-1-s-i}(2^{\mathcal{K}-1})^i$$
$$\times \left[x^{i+\nu} \frac{\Gamma(i+1)}{\Gamma(i+\nu+1)} - \frac{x^{\nu-1}}{\Gamma(\nu)(i+1)} \left(\frac{n}{2^{\mathcal{K}-1}}\right)^{(i+1)} {}_2F_1\left(1-\nu, i+1, i+2; x\left(\frac{n}{2^{\mathcal{K}-1}}\right)\right)\right].$$

Thus, due to Eq. (3.5), we achieve GRLFIO in Eq. (3.1).

4. Description of the numerical method

For finding solution of the mentioned problem in Eqs. (1.1)-(1.3), we expand $\frac{\partial^4 \mathcal{U}(x,t)}{\partial x^2 \partial t^2}$ in terms of GLSFs as

$$\frac{\partial^4 \mathcal{U}(x,t)}{\partial x^2 \partial t^2} \simeq \Psi_{\mathcal{K}_1}^{\mathcal{M}_1 T}(x) U \Psi_{\mathcal{K}_2}^{\mathcal{M}_2}(t). \tag{4.1}$$

Due to the initial conditions (given in Eq. (1.2)) and by integrating two times with respect to t on both sides of the above equation, we derive

$$\frac{\partial^2 \mathcal{U}(x,t)}{\partial x^2} \simeq \Psi_{\mathcal{K}_1}^{\mathcal{M}_1 T}(x) U \tilde{\Psi}_{\mathcal{K}_2}^{\mathcal{M}_2}(t,2) + \mu_0^{''}(x) + t\mu_1^{''}(x), \tag{4.2}$$

and then by integrating one and two times with respect to x, we get the following relations.

$$\frac{\partial \mathcal{U}(x,t)}{\partial x} \simeq \tilde{\Psi}_{\mathcal{K}_1}^{\mathcal{M}_1 T}(x,1) U \tilde{\Psi}_{\mathcal{K}_2}^{\mathcal{M}_2}(t,2) + [\mu_0^{'}(x) - \mu_0^{'}(0)] + t[\mu_1^{'}(x) - \mu_1^{'}(0)] + \left.\frac{\partial \mathcal{U}(x,t)}{\partial x}\right|_{x=0}, \tag{4.3}$$

and

$$\begin{aligned} \mathcal{U}(x,t) \simeq\ & \tilde{\Psi}_{\mathcal{K}_1}^{\mathcal{M}_1 T}(x,2) U \tilde{\Psi}_{\mathcal{K}_2}^{\mathcal{M}_2}(t,2) + [\mu_0(x) - \mu_0(0) - x\mu_0^{'}(0)] \\ & + t[\mu_1(x) - \mu_1(0) - x\mu_1^{'}(0)] + x\left.\frac{\partial \mathcal{U}(x,t)}{\partial x}\right|_{x=0} + \xi_0(t), \end{aligned} \tag{4.4}$$

where $\left.\frac{\partial \mathcal{U}(x,t)}{\partial x}\right|_{x=0}$ is unknown. According to Eq. (1.3), and by integrating Eq. (4.3) with respect to x from 0 to 1, we can compute the unknown term.

$$\begin{aligned} \xi_1(t) - \xi_0(t) \ &\simeq \left(\int_0^1 \tilde{\Psi}_{\mathcal{K}_1}^{\mathcal{M}_1 T}(x,1)dx\right) U \tilde{\Psi}_{\mathcal{K}_2}^{\mathcal{M}_2}(t,2) + \int_0^1 [\mu_0^{'}(x) - \mu_0^{'}(0)]dx \\ &+ t\int_0^1 [\mu_1^{'}(x) - \mu_1^{'}(0)]dx + \left.\frac{\partial \mathcal{U}(x,t)}{\partial x}\right|_{x=0} \\ &= \left(\int_0^1 \tilde{\Psi}_{\mathcal{K}_1}^{\mathcal{M}_1 T}(x,1)dx\right) U \tilde{\Psi}_{\mathcal{K}_2}^{\mathcal{M}_2}(t,2) + [\mu_0(1) - \mu_0(0) - \mu_0^{'}(0)] \\ &+ t[\mu_1(1) - \mu_1(0) - \mu_1^{'}(0)] + \left.\frac{\partial \mathcal{U}(x,t)}{\partial x}\right|_{x=0}, \end{aligned} \tag{4.5}$$

then, we get

$$\frac{\partial \mathcal{U}(x,t)}{\partial x}\bigg|_{x=0} \simeq \xi_1(t)-\xi_0(t)-\left(\int_0^1 \tilde{\Psi}_{\mathcal{K}_1}^{\mathcal{M}_1 T}(x,1)dx\right)U\tilde{\Psi}_{\mathcal{K}_2}^{\mathcal{M}_2}(t,2) - [\mu_0(1)-\mu_0(0)-\mu_0^{'}(0)]-t[\mu_1(1)-\mu_1(0)-\mu_1^{'}(0)]. \quad (4.6)$$

For calculate the terms $\frac{\partial \mathcal{U}(x,t)}{\partial t}$ and $\frac{\partial^\nu \mathcal{U}(x,t)}{\partial t^\nu}$, by deviating of order 1 and ν on both sides of Eq. (4.4) with respect to t, we derive the following relations, respectively:

$$\frac{\partial \mathcal{U}(x,t)}{\partial t} \simeq \tilde{\Psi}_{\mathcal{K}_1}^{\mathcal{M}_1 T}(x,2)U\tilde{\Psi}_{\mathcal{K}_2}^{\mathcal{M}_2}(t,1)+[\mu_1(x)-\mu_1(0)-x\mu_1^{'}(0)]+x\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{U}(x,t)}{\partial x}\bigg|_{x=0}\right)+\xi_0^{'}(t), \quad (4.7)$$

and

$$\frac{\partial^\nu \mathcal{U}(x,t)}{\partial t^\nu} \simeq \tilde{\Psi}_{\mathcal{K}_1}^{\mathcal{M}_1 T}(x,2)U\tilde{\Psi}_{\mathcal{K}_2}^{\mathcal{M}_2}(t,2-\nu)+x\frac{\partial^\nu}{\partial t^\nu}\left(\frac{\partial \mathcal{U}(x,t)}{\partial x}\bigg|_{x=0}\right)+\xi_0^{(\nu)}(t), \quad (4.8)$$

where

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{U}(x,t)}{\partial x}\bigg|_{x=0}\right)\simeq \xi_1^{'}(t)-\xi_0^{'}(t)-\left(\int_0^1 \tilde{\Psi}_{\mathcal{K}_1}^{\mathcal{M}_1 T}(x,1)dx\right)U\tilde{\Psi}_{\mathcal{K}_2}^{\mathcal{M}_2}(t,1)-[\mu_1(1)-\mu_1(0)-\mu_1^{'}(0)],$$

and

$$\frac{\partial^\nu}{\partial t^\nu}\left(\frac{\partial \mathcal{U}(x,t)}{\partial x}\bigg|_{x=0}\right)\simeq \xi_1^{(\nu)}(t)-\xi_0^{(\nu)}(t)-\left(\int_0^1 \tilde{\Psi}_{\mathcal{K}_1}^{\mathcal{M}_1 T}(x,1)dx\right)U\tilde{\Psi}_{\mathcal{K}_2}^{\mathcal{M}_2}(t,2-\nu).$$

At last, by substituting Eqs. (4.1)-(4.8) in the considered problem in Eqs. (1.1)-(1.3), we construct $\mathcal{J}(x,t,U)$

$$\begin{aligned}\mathcal{J}(x,t,U) := & \Bigg[\tilde{\Psi}_{\mathcal{K}_1}^{\mathcal{M}_1 T}(x,2)U\tilde{\Psi}_{\mathcal{K}_2}^{\mathcal{M}_2}(t,2-\nu) \\ &+ x\frac{\partial^\nu}{\partial t^\nu}\left(\frac{\partial \mathcal{U}(x,t)}{\partial x}\bigg|_{x=0}\right)+\xi_0^{(\nu)}(t)\Bigg] \\ &+ \Bigg[\tilde{\Psi}_{\mathcal{K}_1}^{\mathcal{M}_1 T}(x,2)U\tilde{\Psi}_{\mathcal{K}_2}^{\mathcal{M}_2}(t,1)+[\mu_1(x)-\mu_1(0)-x\mu_1^{'}(0)] \\ &+ x\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{U}(x,t)}{\partial x}\bigg|_{x=0}\right)+\xi_0^{'}(t)\Bigg] \\ &- \Bigg[\Psi_{\mathcal{K}_1}^{\mathcal{M}_1 T}(x)U\tilde{\Psi}_{\mathcal{K}_2}^{\mathcal{M}_2}(t,2)+\mu_0^{''}(x)+t\mu_1^{''}(x)\Bigg]-\mathcal{Q}(x,t).\end{aligned}$$

Then, we consider functional $\mathcal{J}^*$, as follows

$$\mathcal{J}^*(U) := min\int_0^1\int_0^1 \mathcal{J}^2(x,t,U)\mathrm{d}x\mathrm{d}t.$$

The necessary conditions of the aforesaid problem to minimize $\mathcal{J}^*(U)$ and evaluate the optimal value of unknown matrix U are

$$\frac{\partial \mathcal{J}^*}{\partial U}=0. \quad (4.9)$$

As a result, by deriving the solution of the above system of algebraic equations using the "Find Root" package in the "Mathematica software", we compute the matrix U. Then, by inserting this into Eq. (4.4), $\mathcal{U}(x,t)$ is achieved.

5. Error analysis

In this part, we propose the error bound of approximation using GLSFs and a bound of residual error of the mentioned problem.

Theorem 5.1. *Assume that Let $\mathcal{U}_{App}(x,t)$ be the GLSFs expansion of the real sufficiently smooth function $\mathcal{U}(x,t)$ on the region $\Delta=[0,1)\times[0,1)$ and $\mathcal{U}_{App}$ is the best approximation of $\mathcal{U}$. So, the following relation shows the error bound of the approximate solution derived by GLSFs:*

$$\|\mathcal{U}(x,t)-\mathcal{U}_{App}(x,t)\|_{L^2(\Delta)} \leq \frac{1}{\mathcal{M}_1!\mathcal{M}_2!\sqrt{2\mathcal{M}_1+1}\sqrt{2\mathcal{M}_2+1}} q_{\mathcal{M}_2}^{\mathcal{M}_1}, \tag{5.1}$$

subject to $q_{\mathcal{M}_2}^{\mathcal{M}_1} \geq \sup_{(x,t)\in\Delta} |D_x^i D_t^j \mathcal{U}(x,t)|$.

Proof. Suppose that $D_x^i D_t^j \mathcal{U} \in \mathcal{C}(\Delta), i=0,1,\cdots,\mathcal{M}_1,\ j=0,1,\cdots,\mathcal{M}_2$, multi-variable Taylor formula is as

$$\mathcal{U}_{\mathcal{M}_2}^{\mathcal{M}_1}(x,t)=\sum_{i=0}^{\mathcal{M}_1-1}\sum_{j=0}^{\mathcal{M}_2-1}\frac{x^i t^j}{i!j!}D_x^i D_t^j \mathcal{U}(x,t)\Big|_{(0,0)},$$

then, we have

$$|\mathcal{U}(x,t)-\mathcal{U}_{\mathcal{M}_2}^{\mathcal{M}_1}(x,t)| \leq \frac{x^{\mathcal{M}_1}t^{\mathcal{M}_2}}{\mathcal{M}_1!\mathcal{M}_2!}\sup_{(x,t)\in\Delta}|D_x^i D_t^j \mathcal{U}(x,t)|. \tag{5.2}$$

Now, we assume $\tilde{\mathcal{U}}(x,t)$ denotes the approximate of $\mathcal{U}(x,t)$ using Lagrange polynomials on the region $\tilde{\Delta} = [\frac{n_1-1}{2^{\mathcal{K}_1-1}}, \frac{n_1}{2^{\mathcal{K}_1-1}}) \times [\frac{n_2-1}{2^{\mathcal{K}_2-1}}, \frac{n_2}{2^{\mathcal{K}_2-1}})$. Herein, we have

$$\begin{aligned}
\|\mathcal{U}(x,t)-\mathcal{U}_{App}(x,t)\|^2_{L^2(\Delta)} &= \|\mathcal{U}(x,t)-\Psi_{\mathcal{K}_1}^{\mathcal{M}_1 T}(x)U\Psi_{\mathcal{K}_2}^{\mathcal{M}_2}(t)\|^2_{L^2(\Delta)}\\
&= \sum_{n_1=1}^{2^{\mathcal{K}_1-1}}\sum_{n_2=1}^{2^{\mathcal{K}_2-1}}\|\mathcal{U}(x,t)-\tilde{\mathcal{U}}(x,t)\|^2_{L^2(\tilde{\Delta})}\\
&\leq \sum_{n_1=1}^{2^{\mathcal{K}_1-1}}\sum_{n_2=1}^{2^{\mathcal{K}_2-1}}\|\mathcal{U}(x,t)-\mathcal{U}_{\mathcal{M}_2}^{\mathcal{M}_1}(x,t)\|^2_{L^2(\tilde{\Delta})}\\
&\leq \sum_{n_1=1}^{2^{\mathcal{K}_1-1}}\sum_{n_2=1}^{2^{\mathcal{K}_2-1}}\int_{\frac{n_1-1}{2^{\mathcal{K}_1-1}}}^{\frac{n_1}{2^{\mathcal{K}_1-1}}}\int_{\frac{n_2-1}{2^{\mathcal{K}_2-1}}}^{\frac{n_2}{2^{\mathcal{K}_2-1}}}\left[\frac{x^{\mathcal{M}_1}t^{\mathcal{M}_2}}{\mathcal{M}_1!\mathcal{M}_2!}\sup_{(x,t)\in\tilde{\Delta}}|D_x^i D_t^j\mathcal{U}(x,t)|\right]^2 dxdt\\
&\leq \int_0^1\int_0^1\left[\frac{x^{\mathcal{M}_1}t^{\mathcal{M}_2}}{\mathcal{M}_1!\mathcal{M}_2!}q_{\mathcal{M}_2}^{\mathcal{M}_1}\right]^2 dxdt\\
&= \frac{1}{(\mathcal{M}_1!)^2(\mathcal{M}_2!)^2(2\mathcal{M}_1+1)(2\mathcal{M}_2+1)}(q_{\mathcal{M}_2}^{\mathcal{M}_1})^2.
\end{aligned}$$

□

Due to Theorem 5.1, it can be seen that by increasing the terms of $\mathcal{M}_1,\mathcal{M}_2$, the approximation $\mathcal{U}_{App}$ converge to the analytical solution.

Theorem 5.2. *Let $D_x^n\mathcal{U}_{App}(x,t)$ is the best approximate of $D_x^n\mathcal{U}(x,t)$ on the region Δ. So the following relation displays the error bound of derivative of the approximation for $\mathcal{M}_1>n$:*

$$\|D_x^n\mathcal{U}(x,t)-D_x^n\mathcal{U}_{App}(x,t)\|_{L^2(\Delta)} \leq \frac{1}{(\mathcal{M}_1-n)!\mathcal{M}_2!\sqrt{2\mathcal{M}_1-2n+1}\sqrt{2\mathcal{M}_2+1}} q_{\mathcal{M}_2}^{\mathcal{M}_1}, \tag{5.3}$$

where $n=1,2,\cdots$.

Proof. Given Eqs. (5.1) and (5.2), we get

$$\begin{aligned}\|D_x^n\mathcal{U}(x,t)-D_x^n\mathcal{U}_{App}(x,t)\|_{L^2(\Delta)} &\le \sum_{n_1=1}^{2^{\mathcal{K}_1-1}}\sum_{n_2=1}^{2^{\mathcal{K}_2-1}}\|D_x^n\mathcal{U}(x,t)-D_x^n\tilde{\mathcal{U}}(x,t)\|^2_{L^2(\tilde{\Delta})}\\ &\le \sum_{n_1=1}^{2^{\mathcal{K}_1-1}}\sum_{n_2=1}^{2^{\mathcal{K}_2-1}}\|D_x^n\mathcal{U}(x,t)-D_x^n\mathcal{U}^{\mathcal{M}_1}_{\mathcal{M}_2}(x,t)\|^2_{L^2(\tilde{\Delta})}\\ &\le \sum_{n_1=1}^{2^{\mathcal{K}_1-1}}\sum_{n_2=1}^{2^{\mathcal{K}_2-1}}\int_{\frac{n_1-1}{2^{\mathcal{K}_1-1}}}^{\frac{n_1}{2^{\mathcal{K}_1-1}}}\int_{\frac{n_2-1}{2^{\mathcal{K}_2-1}}}^{\frac{n_2}{2^{\mathcal{K}_2-1}}}\left[\frac{x^{\mathcal{M}_1-n}t^{\mathcal{M}_2}}{(\mathcal{M}_1-n)!\mathcal{M}_2!}\sup_{(x,t)\in\tilde{\Delta}}|D_x^iD_t^j\mathcal{U}(x,t)|\right]^2dxdt\\ &\le \int_0^1\int_0^1\left[\frac{x^{\mathcal{M}_1-n}t^{\mathcal{M}_2}}{(\mathcal{M}_1-n)!\mathcal{M}_2!}q^{\mathcal{M}_1}_{\mathcal{M}_2}\right]^2dxdt\\ &=\frac{1}{((\mathcal{M}_1-n)!\mathcal{M}_2!)^2(2\mathcal{M}_1-2n+1)(2\mathcal{M}_2+1)}(q^{\mathcal{M}_1}_{\mathcal{M}_2})^2.\end{aligned}$$

□

Corollary 5.3. *Similar to Theorem 5.2, the error bound of derivative of the approximation* $D_t^{n'}\mathcal{U}(x,t)-D_t^{n'}\mathcal{U}_{App}(x,t)$ *for* $\mathcal{M}_2>n'$ *is achieved as follows:*

$$\|D_t^{n'}\mathcal{U}(x,t)-D_t^{n'}\mathcal{U}_{App}(x,t)\|_{L^2(\Delta)}\le\frac{1}{\mathcal{M}_1!(\mathcal{M}_2-n')!\sqrt{2\mathcal{M}_1+1}\sqrt{2\mathcal{M}_2-2n'+1}}q^{\mathcal{M}_1}_{\mathcal{M}_2}, \tag{5.4}$$

where $n'=1,2,\cdots$.

Theorem 5.4. *Suppose that* $D_t^\nu\mathcal{U}_{App}(x,t)$ *is the best approximate of* $D_t^\nu\mathcal{U}(x,t)$ *on the region* Δ. *Then, the error bound of derivative of the approximation of order* $1\le\nu<2$, *respect to* t, *is obtained*

$$\|D_t^{\nu}\mathcal{U}(x,t)-D_t^{\nu}\mathcal{U}_{App}(x,t)\|_{L^2(\Delta)}\le\frac{1}{\mathcal{M}_1!\Gamma(\mathcal{M}_2+1-\nu)\sqrt{2\mathcal{M}_1+1}\sqrt{2\mathcal{M}_2-2\nu+1}}q^{\mathcal{M}_1}_{\mathcal{M}_2}. \tag{5.5}$$

Proof. According to Eqs. (5.1) and (5.2), we get

$$\begin{aligned}\|D_t^\nu\mathcal{U}(x,t)-D_t^\nu\mathcal{U}_{App}(x,t)\|_{L^2(\Delta)} &\le \sum_{n_1=1}^{2^{\mathcal{K}_1-1}}\sum_{n_2=1}^{2^{\mathcal{K}_2-1}}\|D_t^\nu\mathcal{U}(x,t)-D_t^\nu\tilde{\mathcal{U}}(x,t)\|^2_{L^2(\tilde{\Delta})}\\ &\le \sum_{n_1=1}^{2^{\mathcal{K}_1-1}}\sum_{n_2=1}^{2^{\mathcal{K}_2-1}}\|D_t^\nu\mathcal{U}(x,t)-D_t^\nu\mathcal{U}^{\mathcal{M}_1}_{\mathcal{M}_2}(x,t)\|^2_{L^2(\tilde{\Delta})}\\ &\le \sum_{n_1=1}^{2^{\mathcal{K}_1-1}}\sum_{n_2=1}^{2^{\mathcal{K}_2-1}}\int_{\frac{n_1-1}{2^{\mathcal{K}_1-1}}}^{\frac{n_1}{2^{\mathcal{K}_1-1}}}\int_{\frac{n_2-1}{2^{\mathcal{K}_2-1}}}^{\frac{n_2}{2^{\mathcal{K}_2-1}}}\left[\frac{\Gamma(\mathcal{M}_2+1)x^{\mathcal{M}_1}t^{\mathcal{M}_2-\nu}}{\Gamma(\mathcal{M}_2+1-\nu)\mathcal{M}_1!\mathcal{M}_2!}\sup_{(x,t)\in\tilde{\Delta}}|D_x^iD_t^j\mathcal{U}(x,t)|\right]^2dxdt\\ &\le \int_0^1\int_0^1\left[\frac{x^{\mathcal{M}_1}t^{\mathcal{M}_2-\nu}}{\Gamma(\mathcal{M}_2+1-\nu)\mathcal{M}_1!}q^{\mathcal{M}_1}_{\mathcal{M}_2}\right]^2dxdt\\ &=\frac{1}{(\mathcal{M}_1!\Gamma(\mathcal{M}_2+1-\nu))^2(2\mathcal{M}_1+1)(2\mathcal{M}_2-2\nu+1)}(q^{\mathcal{M}_1}_{\mathcal{M}_2})^2.\end{aligned}$$

□

According to the aforesaid theorems, it can be observed that by increasing the terms of $\mathcal{M}_1,\mathcal{M}_2$, the error of derivatives of the approximate solution tends to be zeros.

Residual error. In this subsection, given Eq. (1.1), we present an upper bound of the residual error function. We define

$$\mathcal{R}_{\mathcal{M}_2}^{\mathcal{M}_1}(x,t)=\mathcal{E}(x,t)-\mathcal{E}_{App}(x,t),$$

in which

$$\mathcal{E}(x,t)=D_t^{\nu}\mathcal{U}(x,t)+D_t\mathcal{U}(x,t)-D_x^2\mathcal{U}(x,t)-\mathcal{Q}(x,t),$$

and

$$\mathcal{E}_{App}(x,t)=D_t^{\nu}\mathcal{U}_{App}(x,t)+D_t\mathcal{U}_{App}(x,t)-D_x^2\mathcal{U}_{App}(x,t)-\mathcal{Q}(x,t).$$

Then, using Theorems 5.1-5.4 and Corollary, we have

$$\begin{aligned}
\|\mathcal{R}_{\mathcal{M}_2}^{\mathcal{M}_1}(x,t)\|_{L^2(\Delta)} &\leq \|D_t^{\nu}\mathcal{U}(x,t)-D_t^{\nu}\mathcal{U}_{App}(x,t)\|_{L^2(\Delta)}\\
&+ \|D_t\mathcal{U}(x,t)-D_t\mathcal{U}_{App}(x,t)\|_{L^2(\Delta)}\\
&+ \|D_x^2\mathcal{U}(x,t)-D_x^2\mathcal{U}_{App}(x,t)\|_{L^2(\Delta)}\\
&\leq \frac{1}{\mathcal{M}_1!\Gamma(\mathcal{M}_2+1-\nu)\sqrt{2\mathcal{M}_1+1}\sqrt{2\mathcal{M}_2-2\nu+1}}q_{\mathcal{M}_2}^{\mathcal{M}_1}\\
&+ \frac{1}{\mathcal{M}_1!(\mathcal{M}_2-1)!\sqrt{2\mathcal{M}_1+1}\sqrt{2\mathcal{M}_2-1}}q_{\mathcal{M}_2}^{\mathcal{M}_1}\\
&+ \frac{1}{(\mathcal{M}_1-2)!\mathcal{M}_2!\sqrt{2\mathcal{M}_1-3}\sqrt{2\mathcal{M}_2+1}}q_{\mathcal{M}_2}^{\mathcal{M}_1}.
\end{aligned}$$

6. Numerical results

In the current section, three examples are proposed to examine the accuracy of the numerical solution of the mentioned problem by implementing the present technique. Moreover, the simulations are performed on a personal computer via Mathematica 12.

Example 6.1. *First, we consider the following TFDWE of order ν $(1<\nu\leq 2)$ with the initial and boundary homogeneous conditions [29]*

$$\frac{\partial^{\nu}\mathcal{U}(x,t)}{\partial t^{\nu}}+\frac{\partial\mathcal{U}(x,t)}{\partial t}=\frac{\partial^2\mathcal{U}(x,t)}{\partial x^2}+\mathcal{Q}(x,t),$$

where $\mathcal{Q}(x,t)=(\frac{2t^{2-\nu}}{\Gamma(3-\nu)}+2t)(x-x^2)+2t^2$ is selected so that the exact solution is $\mathcal{U}(x,t)=t^2x(1-x)$. The equation has been solved with various values of $\mathcal{K},\tilde{\mathcal{K}},\mathcal{M},\tilde{\mathcal{M}}$ and ν. The achieved results are shown in Table 2, Figures 3 and 4.

Figure 3 demonstrates the absolute error for $\mathcal{K}=\tilde{\mathcal{K}}=1,\mathcal{M}=\tilde{\mathcal{M}}=1$, $\nu=2$ and $\nu=1.1$. Also, Figure 4 shows the absolute error for $\mathcal{K}=\tilde{\mathcal{K}}=2,\mathcal{M}=\tilde{\mathcal{M}}=2,\nu=1.5$ when we select zeros of shifted Legendre polynomials and $\frac{i}{\mathcal{M}+1},\frac{j}{\tilde{\mathcal{M}}+1}$, $i=1,\cdots\mathcal{M}$, $j=1,\cdots\tilde{\mathcal{M}}$ as nodal points. For more investigation, Table 2 presents a comparison the absolute errors between the result derived by wavelet method with $(k=3,M=3)$ [9] and our method with $\mathcal{K}=\tilde{\mathcal{K}}=\mathcal{M}=\tilde{\mathcal{M}}=2$ in zeros of Legendre polynomials as nodal points. In this table, we achieved more accurate results than the wavelet method [9] with fewer basic functions.

Example 6.2. *Consider the following TFDWE of order ν $(1<\nu\leq 2)$ [2]*

$$\begin{cases}\frac{\partial^{\nu}\mathcal{U}(x,t)}{\partial t^{\nu}}+\frac{\partial\mathcal{U}(x,t)}{\partial t}=\frac{\partial^2\mathcal{U}(x,t)}{\partial x^2}+\mathcal{Q}(x,t),\\ \mathcal{U}(x,0)=\frac{\partial\mathcal{U}(x,0)}{\partial t}=0, & 0\leq x\leq 1,\\ \mathcal{U}(0,t)=t^3,\ \mathcal{U}(1,t)=Exp(1)t^3, & 0\leq t\leq 1.\end{cases}$$

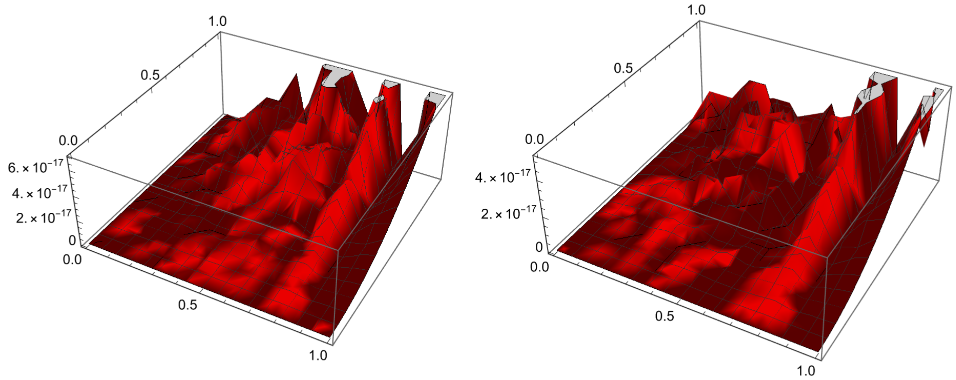

FIGURE 3. Absolute error for $\mathcal{K} = \tilde{\mathcal{K}} = \mathcal{M} = \tilde{\mathcal{M}} = 1$, $\nu = 1.1$ (left) and $\nu = 2$ (right) in Example 6.1.

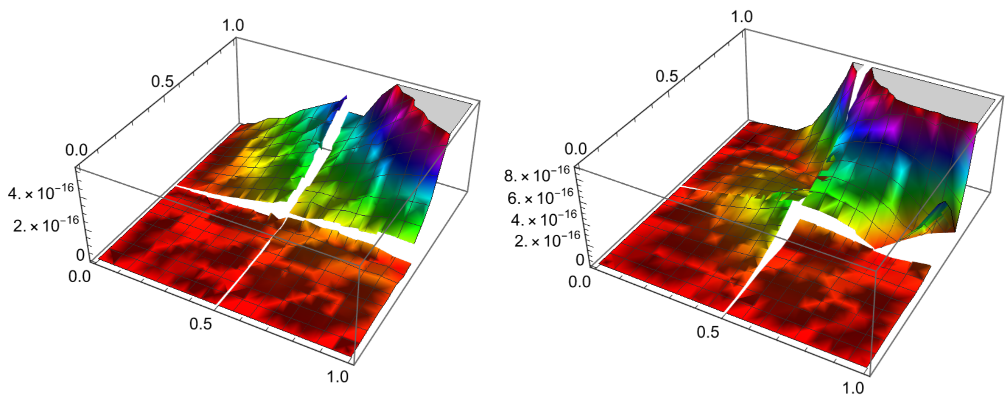

FIGURE 4. The bsolute error for $\mathcal{K} = \tilde{\mathcal{K}} = \mathcal{M} = \tilde{\mathcal{M}} = 2, \nu = 1.5$ and with nodal points: zeros of shifted Legendre polynomials (right) and $\frac{i}{\mathcal{M}+1}, \frac{j}{\tilde{\mathcal{M}}+1},\ i = 1, \cdots \mathcal{M},\ j = 1, \cdots \tilde{\mathcal{M}}$ (left) in Example 6.1.

TABLE 2. The comparison of the absolute errors between the our method ($\mathcal{K} = \tilde{\mathcal{K}} = \mathcal{M} = \tilde{\mathcal{M}} = 2$) and wavelets method ($k = 3, M = 3$) [9] for different values of ν in Example 6.1.

(x,t)	*Wavelets method [9]*		*Present method*	
	$\nu = 1.3$	$\nu = 1.9$	$\nu = 1.3$	$\nu = 1.9$
$(0.1, 0.1)$	$1.3694E^{-5}$	$1.2825E^{-5}$	$4.33681E^{-19}$	$8.02310E^{-18}$
$(0.3, 0.3)$	$2.6323E^{-5}$	$5.3169E^{-5}$	$6.93889E^{-18}$	$1.04083E^{-17}$
$(0.5, 0.5)$	$1.8821E^{-5}$	$6.7208E^{-5}$	$9.71445E^{-17}$	$6.93889E^{-18}$
$(0.7, 0.7)$	$8.1172E^{-6}$	$3.6814E^{-5}$	$2.49800E^{-16}$	$4.77396E^{-15}$
$(0.9, 0.9)$	$3.1435E^{-6}$	$1.2030E^{-5}$	$6.52256E^{-16}$	$1.11577E^{-14}$

TABLE 3. The comparision of the maximum absolute errors in Example 6.2.

Method	*Error*
FPCM [10]	$2.76E^{-2}$
Spectral Tau method [2] ($N = M = 8, \alpha = \beta = 1$)	$2.64E^{-5}$
Spectral Tau method [2] ($N = M = 16, \alpha = \beta = 1$)	$1.25E^{-6}$
Present method ($\mathcal{K} = \tilde{\mathcal{K}} = 1, \mathcal{M} = 5, \tilde{\mathcal{M}} = 2$)	$1.25E^{-7}$

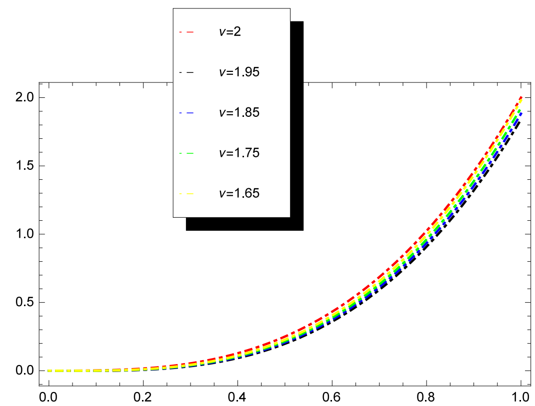

FIGURE 5. The convergence of Example 6.2 at $\mathcal{K} = 3, \mathcal{M} = 5, \tilde{\mathcal{K}} = 1, \tilde{\mathcal{M}} = 2, x = 1$ and different values of ν.

We select $\mathcal{Q}(x,t)$ so that the exact solution of this problem is $\mathcal{U}(x,t) = t^3 Exp(x)$. The comparison of maximum absolute errors obtained by implementing FPCM [10], spectral Tau method [2] and our method in nodal points $\frac{i}{\mathcal{M}+1}, \frac{j}{\tilde{\mathcal{M}}+1}, \; i = 1, \cdots \mathcal{M}, \; j = 1, \cdots \tilde{\mathcal{M}}$ are reported Table 3.

Figure 5 exhibits the convergence of the problem at $\mathcal{K} = 3, \mathcal{M} = 5, \tilde{\mathcal{K}} = 1, \tilde{\mathcal{M}} = 2, x = 1$ and different values of ν.

Example 6.3. *Consider the following TFDWE of order ν $(1 < \nu \leq 2)$ [5]*

$$\frac{\partial^\nu \mathcal{U}(x,t)}{\partial t^\nu} = \frac{\partial^2 \mathcal{U}(x,t)}{\partial x^2} + sin(\pi x), \quad 0 \leq x, t \leq 1,$$

with the initial and boundary homogeneous conditions. $\mathcal{U}(x,t) = \frac{1}{\pi^2}(1 - M_\nu(-\pi^2 t^\nu))sin(\pi x)$ is the exact solution, in which $M_\nu(t)$ denotes the Mittag-Leffler function.

Table 4 displays the achieved results by applying the method with $\mathcal{K} = \tilde{\mathcal{K}} = 1, \mathcal{M} = \tilde{\mathcal{M}} = 5$ and different values of ν and comparing with the exact solution and compact difference method [5], and a fully discrete difference method [25].

Also, we show the approximate solution's graphs and the absolute error for $\nu = 2$ with $\mathcal{K} = \tilde{\mathcal{K}} = 1, \mathcal{M} = \tilde{\mathcal{M}} = 5$ and zeros of Fibonacci polynomials as nodal points in Figure 6.

7. CONCLUSION

A new numerical optimization technique is proposed to evaluate the approximate solution of time-fractional diffusion-wave equations. This method is based on general Lagrange scaling functions. In GLSFs, by selecting various nodal points in Lagrange polynomials, we have various kinds of orthogonal or non-orthogonal Lagrange scaling functions. We compute a general Riemann-Liouville fractional integral operator of GLSFs for these functions, exactly and without considering nodal points. The operator and applying optimization method are used to insert the mentioned

Table 4. The comparison of the achieved results for $\nu = 1.5$ in Example 6.3.

(x,t)	*Compact difference method [5]*	*A fully discrete difference method [25]*	*Present method*	*Exact Solution*
$(\frac{1}{8},1)$	$4.3724E^{-2}$	$4.34346E^{-2}$	$4.34332E^{-2}$	$4.32436E^{-2}$
$(\frac{2}{8},1)$	$8.0607E^{-2}$	$8.02566E^{-2}$	$8.00538E^{-2}$	$7.99519E^{-2}$
$(\frac{3}{8},1)$	$1.0532E^{-1}$	$1.04860E^{-1}$	$1.04595E^{-1}$	$1.04399E^{-1}$
$(\frac{4}{8},1)$	$1.1400E^{-1}$	$1.1500E^{-1}$	$1.13213E^{-1}$	$1.13001E^{-1}$

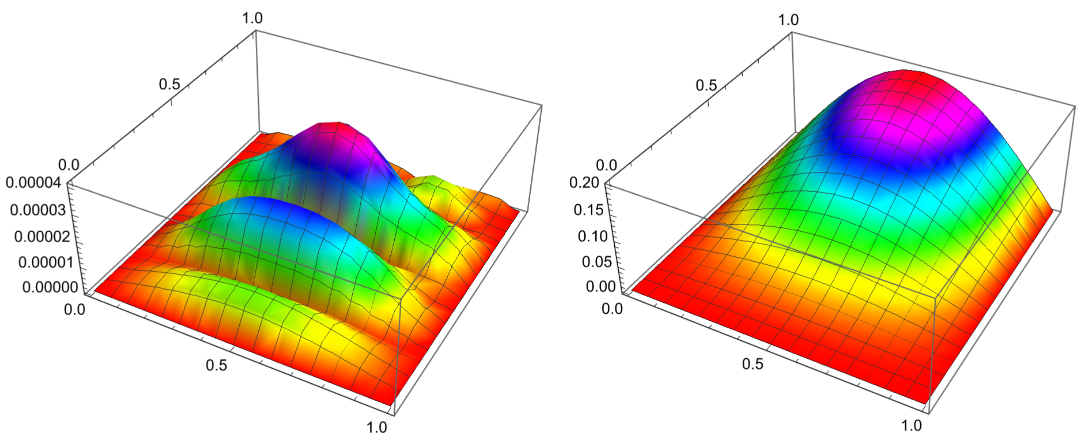

Figure 6. Graphs of the absolute error (left) and approximate solution (right) at $\mathcal{K} = \tilde{\mathcal{K}} = 1, \mathcal{M} = \tilde{\mathcal{M}} = 5$ and $\nu = 2$ in Example 6.3.

problem into a set of algebraic equations. We investigate the error analysis for the method. Some numerical problems are proposed with graphs and tables to examine the efficiency and effectiveness of the proposed algorithm.

Conflict of interest. The authors declare no potential conflict of interests.

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