Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-16
DOI:10.22034/cmde.2024.59451.2531

Exploring novel solutions for the generalized q-deformed wave equation

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Abstract

Our primary goal is to address the q-deformed wave equation, which serves as a mathematical framework for characterizing physical systems with symmetries that have been violated. By incorporating a q-deformation parameter, this equation expands upon the traditional wave equation, introducing non-commutativity and non-linearity to the dynamics of the system. In our investigation, we explore three distinct approaches for solving the generalized q-deformed wave equation: the reduced q-differential transform method (RqDTM) [17], the separation method (SM), and the variational iteration method (VIM). The RqDTM is a modified version of the differential transform method specially designed to handle q-deformed equations. The SM aims to identify solutions that can be expressed as separable variables, while the VIM employs an iterative scheme to refine the solution. We conduct a comparative analysis of the accuracy and efficiency of the solutions obtained through these methods and present numerical results. This comparative analysis enables us to evaluate the strengths and weaknesses of each approach in effectively solving the q-deformed wave equation, providing valuable insights into their applicability and performance. Additionally, this paper introduces a generalization of the q-deformed wave equation, as previously proposed in [13], and investigates its solution using two different analytical methods: RqDTM, SM, and an approximation method known as VIM.

Keywords. q-calculus, q-deformed equation, RqDTM, Separation method, VIM.

2010 Mathematics Subject Classification. 65L05, 34K06, 34K28.

1. Introduction

The propagation of various waves including sound and water waves is described by the wave equation. It occurs in a variety of disciplines such as electromagnetics and fluid dynamics. D'Alembert is credited with the discovery of the one-dimensional wave equation in 1747 [3] and he provided the string's motion model equation in one dimension in 1743 [4]. The wave equation of three-dimensional is discovered by Euler 10 years later [5]. The wave equation was a useful description that encompasses a broad spectrum of events that generally is used to simulate modest oscillations about the equilibrium which is the system will frequently be adequately approximately using the Hooke principle. The wave equation has various applications, not just in fluid dynamics, but also in electromagnetic fields, optics, gravitational physics, heat transfer, etc. The standard form of wave equation takes this form:

$$\frac{\partial^2 \xi}{\partial \mathfrak{T}^2} = c^2 \frac{\partial^2 \xi}{\partial \mathcal{X}^2}, \tag{1.1}$$

where c denotes the constant speed of wave propagation and ξ represents the displacement of the wave as a function of both time $\mathfrak{T}$ and space $\mathcal{X}$, the partial derivatives with respect to $\mathfrak{T}$ and $\mathcal{X}$ describe how the displacement changes with time and with the spatial coordinates respectively.

Fractional calculus has long been regarded as a branch of pure mathematics with no practical applications since its formulation is predicated on the idea of a non-integer order that can be integral or derivative. However, in recent years, the function of fractional calculus has evolved and we find applications of this branch in many fields such as nanofluid

Received:05 December 2023 ; Accepted: 09 February 2024.

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flow [6–8], quantum mechanics, [9] and electrical engineering [10]. The q-derivative, known as the Jackson derivative, is a q-analogue for the ordinary derivative that was first presented with Jackson in the fields of combinatorics and quantum calculus [11].

The q-calculus had over the past twenty years and q-calculus has evolved in the specializations subject and acted as a link between physics and mathematics. Physicists make up the bulk of users of the q-calculus worldwide. The discipline has rapidly grown as a result of applications of fundamental hypergeometric series to a variety of topics including quantum theory [12], number theory [2], and mechanical statistics [14]. Numerous findings from research by the theory for operators for q-calculus in the last years have been used in a variety of fields, including the geometric function theory of complex analysis [15], problems in ordinary fractional calculus [16], optimal control [1], solutions of the q-difference equations [18], q-integral equations [19] and q-transform analysis [20].

The fractional q-calculus can be defined as a q-protraction for basic fractional calculus and it has several applications in the mathematical sciences such as time scale [21, 22] etc.

From this standpoint, some researchers began to develop the wave equation and put it as q-deformed equation in this form [13]:

$$\frac{\partial^2 \xi(\mathfrak{T}, \mathcal{X})}{\partial \mathfrak{T}^2} = c^2 D^2_{q,\mathcal{X}} \xi(\mathfrak{T}, \mathcal{X}). \tag{1.2}$$

The q-deformed wave equation (1.2) is an equation involving q- deformation which is a type of mathematical calculus that extends the ordinary calculus by introducing a parameter q, where it is a deformation parameter which is used for a generalization of various mathematical concepts. The author in [13] has solved (1.2) using (SM).

In this paper, we proposed more generalization of (1.2) in this form:

$$D^2_{q,\mathfrak{T}} \xi(\mathfrak{T}, \mathcal{X}) = c^2 D^2_{q,\mathcal{X}} \xi(\mathfrak{T}, \mathcal{X}), \quad where \quad q \in (0, 1), \tag{1.3}$$

where $\xi(\mathfrak{T}, \mathcal{X})$ represents the displacement of the wave as a function of both time $\mathfrak{T}$ and spatial coordinates $\mathcal{X}$ and c is the wave speed, representing how fast the wave propagates through space, and it is a constant in the equation depends on the properties of the medium through which the wave is traveling, and $D^2_{q,\mathcal{X}}$ is the second q-derivative with respect to $\mathcal{X}$ and $D^2_{q,\mathfrak{T}}$ is the second q-derivative with respect to $\mathfrak{T}$. We solve (1.3) by two analytical methods, (SM) [13] and (RqDTM) respectively. Also, we solved it with an approximation method (VIM) [28].

This paper is organized as follows: In section 2, we present fundamental notations in the q-calculus language. In section 3, we introduce the analytical solution for (1.3) by using the (RqDTM). In section 4, we explain the analytical solution for (1.3) by using (SM). In section 5, we compute the approximation solution for (1.3) by using (VIM). In section 6, we present the numerical results for (1.3). In section 7, we provide graphical illustrations.

2. Preliminaries

We review several fundamental notations in the q-calculus language [23, 24, 27]. The natural number m has the following properties q-deformation for $q \in (0, 1)$:

$$[m]_q = 1 + q + q^2 + ... + q^{m-1}, \quad with \quad [0]_q = 0.$$

We occasionally write $[\infty]_q$ for the upper bound of these numbers: $\frac{1}{(1-q)}$. That is easily obtained

$$[m]_q = \frac{1 - q^m}{1 - q}, \quad q \in (0, 1).$$

For $m \in \mathbb{N}$, this definition applies to any real number ς. In this example, we refer to $[\varsigma]_q$ as a q-real number. The q-binomial coefficients and the q-factorials naturally are defined to be

$$[m]_q! = [1]_q.[2]_q...[m]_q, \quad with \quad [0]_q = 1.$$

At $q \in (0, 1)$ and the analytical function $g : \mathbb{C} \to \mathbb{C}$ specifies operators H and D_q as seen below [24, 25].

$$(Hg)(\mathcal{X}) = \mathcal{X} g(\mathcal{X}),$$

$$(D_q g)(\mathcal{X}) = \begin{cases} \frac{g(\mathcal{X})-g(q\mathcal{X})}{(1-q)\mathcal{X}} & \mathcal{X} \neq 0 \\ g'(0) & \mathcal{X}=0, q=1 \end{cases}$$

$D_{q,\mathcal{X}_1}$ and $D_{q,\mathcal{X}_2}$ are provided by these relations:

$$D_{q,\mathcal{X}_1} f(\mathcal{X}_1,\mathcal{X}_2) = (D_q f(.,\mathcal{X}_2))(\mathcal{X}_1),$$

$$D_{q,\mathcal{X}_2} f(\mathcal{X}_1,\mathcal{X}_2) = (D_q f(\mathcal{X}_1,.))(\mathcal{X}_2).$$

The following are the attributes of the operator D_q:

(1) $\lim\limits_{q\to 1} (D_q \rho)(\mathcal{X}) = \rho'(\mathcal{X})$,
(2) $D_q(\mathcal{X}^\gamma) = [\gamma]_q \mathcal{X}^{\gamma-1}$,
(3) $D_q(f(\mathcal{X})g(\mathcal{X})) = (D_q f)(\mathcal{X})g(\mathcal{X}) + f(q\mathcal{X})(D_q g)(\mathcal{X})$,
(4) $D_q(\frac{f(\mathcal{X})}{g(\mathcal{X})}) = \frac{(D_q f)(\mathcal{X})g(\mathcal{X})-f(\mathcal{X})(D_q g)(\mathcal{X})}{g(\mathcal{X})g(q\mathcal{X})}$,
(5) $D_q f(\mathcal{X}) = \sum_{r=0}^{\infty} \frac{(q-1)^r}{(r+1)!}\mathcal{X}^r \frac{d^{r+1}}{dx^{r+1}} f(\mathcal{X})$,
(6) $D_q^{\digamma} f(\mathcal{X}) = \frac{\mathcal{X}^{-\digamma} q^{-\digamma(\digamma-1)/2}}{(q-1)^{\digamma}} \sum_{\female=0}^{\digamma} \left\{ \begin{array}{c} \digamma \\ \female \end{array} \right\}_q (-1)^{\female} q^{\female(\female-1)/2} f(q^{\digamma-\female}\mathcal{X})$.

It is widely recognized that the operators D_q and H are satisfying the following equation: [25]

$$D_q H - \mathrm{qHD}_q = 1.$$

The q-analogs for the traditional exponential function $e^{\mathcal{X}}$ are denoted by

$$e_q(\mathcal{X}) = \sum_{\iota=0}^{\infty} \frac{\mathcal{X}^\iota}{[\iota]_q!}.$$

As we can see,

$$e_q(\mathcal{X}) = \frac{1}{(1-(1-q)\mathcal{X})_q^\infty},$$

where

$$(1-y)_q^\infty = \prod_{\tau=0}^{\infty}(1-q^\tau y).$$

The following property is shared by q-exponential functions:

$$D_q e_q(\mathcal{X}) = e_q(\mathcal{X}).$$

The q-integration by parts is valid:

$$\int_a^b g(q\mathcal{X}) D_q^{\mathcal{X}} f(\mathcal{X}) d_q\mathcal{X} = f(\mathcal{X})g(\mathcal{X})|_a^b - \int_a^b f(\mathcal{X}) D_q^{\mathcal{X}} g(\mathcal{X}) d_q\mathcal{X} \quad . \tag{2.1}$$

Remark 2.1. *For $q \in (0,1)$, the continuation of the series $e_q(\mathcal{X})$ has a convergence radius $\frac{1}{1-q}$.*

3. Analytical solution of (1.3) by using (RqDTM)

In this section, the (RqDTM) is applied to find the solution of (1.3).

Firstly we explain the (RqDTM) as below:

Step(1):

In this step, we have some concepts for the partial q-derivative, such as:

The partial q-derivative of a continuous function of many real variables $g(\mathcal{X}_1,\mathcal{X}_2,...,\mathcal{X}_n)$ with respect to a variable $\mathcal{X}_i$ is defined by;

$$D_{q,\mathcal{X}_j} g(\mathcal{X}) = \frac{(\epsilon_{q,j} g)(\mathcal{X}) - g(\mathcal{X})}{(q-1)\mathcal{X}_j}, \mathcal{X}_j \neq 0, q \in (0,1). \tag{3.1}$$

$$[D_{q,\mathcal{X}_j} g(\mathcal{X})]_{\mathcal{X}_j=0} = \lim_{\mathcal{X}_j \to 0} D_{q,\mathcal{X}_j} g(\mathcal{X}), \tag{3.2}$$

where $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, ..., \mathcal{X}_n)$ and, $(\epsilon_{q,j} g)(\mathcal{X}) = g(\mathcal{X}_1, \mathcal{X}_2, ..., q\mathcal{X}_j, ..., \mathcal{X}_n)$.

Following that, we acquire the identity $D^k_{q,\mathcal{X}} \equiv \frac{\partial^k_q}{\partial_q \mathcal{X}^k}$ for the k^{th}order q-derivative with respect to $\mathcal{X}^k$.

The (RqDTM) given that all q-differentials of $\xi(\mathcal{X}, \mathfrak{T})$ occur in some vicinity of $\mathfrak{T} = \mathbf{a}$, then allow

$$\Xi_k(\mathcal{X}) = \frac{1}{[k]_q!} \left[\frac{\partial^k_q}{\partial_q \mathfrak{T}^k} \xi(\mathcal{X}, \mathfrak{T}) \right]_{\mathfrak{T}=\mathbf{a}}, \tag{3.3}$$

where $\mathfrak{T}$ is the dimensionality of the spectrum function $\Xi_k(\mathcal{X})$ is the function that has been modified. As a result, the lowercase $\xi(\mathcal{X}, \mathfrak{T})$ denotes the original function, whereas uppercase $\Xi_k(\mathcal{X})$ represents the converted function. The following is an essential definition.

Definition 3.1. *The inverse transform of q-differential for $\Xi_k(\mathcal{X})$ is characterized by;*

$$\xi(\mathcal{X}, \mathfrak{T}) = \sum_{k=0}^{\infty} \Xi_k(\mathcal{X})(\mathfrak{T} - \mathbf{a})^{(k)}. \tag{3.4}$$

Substituting Equations (3.3) in (3.4) we achieve

$$\xi(\mathcal{X}, \mathfrak{T}) = \sum_{k=0}^{\infty} \frac{1}{[k]_q!} \left[\frac{\partial^k_q}{\partial_q \mathfrak{T}^k} \xi(\mathcal{X}, \mathfrak{T}) \right]_{\mathfrak{T}=\mathbf{a}} (\mathfrak{T} - \mathbf{a})^{(k)}. \tag{3.5}$$

In the following theorems, we establish $\mathbf{a} = 0$ so that $(\mathfrak{T} - \mathbf{a})^{(k)} = (\mathfrak{T} - 0)^{(k)} = \mathfrak{T}^k$.

We may deduce the following truth from the linearity of the q-derivative, provided $\gamma(\mathcal{X}, \mathfrak{T}) = \alpha u(\mathcal{X}, \mathfrak{T}) \pm j(\mathcal{X}, \mathfrak{T})$ then $\Gamma_k(\mathcal{X}) = \alpha U_k(\mathcal{X}) \pm \mathcal{J}_k(\mathcal{X})$, α being a constant.

Step(2):

In this step, we will present some important theorems for the (RqDTM) that we use for solving (1.3):

Theorem 3.2. *Given $\gamma(\mathcal{X}, \mathfrak{T}) = \frac{\partial_q}{\partial_q \mathcal{X}} \xi(\mathcal{X}, \mathfrak{T})$ then $\Gamma_k(\mathcal{X}) = \frac{\partial_q}{\partial_q \mathcal{X}} \Xi_k(\mathcal{X})$.*

Proof. See [25]. □

Theorem 3.3. *Given $\gamma(\mathcal{X}, \mathfrak{T}) = \frac{\partial^r_q}{\partial_q \mathfrak{T}^r} \xi(\mathcal{X}, \mathfrak{T})$ then*

$$\Gamma_k(\mathcal{X}) = [k+1]_q [k+2]_q ... [k+r]_q \Xi_{k+r}(\mathcal{X}). \tag{3.6}$$

Proof. See [25]. □

The following lemma will be utilized to establish the next theorem [22]:

Lemma 3.4.

$$g(\mathcal{X}, \mathfrak{T} q^j) = \sum_{i=0}^{j} (-1)^i (1-q)^i \left\{ \begin{array}{c} j \\ i \end{array} \right\}_q q^{i(i-1)/2} \mathcal{X}^i D^i_{q,\mathfrak{T}} g(\mathcal{X}, \mathfrak{T}).$$

Theorem 3.5. *If $\gamma(\mathcal{X}, \mathfrak{T}) = \xi(\mathcal{X}, \mathfrak{T}) \phi(\mathcal{X}, \mathfrak{T})$ then $\Gamma_k(\mathcal{X}) = \sum_{n=0}^{k} \Xi_n(\mathcal{X}) \Phi_{k-n}(\mathcal{X})$.*

Proof.

$$\begin{aligned}
\Gamma_k(\mathcal{X}) &= \frac{1}{[k]_q!}\frac{\partial_q^k}{\partial_q \mathfrak{T}^k}\gamma(\mathcal{X},\mathfrak{T}) = \frac{1}{[k]_q!}\left[D^k_{q,\mathfrak{T}}\xi(\mathcal{X},\mathfrak{T})\phi(\mathcal{X},\mathfrak{T})\right]\\
&= \frac{1}{[k]_q!}\sum_{n=0}^{k}\left\{\begin{array}{c} k\\ n\end{array}\right\}_q D^n_{q,\mathfrak{T}}\xi(\mathcal{X},\mathfrak{T}q^{k-n})D^{k-n}_{q,\mathfrak{T}}\phi(\mathcal{X},\mathfrak{T})\\
&= \sum_{n=0}^{k}\frac{1}{[k]_q!}\cdot\frac{[k]_q!}{[n]_q![k-n]_q!}\left[\textstyle\sum_{i=0}^{k-n}(-1)^i(1-q)^i\left\{\begin{array}{c} k-n\\ i\end{array}\right\}_q q^{i(i-1)/2}\mathcal{X}^i D^{n+i}_{q,\mathfrak{T}}\xi(\mathcal{X},\mathfrak{T}).D^{k-n}_{q,\mathfrak{T}}\phi(\mathcal{X},\mathfrak{T})\right]\\
&= \sum_{n=0}^{k}\frac{1}{[n]_q![k-n]_q!}D^n_{q,\mathfrak{T}}\xi(\mathcal{X},\mathfrak{T})D^{k-n}_{q,\mathfrak{T}}\phi(\mathcal{X},\mathfrak{T})\\
&= \sum_{n=0}^{k}\frac{1}{[n]_q!}D^n_{q,\mathfrak{T}}\xi(\mathcal{X},\mathfrak{T}).\frac{1}{[k-n]_q!}D^{k-n}_{q,\mathfrak{T}}\phi(\mathcal{X},\mathfrak{T}) = \sum_{n=0}^{k}\Xi_n(\mathcal{X})\Phi_{k-n}(\mathcal{X}).
\end{aligned}$$

□

Step(3):
We apply the above theorems to find the (RqDTM) for the partial q-differential equations for solving (1.3). Now, we apply the above steps to (1.3).
Since,

$$D^k_{q,\mathcal{X}} \equiv \frac{\partial_q^k}{\partial_q \mathcal{X}^k} \quad \& \quad D^k_{q,\mathfrak{T}} \equiv \frac{\partial_q^k}{\partial_q \mathfrak{T}^k}. \tag{3.7}$$

Substituting by (3.7) in (1.3) we get,

$$\frac{\partial_q^2}{\partial_q \mathfrak{T}^2}\xi(\mathfrak{T},\mathcal{X}) = c^2\frac{\partial_q^2}{\partial_q \mathcal{X}^2}\xi(\mathfrak{T},\mathcal{X}). \tag{3.8}$$

Taking (RqDTM) for the partial q-differential Equation (3.8), then we get:

$$[k+1]_q[k+2]_q\Xi_{k+2}(\mathcal{X}) = c^2\frac{\partial_q^2}{\partial_q \mathcal{X}^2}\Xi_k(\mathcal{X}), \tag{3.9}$$

where k=0,1,2,... and taking the initial conditions to become;

$$\Xi_0(\mathcal{X}) = e_q\left(\sqrt{M}\mathcal{X}\right)\&\Xi_1(\mathcal{X}) = \sqrt{c^2M}\left(e_q\left(\sqrt{M}\mathcal{X}\right)\right). \tag{3.10}$$

Set k=0 in (3.9) then we get;

$$[1]_q[2]_q\Xi_2(\mathcal{X}) = c^2\frac{\partial_q^2}{\partial_q \mathcal{X}^2}e_q\left(\sqrt{M}\mathcal{X}\right).$$

Hence;

$$\Xi_2(\mathcal{X}) = \frac{(c^2M)\left(e_q\left(\sqrt{M}\mathcal{X}\right)\right)}{q+1}, \tag{3.11}$$

and since, we have:

$$\xi(\mathfrak{T},\mathcal{X}) = \left(\sum_{k=0}^{\infty}\Xi_k(\mathcal{X})\mathfrak{T}^k\right). \tag{3.12}$$

Substituting by (3.10) and (3.11) in (3.12), then we get :

$$\xi(\mathfrak{T},\mathcal{X}) = e_q\left(\sqrt{M}\mathcal{X}\right) + \mathfrak{T}\sqrt{c^2M}\left(e_q\left(\sqrt{M}\mathcal{X}\right)\right) + \frac{\mathfrak{T}^2\left(c^2M\right)\left(e_q\left(\sqrt{M}\mathcal{X}\right)\right)}{q+1} + \dots \quad . \tag{3.13}$$

4. Analytical solution of (1.3) by using (SM)

To find the solution of (1.3) using (SM) we need to apply the following steps of (1.3).

Step(1):

In this step, we suppose that the solution of (1.3) can be expressed as a multiple of two functions say $\mathcal{G}(\mathfrak{T})$ and $\mathcal{H}(\mathcal{X})$, then we obtain two partial q-differential equations.

Step(2): Now let us apply the above step:

Assume that:

$$\xi(\mathfrak{T},\mathcal{X}) = \mathcal{G}(\mathfrak{T})\mathcal{H}(\mathcal{X}).$$

Substituting in (1.3) we get

$$\mathcal{H}(\mathcal{X})D^2_{q,\mathfrak{T}}\mathcal{G}(\mathfrak{T}) = c^2\mathcal{G}(\mathfrak{T})D^2_{q,\mathcal{X}}\mathcal{H}(\mathcal{X}).$$

That is the same as saying,

$$\frac{D^2_{q,\mathfrak{T}}\mathcal{G}(\mathfrak{T})}{c^2\mathcal{G}(\mathfrak{T})} = \frac{D^2_{q,\mathcal{X}}\mathcal{H}(\mathcal{X})}{\mathcal{H}(\mathcal{X})} = M,$$

where M is constant. Then we have,

$$D^2_{q,\mathcal{X}}\mathcal{H}(\mathcal{X}) = M\mathcal{H}(\mathcal{X})\& D^2_{q,\mathfrak{T}}\mathcal{G}(\mathfrak{T}) = c^2M\mathcal{G}(\mathfrak{T}).$$

Hence, while $0 < q < 1$, we can write

$$D_{q,\mathcal{X}}\xi(\mathfrak{T},\mathcal{X}) = \frac{\xi(\mathfrak{T},\mathcal{X}) - \xi(\mathfrak{T},q\mathcal{X})}{(1-q)\mathcal{X}} = \Lambda(\mathfrak{T},\mathcal{X}), \tag{4.1}$$

and

$$D^2_{q,\mathcal{X}}\xi(\mathfrak{T},\mathcal{X}) = D_{q,\mathcal{X}}\Lambda(\mathfrak{T},\chi) = \frac{\Lambda(\mathfrak{T},\mathcal{X}) - \Lambda(\mathfrak{T},q\mathcal{X})}{(1-q)\mathcal{X}}. \tag{4.2}$$

Using equation (4.2) we can obtain

$$\frac{q\mathcal{H}(\mathcal{X}) - (1+q)\mathcal{H}(q\mathcal{X}) + \mathcal{H}(q^2\mathcal{X})}{\mathcal{X}^2(1-q)^2q} = M\mathcal{H}(\mathcal{X}),$$

$$q\mathcal{H}(\mathcal{X}) - \mathcal{H}(q\mathcal{X})(1+q) + \mathcal{H}(q^2\mathcal{X}) = q\mathcal{X}^2(1-q)^2M\mathcal{H}(\mathcal{X}).$$

The above equation gives

$$\left(1-(1-q)^2\mathcal{X}^2M\right)q\mathcal{H}(\mathcal{X}) = \mathcal{H}(q\mathcal{X})(1+q) - \mathcal{H}(q^2\mathcal{X}). \tag{4.3}$$

As a result, we have

$$q\left(1-q^2\mathcal{X}^2(1-q)^2M\right)\mathcal{H}(q\mathcal{X}) = (1+q)\mathcal{H}(q^2\chi) - \mathcal{H}(q^3\mathcal{X}),$$

$$\vdots$$

$$q\left(1-q^{2n}\mathcal{X}^2(1-q)^2M\right)\mathcal{H}(q^n\mathcal{X}) = (1+q)\mathcal{H}(q^{n+1}\mathcal{X}) - \mathcal{H}(q^{n+2}\mathcal{X}),$$

$$q\left(1-q^{2n+2}\mathcal{X}^2(1-q)^2M\right)\mathcal{H}(q^{n+1}\mathcal{X}) = (1+q)\mathcal{H}(q^{n+2}\mathcal{X}) - \mathcal{H}(q^{n+3}\mathcal{X}).$$

Then, we deduce that

$$\begin{aligned} q(1-\alpha_{\mathcal{X}})\mathcal{H}(\mathcal{X}) + q\mathcal{H}(q\mathcal{X}) - q^3\alpha_{\mathcal{X}}\mathcal{H}(q\mathcal{X})... + q\mathcal{H}(q^n\mathcal{X}) - q^{2n+1}\alpha_{\mathcal{X}}\mathcal{H}(q^n\mathcal{X}) + q\mathcal{H}(q^{n+1}\mathcal{X}) \\ - q^{2n+3}\alpha_{\mathcal{X}}\mathcal{H}(q^{n+1}\mathcal{X}) = \mathcal{H}(q\mathcal{X}) + q\mathcal{H}(q\mathcal{X}) - \mathcal{H}(q^2\mathcal{X}) + \mathcal{H}(q^2\mathcal{X}) + q\mathcal{H}(q^2\mathcal{X}) \\ - \mathcal{H}(q^3\mathcal{X})... + \mathcal{H}(q^{n+1}\mathcal{X}) + q\mathcal{H}(q^{n+1}\mathcal{X}) - \mathcal{H}(q^{n+2}\mathcal{X}) \\ + \mathcal{H}(q^{n+2}\mathcal{X}) + q\mathcal{H}(q^{n+2}\mathcal{X}) - \mathcal{H}(q^{n+3}\mathcal{X}). \end{aligned}$$

Implies that

$$(1-\alpha_{\mathcal{X}})\mathcal{H}(\mathcal{X})q - (q^3\mathcal{H}(q\mathcal{X}))\alpha_{\mathcal{X}} + ... + \mathcal{H}(q^{n+1}\mathcal{X})q^{2n+3} = \mathcal{H}(q\mathcal{X}) + q\mathcal{H}(q^{n+2}\mathcal{X}) - \mathcal{H}(q^{n+3}\mathcal{X}), \tag{4.4}$$

where

$$\alpha_{\mathcal{X}} = \mathcal{X}^2(1-q)^2M.$$

Now, letting $\mathcal{H}(\mathcal{X})$ given by

$$\mathcal{H}(\mathcal{X}) = \sum_{i=0}^{\infty} b_i\mathcal{X}^i.$$

Then, we get

$$\begin{aligned} &\alpha_{\mathcal{X}}\left(q^3\textstyle\sum_{i=0}^{\infty} b_iq^i\mathcal{X}^i + ... + q^{2n+3}\sum_{i=0}^{\infty} b_iq^{(n+1)i}\mathcal{X}^i\right) \\ &= q^3(1-q)^2M\left(\textstyle\sum_{i=0}^{\infty}(1+q^i+...+q^{(i+2)n})b_iq^i\mathcal{X}^{i+2}\right) \\ &= q^3(1-q)^2M\sum_{i=0}^{\infty} b_iq^i\left(\tfrac{1-(q^{2+i})^{n+1}}{1-q^{2+i}}\right)\mathcal{X}^{i+2} = \beta, \end{aligned}$$

when $n \to \infty$, the last term provides

$$\beta = q^3(1-q)^2M\sum_{i=0}^{\infty} b_{i-2}\frac{q^i}{1-q^i}\mathcal{X}^i. \tag{4.5}$$

Then, Equation (4.4) and Equation (4.5) yield

$$\left(1-(1-q)^2M\mathcal{X}^2\right)q\sum_{i=0}^{\infty} b_i\mathcal{X}^i - \beta = \left(\textstyle\sum_{i=0}^{\infty} b_iq^i\mathcal{X}^i\right) + b_0(q-1).$$

Therefore, we get

$$\left(\textstyle\sum_{i=0}^{\infty} qb_i\mathcal{X}^i\right) - qM(1-q)^2\sum_{i=0}^{\infty} b_i\mathcal{X}^{i+2} - \beta = \left(\textstyle\sum_{i=0}^{\infty} b_iq^i\mathcal{X}^i\right) + (q-1)b_0,$$

which results

$$\begin{aligned} qb_0 + qb_1\mathcal{X} + \left(\textstyle\sum_{i=0}^{\infty} qb_i\mathcal{X}^i\right) - q(1-q)^2M\left(\textstyle\sum_{i=0}^{\infty} b_{i-2}\mathcal{X}^i\right) - \beta = \\ b_0 + qb_1\mathcal{X} + \left(\textstyle\sum_{i=0}^{\infty} q^ib_i\mathcal{X}^i\right) + (q-1)b_0. \end{aligned}$$

This means that

$$qb_i - q(1-q)^2Mb_{i-2} - q(1-q)^2M\frac{q^i}{1-q^i}b_{i-2} = b_iq^i.$$

Hence, we obtain

$$q(1-q^{i-1})b_i = q(1-q)^2b_{i-2}M\left(1+\tfrac{q^i}{1-q^i}\right).$$

From the above equation we deduced that,

$$b_i q[i-1]_q = M\left(\frac{1-q^i+q^i}{1-q^i}\right) b_{i-2} q(1-q).$$

This implies that

$$[i-1]_q b_i = M\frac{(1-q)}{1-q^i} b_{i-2}.$$

Which results in

$$[i-1]_q b_i = M\frac{1}{[i]_q} b_{i-2}.$$

As a result, we receive

$$b_i = \frac{M}{[i]_q[i-1]_q} b_{i-2}, i \geq 2. \tag{4.6}$$

If $M > 0$, then

$$b_i = \frac{(\sqrt{M})^i}{[i]_q!}.$$

Therefore, we obtain

$$b_{i+2} = \frac{(\sqrt{M})^{i+2}}{[i+2]_q!} = \frac{(\sqrt{M})^i}{[i]_q!}.\frac{(\sqrt{M})^2}{[i+2]_q[i+1]_q} = b_i\frac{M}{[i+2]_q[i+1]_q}.$$

This confirms Equation (4.6) then, we have

$$\mathcal{H}(\mathcal{X}) = \sum_{i=0}^{\infty} b_i \mathcal{X}^i = \sum_{i=0}^{\infty} \frac{(\sqrt{M}\mathcal{X})^i}{[i]_q!} = e_q(\mathcal{X}\sqrt{M}),$$

when $M < 0$, we consider

$$b_i = \frac{(j\sqrt{-M})^i}{[i]_q!}, j = \sqrt{-1}.$$

As a result, we get

$$b_{i+2} = \frac{(j)^{i+2}(\sqrt{-M})^{i+2}}{[i+2]_q!} = \frac{(j\sqrt{-M})^i}{[i]_q!}.\frac{j^2(\sqrt{-M})^2}{[i+2]_q[i+1]_q} = b_i\frac{M}{[i+2]_q[i+1]_q}.$$

This confirms Equation (4.6) then, we get

$$\mathcal{H}(\mathcal{X}) = \sum_{i=0}^{\infty} b_i \mathcal{X}^i = e_q\Big(j\mathcal{X}\sqrt{-M}\Big).$$

Similarly, we get:

$$\mathcal{G}(\mathfrak{T}) = e_q\left(\mathfrak{T}\sqrt{c^2 M}\right),$$

where $M > 0$,

$$\mathcal{G}(\mathfrak{T}) = e_q\left(j\mathfrak{T}\sqrt{c^2(-M)}\right),$$

where $M < 0$.

Hence,

$$\xi(\mathfrak{T}, \mathcal{X}) = \left(e_q\left(\sqrt{M}\mathcal{X}\right)\right)\left(e_q\left(\mathfrak{T}\sqrt{c^2 M}\right)\right). \tag{4.7}$$

Where $M > 0$,

$$\xi(\mathfrak{T},\mathcal{X}) = \left(e_q\left(j\mathfrak{T}\sqrt{c^2(-M)}\right)\right)\left(e_q\left(j\sqrt{-M}\mathcal{X}\right)\right).$$

Where $M < 0$.

But from the definition of q-analogue of the classical exponential function, then Equation (4.7) can be expressed as:

$$\xi(\mathfrak{T},\chi) = e_q\left(\sqrt{M}\mathcal{X}\right) + \mathfrak{T}\sqrt{c^2M}\left(e_q\left(\sqrt{M}\mathcal{X}\right)\right) + \frac{\mathfrak{T}^2\left(c^2M\right)\left(e_q\left(\sqrt{M}\mathcal{X}\right)\right)}{q+1} + \dots . \tag{4.8}$$

Then, we deduced that Equation (4.8) and Equation (3.13) is the same, that mean the analytical solution by (RqDTM) that is given in Equation (3.13) is equivalent to the solution by (SM) which is given in (4.7).

5. Approximation solution for (1.3) by using (VIM)

The relation of (VIM) can be writen as: [28]

$$\xi_{n+1}(\mathcal{X},\mathfrak{T}) = \xi_n(\mathcal{X},\mathfrak{T}) + \int_0^{\mathfrak{T}} \lambda(\mathfrak{T},q\tau)\left[\frac{\partial_q^2}{\partial_q\tau^2}\xi_n(\mathcal{X},\tau) - \frac{\partial_q^2}{\partial_q\chi^2}\xi_n(\mathcal{X},\tau)\right]d_q\tau. \tag{5.1}$$

Taking the variational derivative δ for the Equation (5.1), we have

$$\delta\xi_{n+1}(\mathcal{X},\mathfrak{T}) = \delta\xi_n(\mathcal{X},\mathfrak{T}) + \delta\int_0^{\mathfrak{T}} \lambda(\mathfrak{T},q\tau)\frac{\partial_q^2\xi_n(\mathcal{X},\tau)}{\partial_q\tau^2}d_q\tau, \tag{5.2}$$

but we have from q-integration by parts (2.1),

$$\delta\xi_{n+1}(\mathcal{X},\mathfrak{T}) = \Big(1+\lambda(\mathfrak{T},\tau)\Big)\delta\xi_n(\mathcal{X},\mathfrak{T}) - \int_0^{\mathfrak{T}} \delta\xi_n(\mathcal{X},\tau)\frac{\partial_q^2\lambda(\mathfrak{T},\tau)}{\partial_q\tau^2}d_q\tau. \tag{5.3}$$

The fixed part of the functional (5.3) , *i.e.* $\Big(\delta\xi_{n+1}(\mathcal{X},\mathfrak{T}) = 0\Big)$, the Lagrange multiplier is subject to certain conditions, including

$$\begin{cases} 1+\lambda(\mathfrak{T},\tau) = 0 \\ \frac{\partial_q^2\lambda(\mathfrak{T},\tau)}{\partial_q\tau^2} = 0. \end{cases} \tag{5.4}$$

From (5.4) the Lagrange multiplier can be calculated as $\lambda(\mathfrak{T},\tau) = 1$, and substituting by this equation in (5.1), we get the variational iteration form which can be expressed as

$$\begin{cases} \xi_{n+1}(\mathcal{X},\mathfrak{T}) = \xi_n(\mathcal{X},\mathfrak{T}) - \int_0^{\mathfrak{T}}\left[\frac{\partial_q^2}{\partial_q\tau^2}\xi_n(\mathcal{X},\tau) - \frac{\partial_q^2}{\partial_q\mathcal{X}^2}\xi_n(\mathcal{X},\tau)\right]d_q\tau \\ \xi_0(\mathcal{X},\mathfrak{T}) = e_q(\sqrt{M}\mathcal{X}). \end{cases} \tag{5.5}$$

By applying iterations, we obtain the following approximations:

$$\xi_0(\mathcal{X},\mathfrak{T}) = e_q(\sqrt{M}\mathcal{X}),$$

$$\xi_1(\mathcal{X},\mathfrak{T}) = e_q(\sqrt{M}\mathcal{X}) + \frac{Me_q(\sqrt{M}\mathcal{X})\mathfrak{T}}{[1]_q!},$$

$$\xi_2(\mathcal{X},\mathfrak{T}) = e_q(\sqrt{M}\mathcal{X}) + \frac{Me_q(\sqrt{M}\mathcal{X})\mathfrak{T}}{[1]_q!} + \frac{M^2e_q(\sqrt{M}\mathcal{X})\mathfrak{T}^2}{[2]_q!},\dots.$$

$$\begin{aligned}\xi_n(\mathcal{X},\mathfrak{T}) &= \left[1+\frac{M\mathfrak{T}}{[1]_q!}+\frac{M^2\mathfrak{T}^2}{[2]_q!}+\frac{M^3\mathfrak{T}^3}{[3]_q!}+....\right]e_q(\sqrt{M}\mathcal{X})\\ &= \left(\sum_{i=0}^{n}\frac{(M\mathfrak{T})^i}{[i]_q!}\right)\left(e_q(\sqrt{M}\mathcal{X})\right).\end{aligned} \tag{5.6}$$

Equation (5.6) represents the approximation solution of (1.3).

6. Numerical results

In this section, we are dedicated to showcasing the numerical outcomes for Equation (1.3) through the solutions obtained in equation (5.6) and Equation (4.7). Notably, we employed different values for the parameters $\mathcal{X}$, q, and t to explore the impact of varying these parameters on the results.

The obtained numerical results, which provide valuable insights into the problem at hand, are presented in a tabular format called Table 1 and visually represented through Figure 1 at different values for $\mathcal{X}$ and $M = 0.1, c = 1, \mathfrak{T} = 0.01, q = 0.0001, \mathcal{X} \in [0, 1]$. In addition, Figure 2 presents the absolute error in this case. Furthermore, if we change the domain to $\mathcal{X} \in [-10, 10]$ we get a waveform that appears in Figure 3. These representations help to provide a clear and comprehensive overview of the data.

Table 1. A comparison between the numerical results and the analytical solution at $M = 0.1, c = 1, \mathfrak{T} = 0.01, q = 0.0001, \mathcal{X} \in [0, 1]$.

$\mathcal{X}$	Numerical solutions (VIM)	Analytical solutions (RqDTM & SM)	Abs. error
0.1	1.03369	1.03593	2.2422 E-3
0.2	1.06858	1.07090	2.3179 E-3
0.3	1.10592	1.10832	2.3988 E-3
0.4	1.14595	1.14844	2.4857 E-3
0.5	1.18899	1.19157	2.5790 E-3
0.6	1.23540	1.23808	2.6797 E-3
0.7	1.28557	1.28836	2.2788 E-3
0.8	1.33998	1.34289	2.9066 E-3
0.9	1.39921	1.40225	3.0350 E-3

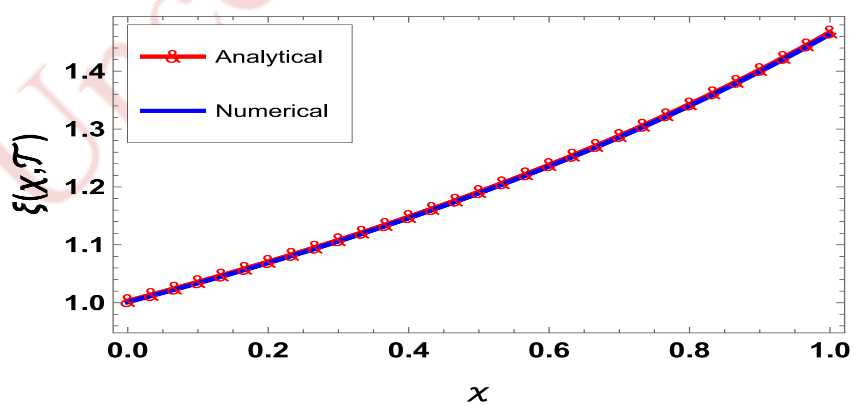

Figure 1. The graph shows the relation between the numerical and analytical solutions for (1.3) at $M = 0.1, c = 1, \mathfrak{T} = 0.01, q = 0.0001, \mathcal{X} \in [0, 1]$.

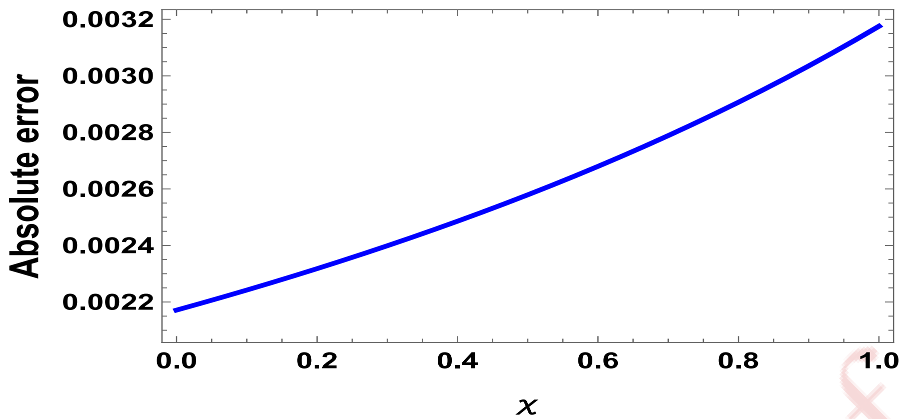

FIGURE 2. The absolute error between the analytical (4.7) and numerical (5.6) solutions at $M = 0.1, c = 1, \mathfrak{T} = 0.01, q = 0.0001, \mathcal{X} \in [0, 1]$

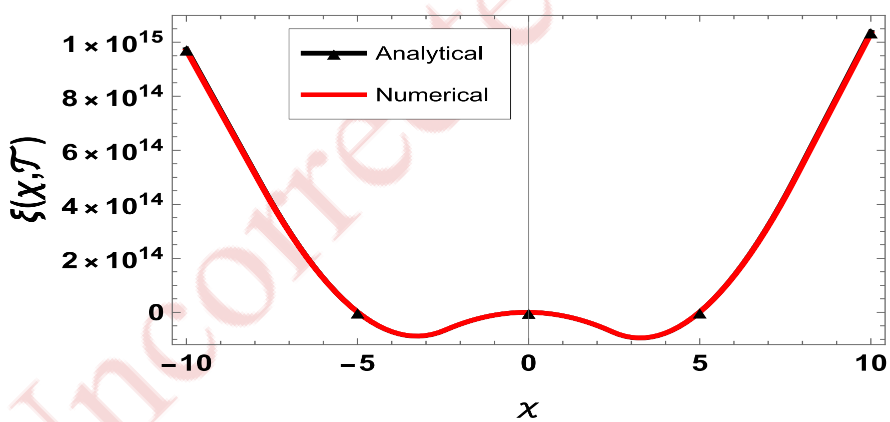

FIGURE 3. The graph shows the comparison between the numerical and analytical solutions for (1.3) at $M = 0.1, c = 1, \mathfrak{T} = 0.0001, q = 0.0001, \mathcal{X} \in [-10, 10]$.

From the above results, we investigate that the analytical methods and the approximate method give related results, and the maximum error between them is 3.0350×10^{-3}.

Furthermore, we continued the analysis by presenting additional results in Table 2 at different values for q and $M = 0.1, c = 1, \mathfrak{T} = 0.01, \mathcal{X} = 0.3, q \in [0, 1]$. In addition, Figure 4 presents the solutions at at different values for q. If we change the domain to $\mathcal{X} \in [-10, 10]$, we get a waveform that appears in Figure 5.

TABLE 2. A comparison between the numerical results and the analytical solution at $M = 0.1, c = 1, \mathfrak{T} = 0.01, \mathcal{X} = 0.3, q \in [0, 1]$.

q	Numerical solutions (VIM)	Analytical solutions (RqDTM & SM)	Abs. error
0.1	1.10492	1.10732	2.3958 E-3
0.2	1.10409	1.10649	2.3932 E-3
0.3	1.10340	1.10579	2.3911 E-3
0.4	1.10281	1.10520	2.3893 E-3
0.5	1.10231	1.10470	2.3877 E-3
0.6	1.10188	1.10427	2.3863 E-3
0.7	1.10150	1.10389	2.3852 E-3
0.8	1.10117	1.10355	2.3841 E-3
0.9	1.10088	1.10326	2.3832 E-3

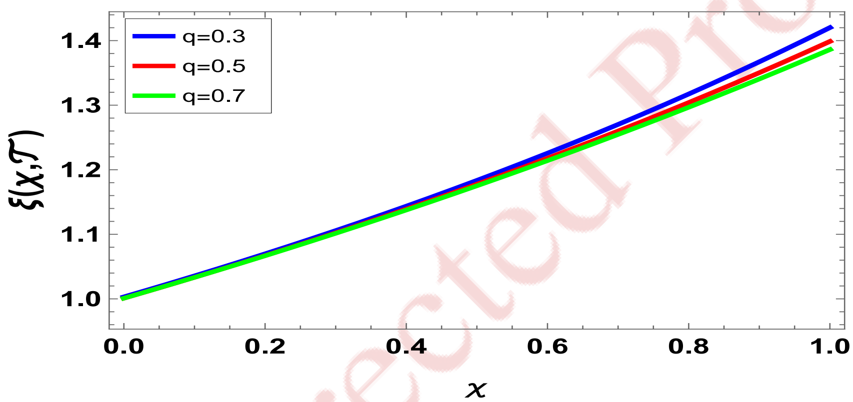

FIGURE 4. The graph shows the effect of q on the numerical solution for Equation (5.6) at $M = 0.1, c = 1, \mathfrak{T} = 0.01, \mathcal{X} \in [0, 1]$.

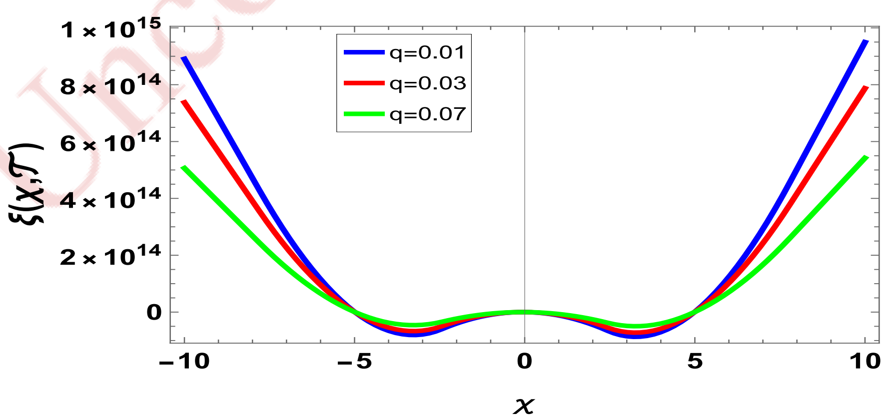

FIGURE 5. The graph shows the effect of q on the numerical solution for Equation (5.6) at $M = 0.1, c = 1, \mathfrak{T} = 0.0001, \mathcal{X} \in [-10, 10]$.

From the above results, we investigate that the changing of parameter q not effect by big form on results, and the maximum error between them is 2.3958×10^{-3}.

Finally, we proceeded to present the remaining set of results in Table 3 at different values for $\mathfrak{T}$ and $M = 0.1, c = 1, q = 0.0001, \mathcal{X} = 0.3, \mathfrak{T} \in [0, 1]$. In addition, Figure 6 presents the solutions at at different values for $\mathfrak{T}$. If we change the domain to $\mathcal{X} \in [-10, 10]$, we get a waveform that appears in Figure 7.

TABLE 3. A comparison between the numerical results and the analytical solution at $M = 0.1, c = 1, q = 0.0001, \mathcal{X} = 0.3, \mathfrak{T} \in [0, 1]$.

$\mathfrak{T}$	Numerical solutions	Analytical solutions	Abs.error
0.01	1.10592	1.10832	2.3988 E-3
0.02	1.10702	1.11184	4.8178 E-3
0.03	1.10813	1.11539	7.2571 E-3
0.04	1.10925	1.11896	9.7169 E-3
0.05	1.11036	1.12256	1.2197 E-2
0.06	1.11148	1.12618	1.4698 E-2
0.07	1.11260	1.12982	1.7221 E-2
0.1	1.11597	1.14089	2.4918 E-2
0.3	1.13898	1.22061	8.1626 E-2
0.5	1.16296	1.31230	1.4934 E-1
0.7	1.18797	1.41889	2.3092 E-1
0.9	1.21408	1.54432	3.3024 E-1

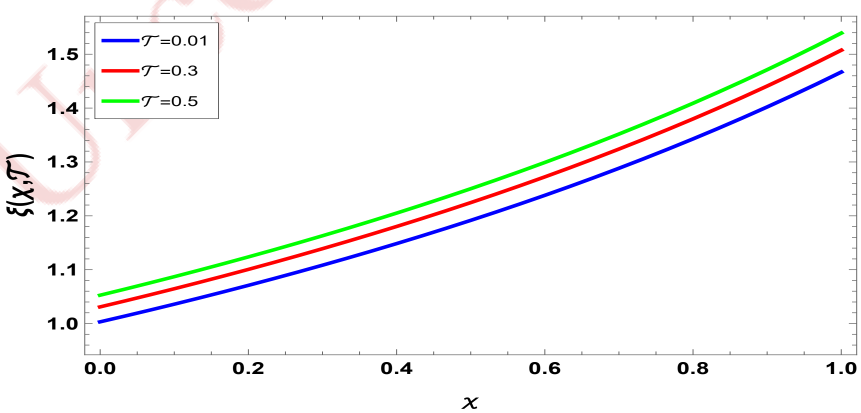

FIGURE 6. The graph shows the effect of changing $\mathfrak{T}$ on the numerical solution for Equation (5.6) at $M = 0.1, c = 1, q = 0.0001, \mathcal{X} \in [0, 1]$

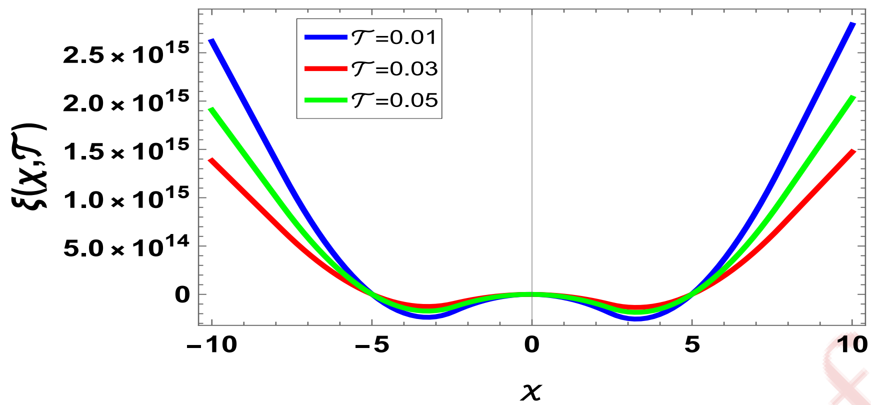

FIGURE 7. The graph shows the effect of changing $\mathfrak{T}$ on the numerical solution for Equation (5.6) at $M = 0.1, c = 1, q = 0.0001, \mathcal{X} \in [-10, 10]$

From the above results, we investigate the effect of changing parameter $\mathfrak{T}$ by big form on results, and the maximum error is 3.3024×10^{-1}. Concluding the section with a thorough exploration of the numerical outcomes obtained under various parameter configurations. By presenting these results in both tabular and graphical formats, we aim to enhance the readers' understanding and facilitate the interpretation of the findings.

7. Discussion

The discussion of our paper revolves around the investigation of three distinct approaches for solving the generalized q-deformed wave equation: the reduced q-differential transform method (RqDTM), the separation method (SM), and the variational iteration method (VIM). These methods were chosen due to their relevance and potential for providing solutions to the q-deformed wave equation, which serves as a mathematical framework for describing physical systems with violated symmetries. By incorporating a q-deformation parameter, the q-deformed wave equation extends the conventional wave equation, introducing non-commutativity and non-linearity into the system dynamics. This expansion allows us to capture and model physical systems that exhibit these characteristics. The RqDTM, specifically designed for q-deformed equations, provides a modified version of the differential transform method tailored to handle the complexities of the q-deformed wave equation. On the other hand, the SM seeks solutions expressed in separable variables, while the VIM utilizes an iterative scheme to refine the solution. In our comparative analysis, we evaluate the accuracy and efficiency of the solutions obtained through these three methods. By presenting numerical results, we provide a quantitative assessment of their performance. This assessment enables us to identify the strengths and weaknesses of each approach in effectively solving the q-deformed wave equation. We gain valuable insights into the applicability and performance of the RqDTM, SM, and VIM in addressing the challenges posed by the q-deformation parameter. Our study contributes to the understanding of the q-deformed wave equation by exploring and comparing three different approaches: RqDTM, SM, and VIM. The comparative analysis sheds light on the strengths and weaknesses of each method in terms of accuracy and efficiency. Additionally, the investigation of a generalized q-deformed wave equation expands the scope of our study, further enriching our understanding of the problem and the methods used to solve it.

8. Conclusion

This research has made significant contributions to the study of the q-deformed wave equation and its solutions by employing various mathematical methods. Firstly, we introduced the reduced q-differential transform method (RqDTM) as an analytical approach to solving Equation (1.3). By applying this method, we derived an explicit expression that accurately describes the behavior of the q-deformed wave equation. The RqDTM demonstrated its effectiveness as a powerful tool for obtaining analytical solutions within the q-calculus framework. Additionally, we explored the separation method (SM) as another analytical approach to solving Equation (1.3). By employing the

SM, we obtained an analytical solution that represented the q-deformed wave equation in terms of separable variables. This technique allowed us to simplify the problem and obtain a more manageable solution. Furthermore, we presented the variational iteration method (VIM) as an approximation technique for solving Equation (1.3). The iterative nature of VIM enabled us to progressively refine the initial guess, providing an approximate solution to the q-deformed wave equation. The effectiveness of VIM was demonstrated by its ability to closely approximate the exact solution. To validate the solutions obtained through these methods, we conducted numerical simulations of Equation (1.3). These simulations involved solving the equation for various parameter values and assessing the accuracy of the solutions. A comprehensive comparison between the analytical and numerical solutions was performed, highlighting the agreement and discrepancies between the two approaches. To enhance the understanding of the solutions and their implications, visual illustrations were provided. These graphical representations of the solutions offered insights into the behavior and characteristics of the q-deformed wave equation under different parameter values. The visual aids served as additional support for our findings and conclusions.

References

[1] W. Abdi, *On certain q-difference equations and q-Laplace transforms*, Proceedings of the Indian National Science Academy, *28* (1992), 1-15.

[2] F. Ali, S. Majeed, and A. Imtiaz, *Magnetohydrodynamic Blood Flow in a Cylindrical Tube with Magnetic Particles: A Time Fractional Model*, Mathematical Problems in Engineering, (2021), 6624912.

[3] M. Ali, M. Sharma, and R. Jain, *An Application of Fractional Calculus in Electrical Engineering*, Advanced Engineering Technology and Application, *5* (2016), 41-45.

[4] D. Anderson and R. Avery, *Fractional-order boundary value problem with Sturm-Liouville boundary conditions*, Electronic Journal of Differential Equations, *29* (2015), 1-10.

[5] O. Arqub and B. Maayah, *Adaptive the Dirichlet model of mobile/immobile advection/dispersion in a time-fractional sense with the reproducing kernel computational approach: Formulations and approximations*, International Journal of Modern Physics B, *37* (2023), 1-18.

[6] O. Arqub, H. Alsulami, and M. Alhodaly, *Numerical Hilbert space solution of fractional Sobolev equation in (1+1)-dimensional space*, Mathematical Sciences, (2022), 1-12.

[7] F. Atici and P. Eloe, *Fractional q-Calculus on a time scale*, Journal of Nonlinear Mathematical Physics, *14* (2007), 341352.

[8] H. Badawi, O. Arqub, and N. Shawagfeh, *Well-posedness and numerical simulations employing Legendre-shifted spectral approach for CaputoFabrizio fractional stochastic integrodifferential equations*, International Journal of Modern Physics C, *34* (2023).

[9] J. Blackledge, *Quantum Mechanics and Control Using Fractional Calculus: A Study of the Shutter Problem for Fractional Quantum Fields*, International Applied Mechanics, *3* (2022), 413463.

[10] J. Brown, *Calculation of a Constant Q Spectral Transform*, Journal of the Acoustical Society of America, *89* (1991), 425-434.

[11] A. Oliveira, *DAlembert and the Wave Equation: Its Disputes and Controversies*, Advances in Historical Studies, *9* (1747).

[12] DAlembert and J. Le R, *Trait de dynamique*, Gauthier-Villars et Cie. ditions, *9* (1743), 1702-2003.

[13] M. Elamin, N. Khalifa, and S. Altoum, *The Q-Space Deformed Wave Equation*, Complexity, (2022), 8753231.

[14] L. Euler, *Remarques sur les mmoires prcdents de M*, Bernoulli. ibd, *9* (1753), 196-222.

[15] G. Gasper and M. Rahman, *Basic hypergeometric series*, Encyclopedia Of Mathematics And Its Application, *35* (1990), 1-428.

[16] H. Jackson, *q-Difference equations*, American Journal of Mathematics, *32* (1990), 305-314.

[17] H. Jafari, A. Haghbin, S. Hesam, and D. Baleanu, *Solving Partial q-Differential Equations within Reduced q-Differential Transformation Method*, Romanian Journal of Physics, *59* (2014), 399-407.

[18] M. Jleli, M. Mursaleen, and B. Samet, *Q-INTEGRAL EQUATIONS OF FRACTIONAL ORDERS*, ELECTRONIC JOURNAL OF DIFFERENTIAL EQUATIONS, *17* (2016), 1-14.

[19] R. Johal, *q calculus and entropy in nonextensive statistical physics*, American Physical Society, *58* (1998), 4147-4151.

[20] A. Lavagno and G. Gervino, *Quantum mechanics in q-deformed calculus*, Journal of Physics: Conference Series, *174* (2009), 012071.

[21] H. Leeuwen and H. Maassen, *A q-deformation of the Gauss distribution*, Journal of mathematical physic, *36* (1995), 4743-4756.

[22] B. Maayah, A. Moussaoui, S. Bushnaq, and O. Arqub, *The multistep Laplace optimized decomposition method for solving fractional-order coronavirus disease model (COVID-19) via the Caputo fractional approach*, Nonlinear Engineering Modeling and Application, *55* (2022), 963977.

[23] G. Mussardo, G. Giudici, and J. Viti, *The Coprime Quantum Chain*, Journal of Statistical Mechanics: Theory and Experiment, (2017), 033104.

[24] J. Nasir, S. Qaisar, S. Butt, K. Khan, and R. Mabela, *Some Simpsons RiemannLiouville Fractional Integral Inequalities with Applications to Special Functions*, Journal of Function Spaces, (2022), 2113742.

[25] M. Sadik and B. Orie, *Application of q-Calculus to the Solution of Partial q-Differential Equations*, Applied Mathematics, *12* (2021), 669-678.

[26] N. Sweilam, A. Nagy, and T. Al-Ajami, *Numerical solutions of fractional optimal control with CaputoKatugampola derivative*, Advances in Difference Equations, (2021), 1-16.

[27] Y. Wang, *Solutions of complex difference and q-difference equations*, Advances in Difference Equations, *98* (2016), 1-22.

[28] G. Wu, *VARIATIONAL ITERATION METHOD FOR THE q-DIFFUSION EQUATIONS ON TIME SCALES*, Heat Transfer Research, *44* (2013), 393-398.