Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 13, No. 1, 2025, pp. 282-293
DOI:10.22034/cmde.2024.58936.2498

Solving a class of Volterra integral equations with M-derivative

Mousa Ilie[1,*], Ali Khoshkenar[1], and Asadollah Torabi Giklou[2]

[1]Department of Mathematics, Rasht Branch, Islamic Azad University, Rasht, Iran.

[2]Department of Applied Mathematics, Parsabad Moghan Branch, Islamic Azad University, Parsabad Moghan, Iran.

Abstract

In this current article, the well-known Neumann method for solving the time M-fractional Volterra integral equations of the second kind is developed. In the several theorems, existence and uniqueness of the solution and convergence of the proposed approach are also studied. The Neumann method for this class of the time M-fractional Volterra integral equations has been called the M-fractional Neumann method (MFNM). The results obtained demonstrate the efficiency of the proposed method for the time M-fractional Volterra integral equations. Several illustrative numerical examples have presented the ability and adequacy of the MFNM for a class of fractional integral equations.

Keywords. Local M-fractional integral, M-fractional Volterra integral equations, M-fractional Neumann method, Existence and uniqueness of solution, Theorem of convergence.

2000 Mathematics Subject Classification. [2020] 45Axx-45Dxx-45Exx.

1. Introduction

Thanks to the efforts of mathematicians over the past few decades, fractional calculus is as well known to everyone as ordinary calculus. Famous mathematicians who have made significant efforts in this field include Riemann, Liouville, Grenville, Caputo, and other mathematicians are cited [8]. All fractional derivatives are generally divided into local and non-local categories. However, some of those definitions take care of drawbacks that caused their application to confront difficulties such as satisfying the derivative product rule, the derivative quotient rule, and the chain rule. In 2017, Sousa et al. introduced an M-fractional derivative involving a Mittag-Leffler function with one parameter that also satisfies the properties of integer-order calculus [12, 24]. In this sense, Sousa and Oliveira introduced a truncated M-fractional derivative type that unifies four existing fractional derivative types mentioned above and which also satisfied the classical properties of integer-order calculus [25]. The truncated M-fractional derivative is one of the types of local fractional derivatives, so in the mode of derivation from the positive integer order, it completely follows the ordinary derivative, and in the fractional mode, it has almost all the properties of the ordinary derivative. Many researchers have used the M-fraction derivative in their research [15, 17].

Definition 1.1. Given a function $f : [0, \infty) \to \mathbb{R}$. Then the truncated M-fractional derivative of f of order α is defined by

$$ {}_i\mathcal{D}_M^{\alpha,\beta} f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t_i\mathbb{E}_\beta(\varepsilon t^{-\alpha})\right) - f(t)}{\varepsilon}, \tag{1.1}$$

for all $t > 0$, $\alpha \in (0, 1)$, where ${}_i\mathbb{E}_\beta(\cdot)$, $\beta > 0$ is the Mittag-Leffler function with one parameter as defined by in Eq. (1.1) [25].

Received: 22 October 2023 ; Accepted: 27 April 2024.
* Corresponding author. Email: ilie@iaurasht.ac.ir, mousailie52@gmail.com .

Note that if f is α- differentiable in some $(0, a)$, $a \geq 0$, and $\lim_{t\to 0^+} {}_i\mathcal{D}_M^{\alpha,\beta} f(t)$ exists, then one can define [25]

$$ {}_i\mathcal{D}_M^{\alpha,\beta} f(0) = \lim_{t\to 0^+} {}_i\mathcal{D}_M^{\alpha,\beta} f(t). $$

Definition 1.2. Given a function $f : [0, \infty) \to \mathbb{R}$, $a \geq 0$. Then local M-fractional integral of f order α is defined by

$$ {}_M\mathcal{T}_a^{\alpha,\beta} f(t) = \Gamma(\beta + 1) \int_a^t \frac{f(s)}{s^{1-\alpha}} ds \tag{1.2} $$

, where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1)$, and $\beta > 0$ [25]. One of the well results is the following [24, 25]. Let $\alpha \in (0, 1)$, and f be α-differentiable at a point $t > 0$, then

H: (Invers theorem) ${}_i\mathcal{D}_M^{\alpha,\beta} \left({}_M\mathcal{T}_a^{\alpha,\beta} f(t)\right) = f(t)$,

I: (Fundamental theorem of calculus) ${}_M\mathcal{T}_a^{\alpha,\beta} \left({}_i\mathcal{D}_M^{\alpha,\beta} f(t)\right) = f(t) - f(0)$.

Most phenomena in our real world are described by time fractional functional equations (FFEs). Although having the exact solution of FFEs in analyzing the phenomena is important, there are many FFEs that cannot be resolved accurately. Due to this fact, finding approximate solutions to time-fractional functional equations is clearly required. In recent years, many effective methods have been proposed for approximate solutions of time-fractional integral equations [5–11].

Özturk used examination of Sturm-Liouville problem with proportional derivatives in control theory [20]. Duran et al. obtained the discussion of numerical and analytical techniques for the emerging fractional order Murnaghan model in materials science [10]. Tariq et al. found some integral inequalities via a new family of preinvex functions. Bonayah et al. created a robust study on the listeriosis disease by adopting fractal-fractional operators [23]. Bonyah et al. proposed a robust study on the listeriosis disease by adopting fractal-fractional operators.[7]. Zada et al. obtained new approximate-analytical solutions to partial differential equations via auxiliary function method [7]. Youssri et al. used numerical spectral Legandre-Galerkin algorithm for solving time fractional Telegraph equation [27]. Hafez et al. proposed a shifted Jacobi collocation scheme for multidimensional time-fractional order telegraph equation [13]. Atta et al. found advanced shifted first-kind Chebyshev collocation approach for solving the nonlinear time-fractional partial integro-differential equation with a weakly singular kernel [6]. Abdelghany et al. gained a Tau approach for solving time-fractional heat equation based on the shifted sixth-kind chebyshev polynomials [1]. Youssri et al. obtained Petrov-Galerkin lucas polynomials procedure for the time-fractional diffusion equation [28]. Moustafa et al. used explicit Chebyshev PetrovGalerkin scheme for time-fractional fourth-order uniform EulerBernoli pinned beam equation [19].

In this study, the Neumann method is advanced for time M-fractional Volterra integral equations of the second kind, and convergence of the proposed approach is studied and this method is utilized to find an approximate solution of the time M-fractional Volterra integral equations.

The organization of the paper is as follows: In sections 2 and 3, several primary definitions and essential concepts and convergence study related to the local time M-fractional Volterra integral equations of the second kind are given. In section 4, the local M-fractional Neumann method is presented to solve a class of fractional integral equations. In section 5, several illustrative examples are provided to show the efficiency of the method. Finally, the conclusion is appointed in section 6.

2. The several primary definitions and essential concepts

Suppose the local time M-fractional Volterra integral equations (MFVIEs) as the following form

$$ f(t) = g(t) + \eta {}_M\mathcal{T}_a^{\alpha,\beta}(K(t, s)f(s)), \qquad \forall \alpha \in (0, 1), \beta > 0, \tag{2.1} $$

where g and K are known functions, η and a are constants, and f is an unknown function [9–14]. Applying the local M-fractional integral definition on Eq. (2.1), results in

$$ f(t) = g(t) + \eta \int_a^t \frac{\Gamma(\beta + 1)K(t, s)f(s)}{s^{1-\alpha}} ds. \tag{2.2} $$

By considering

$$K_\beta^\alpha(t,s) = \frac{\Gamma(\beta+1)K(t,s)}{s^{1-\alpha}}, \tag{2.3}$$

as the new Volterra kernel, and substituting (2.3) into (2.2), we obtain

$$f(t) = g(t) + \eta \int_a^t K_\beta^\alpha(t,s) f(s) ds. \tag{2.4}$$

According to Eq. (2.4), the operator form of MFVIEs (2.1), can be denoted as follows

$$f = g + \eta K_\beta^\alpha f, \qquad \forall \alpha \in (0,1), \beta > 0, \tag{2.5}$$

or

$$L_\beta^\alpha f = (I - \eta K_\beta^\alpha) f = g, \qquad \forall \alpha \in (0,1), \beta > 0, \tag{2.6}$$

Definition 2.1. Lets consider $\eta = \eta_0, \alpha = \alpha_0, \beta_0 = \beta$, and $(L_{\beta_0}^{\alpha_0})^{-1}$ as an $\mathcal{L}^2$ operator exists and satisfies

$$(L_{\beta_0}^{\alpha_0})^{-1} L_{\beta_0}^{\alpha_0} = L_{\beta_0}^{\alpha_0} (L_{\beta_0}^{\alpha_0})^{-1} = I, \tag{2.7}$$

then η_0 is called a regular value of the local M-fractional operator $K_{\beta_0}^{\alpha_0}$ [9–14].

Theorem 2.2. *If for a given $\alpha = \alpha_0$, $\beta = \beta_0$, and $\eta = \eta_0$, the operator $(L_{\beta_0}^{\alpha_0})^{-1}$ exists, then it is unique [9–14].*

Proof. Suppose that $(L_{\beta_0}^{\alpha_0})^{-1}$ and $(\tilde{L}_{\beta_0}^{\alpha_0})^{-1}$ are two $\mathcal{L}^2$ operators that satisfy Eq. (2.3), and let

$$H = (L_{\beta_0}^{\alpha_0})^{-1} - (\tilde{L}_{\beta_0}^{\alpha_0})^{-1}.$$

Regarding Eq. (2.7), one has

$$\begin{aligned} &(L_{\beta_0}^{\alpha_0})^{-1} L_{\beta_0}^{\alpha_0} = L_{\beta_0}^{\alpha_0} (L_{\beta_0}^{\alpha_0})^{-1} = I, \\ &(\tilde{L}_{\beta_0}^{\alpha_0})^{-1} L_{\beta_0}^{\alpha_0} = L_{\beta_0}^{\alpha_0} (\tilde{L}_{\beta_0}^{\alpha_0})^{-1} = I, \end{aligned} \tag{2.8}$$

and subtracting these two relations results in

$$H L_{\beta_0}^{\alpha_0} = L_{\beta_0}^{\alpha_0} H = 0. \tag{2.9}$$

applying Eq. (2.9) by the local M-fractional fractional operator $(L_{\beta_0}^{\alpha_0})^{-1}$ and regarding Eq. (2.8), we get $H = 0$. □

Theorem 2.3. *If η is a regular value of the local M-fractional fractional operator K_β^α, with inverse the local M-fractional fractional operator $(L_\beta^\alpha)^{-1}$, then for any $\mathcal{L}^2$ function g, Eq. (2.6) has a unique $\mathcal{L}^2$ solution, say, f, satisfying [9–14].*

$$f = (L_\beta^\alpha)^{-1} g. \tag{2.10}$$

Proof. By Substitution of Eq. (2.10) into Eq. (2.2), we have

$$L_\beta^\alpha (L_\beta^\alpha)^{-1} g = g, \tag{2.11}$$

and since $L_\beta^\alpha (L_\beta^\alpha)^{-1} = I$, thus the function f, defined by Eq. (2.10), is a solution of Eq. (2.6). To show the uniqueness, lets f_1 and f_2 be two different solutions of (2.6), then

$$L_\beta^\alpha (f_1 - f_2) = 0,$$

hence

$$(L_\beta^\alpha)^{-1} L_\beta^\alpha (f_1 - f_2) = 0.$$

So

$$f_1 = f_2,$$

which completes the proof. If η is a regular value of the local M-fractional operator K_β^α, then the Eq. (2.6) has a unique solution

$$f = (L_\beta^\alpha)^{-1} g = (I - \eta K_\beta^\alpha)^{-1} g.$$

So

$$(L_\beta^\alpha)^{-1} = (I - \eta K_\beta^\alpha)^{-1} = I + \eta K_\beta^\alpha + (\eta K_\beta^\alpha)^2 + (\eta K_\beta^\alpha)^3 + (\eta K_\beta^\alpha)^4 + \cdots,$$

$$(L_\beta^\alpha)^{-1} = I + \sum_{n=1}^{\infty} (\eta K_\beta^\alpha)^n, \qquad \forall \alpha \in (0,1), \beta > 0, \tag{2.12}$$

where Eq. (2.12) is called the local M-fractional Neumann series for the invers the local M-fractional operator $(L_\beta^\alpha)^{-1}$. We set

$$\begin{aligned}
f_0 &= g,\\
f_1 &= g + \eta K_\beta^\alpha f_0 = g + \eta K_\beta^\alpha g,\\
f_2 &= g + \eta K_\beta^\alpha f_1 = g + \eta K_\beta^\alpha g + (\eta K_\beta^\alpha)^2 g,\\
f_3 &= g + \eta K_\beta^\alpha f_2 = g + \eta K_\beta^\alpha g + (\eta K_\beta^\alpha)^2 g + (\eta K_\beta^\alpha)^3 g,\\
&\vdots
\end{aligned}$$

so, the nth approximation to f, can be presented as below

$$f_n = g + \eta K_\beta^\alpha f_{n-1} = g + \sum_{i=1}^{n} \eta (\eta K_\beta^\alpha)^i g.$$

Therefore, if the sequence of functions f_n have a limit as n, tends to infinity, then

$$f = \lim_{n\to\infty} f_n = g + \sum_{i=1}^{\infty} (\eta K_\beta^\alpha)^i g, \qquad \forall \alpha \in (0,1), \beta > 0, \tag{2.13}$$

where Eq. (2.13) is called the local M-fractional Neumann series for the solution x of MFVIEs (2.3) [14, 26]. ☐

3. The convergence study

Theorem 3.1. *The local M-fractional Neumann series* (2.12)*, for* $(L_\beta^\alpha)^{-1}\alpha \in (0,1)$ *and* $\beta > 0$*, is strong convergence if* $\|\eta K_\beta^\alpha\| < 1, \forall \alpha \in (0,1), \beta > 0$*,* [9–14]*.*

Proof. Assume that $\alpha \in (0,1), \beta > 0$ is given and considered as a constant throughout the proof. Define

$$S_n = \sum_{i=0}^{n} \eta (K_\beta^\alpha)^i, \tag{3.1}$$

and take $n > m$. Regarding Eq. (3.1), we have

$$\|S_n - S_m\| \leq \sum_{i=m+1}^{n} \|\eta K_\beta^\alpha\|^i = \frac{\|\eta K_\beta^\alpha\|(\|\eta K_\beta^\alpha\|^m - \|\eta K_\beta^\alpha\|^n)}{1 - |\eta K_\beta^\alpha|}. \tag{3.2}$$

Since $\|\eta K_\beta^\alpha\| < 1$, thus

$$\lim_{n\to\infty} \|\eta K_\beta^\alpha\|^n = 0, \tag{3.3}$$

by considering Eqs. (3.2) and (3.3), we derive

$$\lim_{n,m\to\infty} \|S_n - S_m\| = 0. \tag{3.4}$$

So, the sequence S_n is a Cauchy sequence, so the limit S_n exists.

Now, lets consider the residual R_n as the following form

$$R_n = I - (I - \eta K^{\alpha}_{\beta}) S_n. \tag{3.5}$$

Setting Eq. (3.1) in Eq. (3.5), results in

$$R_n = \eta K^{\alpha}_{\beta})^{n+1},$$
$$\|R_n\| \leq |\eta K^{\alpha}_{\beta}|^{n+1}.$$

Since $\|\eta K^{\alpha}_{\beta}\| < 1$, therefore

$$\lim_{n\to\infty} \|R_n\| = 0.$$

Then, the local M-fractional operator $(L^{\alpha}_{\beta})^{-1}$ is a right inverse of L^{α}_{β}, a similar proof shows that it is also a left inverse of the local M-fractional operator L^{α}_{β}. □

Lemma 3.2. *Whenever K^{α}_{β} is an $\mathcal{L}^2$ the local M-fractional Volterra operator for a given $\alpha \in (0,1), \beta > 0$, and $b > a$, then*

$$|(K^{\alpha}_{\beta})^{n+1}(t,s)| \leq \frac{\|K^{\alpha}_{\beta}\|^{n+1}_E}{[(n-1)!]^{\frac{1}{2}}} K_1(t) K_2(s),$$

where $K_1(t) = [\int_a^t |K^{\alpha}_{\beta}(t,s)|^2 ds]^{\frac{1}{2}}$, and $K_2(s) = [\int_s^b |K^{\alpha}_{\beta}(t,s)|^2 dt]^{\frac{1}{2}}$.

Proof. For $\alpha, \beta = 1$, refer to books [9–26]. □

Theorem 3.3. *If K^{α}_{β} is an $\mathcal{L}^2$ the local M-fractional Volterra operator for a given $\alpha \in (0,1), \beta > 0$, the local M-fractional Neumann series (2.12), converges strongly for all η to the inverse the local M-fractional operator of K^{α}_{β} [9–14].*

Proof: According to Eq. (3.3), for $n > m$, we obtain

$$\|S_n - S_m\|_E \leq \sum_{i=m+1}^{n} \|\eta K^{\alpha}_{\beta}\|^i_E. \tag{3.6}$$

But from Lemma 3.2, and Euclidean norm, we get

$$\|\eta K^{\alpha}_{\beta}\|^i_E \leq |\eta|^i \frac{\|K^{\alpha}_{\beta}\|^i_E}{[(i-2)!]^{\frac{1}{2}}},$$

and hence, for all η,

$$\lim_{i\to\infty} |\eta K^{\alpha}_{\beta}|^i_E = 0. \tag{3.7}$$

By considering Eqs. (3.6) and (3.7), we persuade the sequence S_n is Cauchy, so the local M-fractional Neumann series (2.12), is strong convergence for all η to the inverse the local M-fractional operator of K^{α}_{β}

4. The summary of local M-fractional Neumann method (MFNM) for applying in MFVIEs

Suppose local time M-fractional Volterra integral equations of the second kind as follows form

$$f(t) = g(t) + \eta\, {}_M\mathcal{T}^{\alpha,\beta}_a (K(t,s) g(s)), \;\; \forall \alpha \in (0,1), \beta > 0,$$

where g, K are known functions and η, a are constants and f an unknown function. We define

$$\begin{aligned}
f_0(t) &= g(t),\\
f_1(t) &= g(t) + \eta {}_M\mathcal{T}_a^{\alpha,\beta}(K(t,s)f_0(s))\\
&= g(t) + \eta {}_M\mathcal{T}_a^{\alpha,\beta}(K(t,s)g(s)),\\
f_2(t) &= g(t) + \eta {}_M\mathcal{T}_a^{\alpha,\beta}(K(t,s)f_1(s))\\
&= g(t) + \eta {}_M\mathcal{T}_a^{\alpha,\beta}(K(t,s)g(s)) + \eta^2 {}_M\mathcal{T}_a^{\alpha,\beta}(K(t,s) {}_M\mathcal{T}_a^{\alpha,\beta}(K(s,s_1)g(s_1))),\\
&\vdots
\end{aligned}$$

moreover, the nth approximations f_n, to x, will be as

$$\begin{aligned}
f_n(t) &= g(t) + \eta {}_M\mathcal{T}_a^{\alpha,\beta}(K(t,s)f_{n-1}(s))\\
&= g(t) + \eta {}_M\mathcal{T}_a^{\alpha,\beta}\left(K(t,s)g(s)\right) + \eta^2 {}_M\mathcal{T}_a^{\alpha,\beta}\left(K(t,s) {}_M\mathcal{T}_a^{\alpha,\beta}(K(s,s_1)g(s_1))\right) + \cdots\\
&+ \eta^n {}_M\mathcal{T}_a^{\alpha,\beta}\left(K(t,s) {}_M\mathcal{T}_a^{\alpha,\beta}\left(K(s,s_1)\cdots({}_M\mathcal{T}_a^{\alpha,\beta}(K(s_{n-1},s_n)g(s_n)\right)\right).
\end{aligned}$$

The solution of MFVIEs is

$$f(t) = \lim_{n\to\infty} f_n(t).$$

5. Examples

In this section, the several illustrative examples are provided to demonstrate the efficiency of the method in solving the local time M-fractional Volterra integral equations of second kind.

Example 5.1. Consider the following local time M-fractional Volterra integral equation

$$f(t) = 1 - {}_M\mathcal{T}_0^{\alpha,\beta}((t-s)f(s)),\ \forall\alpha\in(0,1),\beta>0, \tag{5.1}$$

where the exact solution of this MFVIE (5.1), for non-fractional case is as follows [21]

$$f(t) = \cos(t).$$

According to the proposed the local M-fractional Neumann method, we have

$$\begin{aligned}
f_0(t) =& 1,\\
f_1(t) =& 1 - {}_M\mathcal{T}_a^{\alpha,\beta}(t-s),\\
f_2(t) =& 1 - {}_M\mathcal{T}_a^{\alpha,\beta}(t-s) + {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s) {}_M\mathcal{T}_a^{\alpha,\beta}(s-s_1)\right),\\
f_3(t) =& 1 - {}_M\mathcal{T}_a^{\alpha,\beta}(t-s) + {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s) {}_M\mathcal{T}_a^{\alpha,\beta}(s-s_1)\right)\\
&- {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s) + {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s) {}_M\mathcal{T}_a^{\alpha,\beta}(s-s_1)\right)\right).\\
&\vdots
\end{aligned} \tag{5.2}$$

By solving this sequence of integral equations, the solution of Eq. (5.2), can be obtained as the following form

$$\begin{aligned}
f_0(t) =& 1,\\
f_1(t) =& 1 - \frac{\Gamma(\beta+1)}{\alpha(1+\alpha)} t^{1+\alpha},\\
&\vdots
\end{aligned}$$

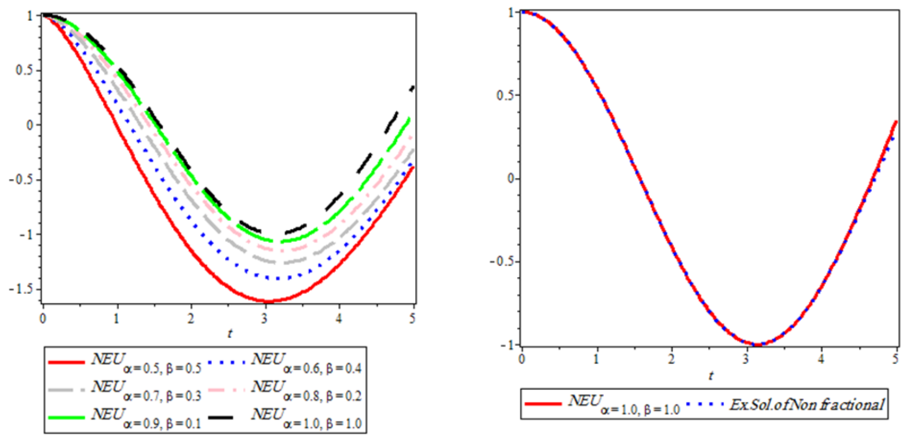

FIGURE 1. The 7th-order approximation of MFNM for different values α and β (left) and for $\alpha, \beta = 1$, versus exact solution of Non-fractional Volterra integral equation (right).

The seven-terms approximate solutions of Eq. (5.1), for different α, β, will be obtained, as follows

$$
\begin{aligned}
f_6(t) =& 1 - \frac{\Gamma(\beta+1)}{\alpha(1+\alpha)} t^{1+\alpha} + \frac{\Gamma(\beta+1)^2}{\alpha(1+2\alpha)(1+\alpha)^2} \frac{t^{2+2\alpha}}{2!} \\
&- \frac{\Gamma(\beta+1)^3}{\alpha(1+2\alpha)(2+3\alpha)(1+\alpha)^3} \frac{t^{3+3\alpha}}{3!} + \frac{\Gamma(\beta+1)^4}{\alpha(1+2\alpha)(2+3\alpha)(3+4\alpha)(1+\alpha)^4} \frac{t^{4+4\alpha}}{4!} \\
&- \frac{\Gamma(\beta+1)^5}{\alpha(1+2\alpha)(2+3\alpha)(3+4\alpha)(4+5\alpha)(1+\alpha)^5} \frac{t^{5+5\alpha}}{5!} \\
&+ \frac{\Gamma(\beta+1)^6}{\alpha(1+2\alpha)(2+3\alpha)(3+4\alpha)(4+5\alpha)(5+6\alpha)(1+\alpha)^6} \frac{t^{6+6\alpha}}{6!}.
\end{aligned}
$$

In Figure 1, the seventh-order approximate solution of Local M-fractional Volterra integral equation for different Values α, β, and exact solution for $\alpha, \beta = 1$ are plotted

Example 5.2. Consider the following local time M-fractional Volterra integral equation

$$
f(t) = 2 + t^2 + {}_M\mathcal{T}_{\alpha}^{\alpha,\beta}((t-s)f(s)), \ \forall \alpha \in (0,1), \beta > 0, \tag{5.3}
$$

where for $\alpha = 1, \beta = 1$, the exact solution of Eq. (5.3) is as follows [21]

$$
f(t) = 4\cosh(t) - 2.
$$

Applying to the proposed local M-fractional Neumann method, results in

$$
\begin{aligned}
f_0(t) =& 2 + t^2, \\
f_1(t) =& 2 + t^2 + {}_M\mathcal{T}_a^{\alpha,\beta}((t-s)(2+s^2)), \\
f_2(t) =& 2 + t^2 + {}_M\mathcal{T}_a^{\alpha,\beta}((t-s)(2+s^2)) + {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s)_M\mathcal{T}_a^{\alpha,\beta}((s-s_1)(2+s_1^2))\right), \\
f_3(t) =& 2 + t^2 + {}_M\mathcal{T}_a^{\alpha,\beta}((t-s)(2+s^2)) + {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s)_M\mathcal{T}_a^{\alpha,\beta}((s-s_1)(2+s_1^2))\right) \\
&+ {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s) + {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s)_M\mathcal{T}_a^{\alpha,\beta}((s-s_1)(2+s_1^2))\right)\right), \\
&\vdots
\end{aligned} \tag{5.4}
$$

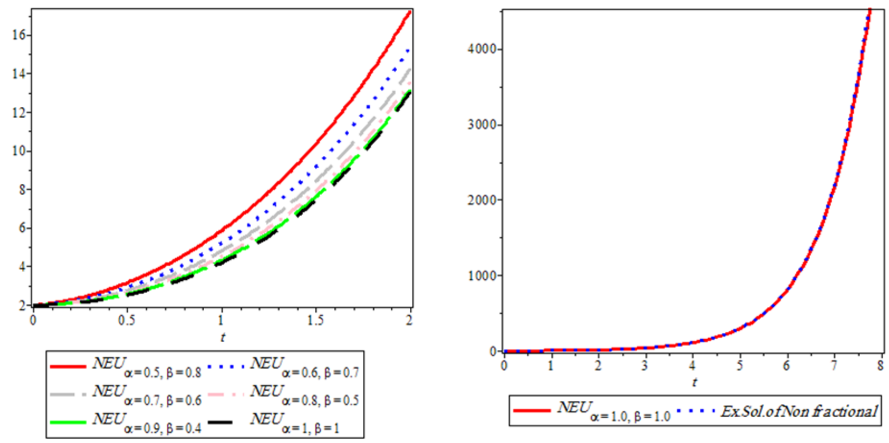

FIGURE 2. The 7th-order approximation of MFNM for different values α and β (left) and for $\alpha, \beta = 1$, versus exact solution of Non-fractional Volterra integral equation (right).

The corresponding solutions of these sequences (5.4) are as below

$$
\begin{aligned}
f_0(t) =& 2 + t^2, \\
f_1(t) =& 2 + t^2 + \frac{\Gamma(\beta+1)}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} t^{\alpha}(t^3\alpha^2 + t^3\alpha + 2t\alpha^2 + 10t\alpha + 12t), \\
f_2(t) =& 2 + t^2 + \frac{\Gamma(\beta+1)}{\alpha(\alpha+1)(\alpha+2)(\alpha+3)} t^{\alpha}(t^3\alpha^2 + t^3\alpha + 2t\alpha^2 + 10t\alpha + 12t) \\
&+ \frac{\Gamma(\beta+1)^2}{2\alpha(\alpha+1)(\alpha+2)(2\alpha+3)(\alpha^3+6\alpha^2+11\alpha+6)} t^{1+2\alpha}(2t^3\alpha^4 + 5t^3\alpha^3 + 4t^3\alpha^2 \\
&+ 4t\alpha^4 + t^3\alpha + 34t\alpha^3 + 106t\alpha^2 + 144t\alpha + 72t), \\
&\vdots
\end{aligned}
$$

In Figure 2, the seventh-order approximate solution of Local M-fractional Volterra integral equation for different Values α, β, and exact solution for $\alpha, \beta = 1$ are plotted.

Example 5.3. Consider the following MFVIE

$$
f(t) = \exp(t) + {}_M\mathcal{T}_0^{\alpha,\beta}(\exp((t-s)f(s)), \qquad \forall \alpha \in (0,1), \beta > 0, \tag{5.5}
$$

whit the exact solution of this local M-fractional Volterra integral Equation (5.5), for non-fractional case is as follows [21]

$$
f(t) = \exp(2t).
$$

According to the MFNM approach, we have

$$
\begin{aligned}
f_0(t) =& \exp(t), \\
f_1(t) =& \exp(t) + {}_M\mathcal{T}_a^{\alpha,\beta}(\exp(t-s)\exp(s)) \\
f_2(t) =& \exp(t) + {}_M\mathcal{T}_a^{\alpha,\beta}(\exp(t-s)\exp(s)) + {}_M\mathcal{T}_a^{\alpha,\beta}\left(\exp(t-s)_M\mathcal{T}_a^{\alpha,\beta}\left(\exp(s-s_1)\exp(s_1)\right)\right), \\
f_3(t) =& \exp(t) + {}_M\mathcal{T}_a^{\alpha,\beta}(\exp(t-s)\exp(s)) + {}_M\mathcal{T}_a^{\alpha,\beta}\left(\exp(t-s)_M\mathcal{T}_a^{\alpha,\beta}\left(\exp(s-s_1)\exp(s_1)\right)\right) \\
&+ {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s)_M\mathcal{T}_a^{\alpha,\beta}\left((s-s_1)\exp(s_1)\right)\right) \\
&+ {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s)_M\mathcal{T}_a^{\alpha,\beta}\left((s-s_1)_M\mathcal{T}_a^{\alpha,\beta}((s_1-s_2)\exp(s_2))\right)\right),
\end{aligned} \tag{5.6}
$$

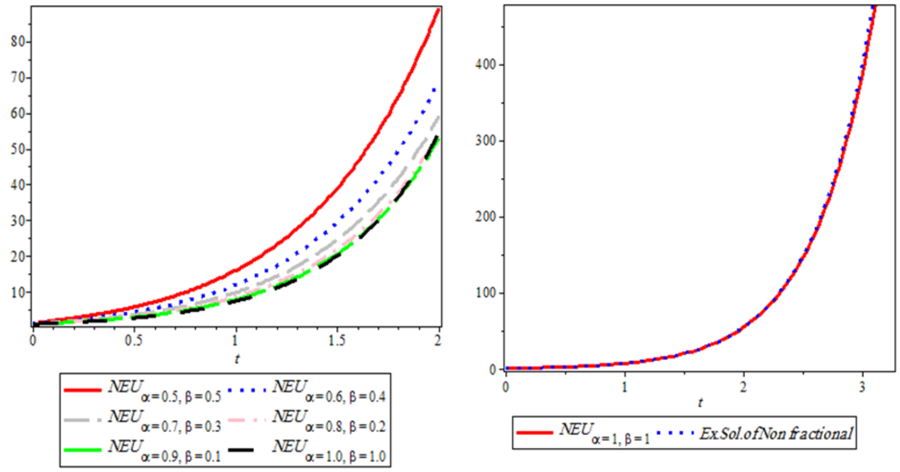

Figure 3. The 7th-order approximation of MFNM for different values α and β (left) and for $\alpha, \beta = 1$, versus exact solution of Non-fractional Volterra integral equation (right).

By solving this sequence of integral Equation (5.8), seventh-order approximation of Eq. (5.5) is

$$\begin{aligned} f_6(t) =& e^t\Big[1 + \frac{\Gamma(\beta+1)}{1!}(\frac{t^\alpha}{\alpha}) + \frac{\Gamma(\beta+1)^2}{2!}(\frac{t^\alpha}{\alpha})^2 + \frac{\Gamma(\beta+1)^3}{3!}(\frac{t^\alpha}{\alpha})^3 + \frac{\Gamma(\beta+1)^4}{4!}(\frac{t^\alpha}{\alpha})^4 \\ &+ \frac{\Gamma(\beta+1)^5}{5!}(\frac{t^\alpha}{\alpha})^5 + \frac{\Gamma(\beta+1)^6}{6!}(\frac{t^\alpha}{\alpha})^6\Big]. \end{aligned}$$

In Figure 3, the seventh-order approximate solution of Local M-fractional Volterra integral equation for different Values α, β, and exact solution for $\alpha, \beta = 1$ are plotted.

Example 5.4. Consider the time fractional integral equation as follows

$$f(t) = 3\sin(2t) - {}_M\mathcal{T}_a^{\alpha,\beta}((t-s)f(s)), \qquad \forall\alpha \in (0,1), \beta > 0, \tag{5.7}$$

with the exact solution of this local M-fractional Volterra integral equation for non-fractional case is [21]

$$f(t) = 4\sin(2t) - 2\sin(t).$$

By using the proposed MFNM approach, we gain

$$\begin{aligned} f_0(t) =& 3\sin(2t), \\ f_1(t) =& 3\sin(2t) - {}_M\mathcal{T}_a^{\alpha,\beta}((t-s)3\sin(2s)), \\ f_2(t) =& 3\sin(2t) + {}_M\mathcal{T}_a^{\alpha,\beta}((t-s)3\sin(2s)) \\ &+ {}_M\mathcal{T}_a^{\alpha,\beta}\left((t-s)_M\mathcal{T}_a^{\alpha,\beta}((s-s_1)3\sin(2s_1))\right). \\ &\vdots \end{aligned} \tag{5.8}$$

By solving above sequences of integral equation, the second-order approximate solution of Eq. (5.8), can be obtained that in Figure 4, for different Values α, β, and exact solution for $\alpha, \beta = 1$ are plotted.

6. Conclusion

In the course of the present investigation, was presented the solving for a class of fractional Volterra integral equations in the sense of the truncated M-fractional derivative. For this aim, the well-recognized Neumann method was successfully expanded and the several theorems related to conditions for existence and uniqueness and also sufficient condition for convergence of solution were proved. The proposed method was called the M-fractional Neumann method

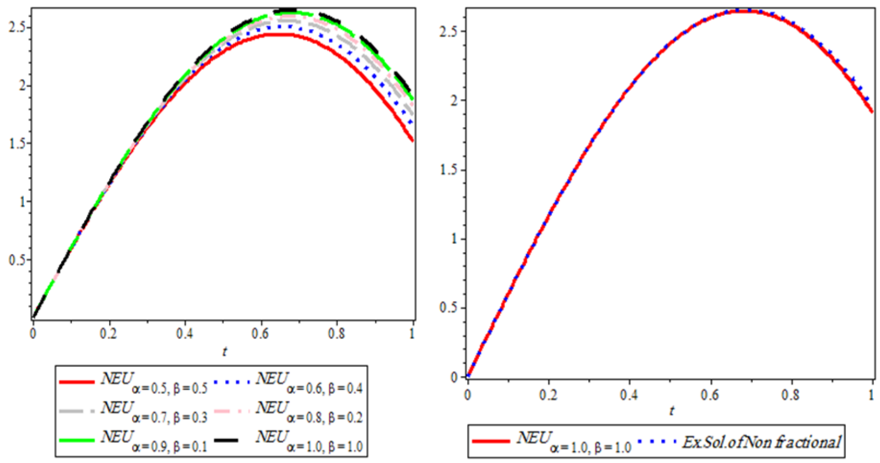

FIGURE 4. The 7th-order approximation of MFNM for different values α and β (left) and for $\alpha, \beta = 1$, versus exact solution of Non-fractional Volterra integral equation (right).

(MFNM). Since for $\alpha = 1$, and $\beta = 1$, MFVIEs are changed into a Volterra integral equations, thus not unexpected that M-fractional Neumann method have had the same accuracy and efficiency the Neumann method for Volterra integral equations. The several illustrative examples also were presented, corroborating the satisfactory implementation of the method in solving the local M-fractional Volterra integral equations. In this study, the norm $\| \cdot \|_2$, was utilized.

Availability of data and material

Not applicable

Competing interests

Not applicable

Funding

Not applicable

Authors' contributions

All authors contributed relatively equally to the production of the article.

Acknowledgements

We would like to express my gratitude for the attention of all respected reviewers to this manuscript

References

[1] E. M. Abdelghany, W. M. Abd-Elhameed, G. M. Moatimid, Y. H. Youssri, and A. G. Atta, *A Tau Approach for Solving Time-Fractional Heat Equation Based on the Shifted Sixth-Kind Chebyshev Polynomials*, Symmetry, *15*(594) (2023).

[2] W. A. Ahmood and A. Kilicman, *Solutions of linear multi-dimensional fractional order Volterra integral equations*, Journal of Theoretical and Applied Information Technology, *89* (2016), 381-388.

[3] R. Agarwal, S. Jain, and R. P. Agarwal, *Solution of fractional Volterra integral equation and non-homogeneous time fractional heat equation using integral transform of pathway type*, Progress in Fractional Differentiation and Applications, *1* (2015), 145-155.

[4] A. Asanov, R. Almeida, and A. B. Malinowska, *Fractional differential equations and VolterraStieltjes integral equations of the second kind*, Computational and Applied Mathematics, *38*(160) (2019).

[5] A. Atangana and N. Bildik, *Existence and numerical solution of the Volterra fractional integral equations of the second kind*, Mathematical Problems in Engineering, (2013).

[6] A. G. Atta and Y. H. Youssri, *Advanced shifted first-kind Chebyshev collocation approach for solving the nonlinear time-fractional partial integro-differential equation with a weakly singular kernel*, Computational & amp; Applied Mathematics, *41*(381) (2022).

[7] E. Bonyah, M. Yavuz, D. Baleanu, and S. Kumar, *A robust study on the listeriosis disease by adopting fractal-fractional operators*, Alexandria Engineering Journal, *61*(3) (2022), 2016-202.

[8] E. C. De Oliveira and J. A. Tenreiro Machado, *A review of definitions for fractional derivatives and integral*, Mathematical Problems in Engineering, (2014), 238459.

[9] L. M. Delves and J. L. Mohamed, *Computational methods for integral equations*, Cambridge University Press, 1985.

[10] S. Duran, H. Durur, M. Yavuz, and A. Yokus, *Discussion of numerical and analytical techniques for the emerging fractional order murnaghan model in materials science*, Optical and Quantum Electronic, *55*(571) (2023).

[11] S. Esmaeili, M. Shamsi, and M. Dehghan, *Numerical solution of fractional differential equations via a Volterra integral equation approach*, Open Physics, *11*(10) (2013), 1470-1481.

[12] R. Goreno, A. A. Kilbas, F. Mainardi, and S. V. Rogosin, *Mittag-Leffler Functions, Related Topics and Applications*, Springer, Berlin, 2014.

[13] R. M. Hafez and Y. H. Youssri, *Shifted Jacobi collocation scheme for multidimensional time-fractional order telegraph equation*, Iranian Journal of Numerical Analysis and Optimization, *10*(1-17) (2020), 195-225.

[14] M. Ilie, J. Biazar, and Z. Ayati, N*eumann method for solving conformable fractional Volterra integral equations*, Computational Methods for Differential Equations, *8*(1) (2020), 54-68.

[15] M. Ilie and A. Khoshkenar, *Resonant solitons solutions to the time M-fractional Schrdinger equation*, Iranian Journal of Optimization, *13*(3) (2022), 197-210.

[16] M. Ilie and A. Khoshkenar, *A novel study on nonlinear fractional differential equations: general solution*, Iranian Journal of Optimization, *14*(4) (2023).

[17] A. Khoshkenar, M. Ilie, K. Hosseini, D. Baleanu, S. Salahshour, and J. R. Lee, *Further studies on ordinary differential equations involving the M-fractional derivative*, AIMS MATHEMATICS, *7*(6) (2023), 10977-10993.

[18] M. Mohammad, and A. Trounev, *Fractional nonlinear VolterraFredholm integral equations involving Atangana-Baleanu fractional derivative: framelet applications*, Advances in Difference Equations, *618* (2020).

[19] M. Moustafa, Y. H. Youssri, and A. G. Atta, *quot;Explicit Chebyshev Petrov-Galerkin scheme for time-fractional fourth-order uniform EulerBernoulli pinnedpinned beam equation*, Nonlinear Engineering, *12*(1) (2023), 20220308.

[20] B. A. Özturk, *Examination of Sturm-Liouville problem with proportional derivative in control theory*, Mathematical Modelling and Numerical Simulation with Applications, *3*(4) (2023), 335-35.

[21] G. F. Simmons, *Differential Equations whit Applications and Historical Notes*, McGraw-Hill, Inc. New York, 1974.

[22] F. Smithies, *Integral equations*, Cambridge University Press, 1958.

[23] M. Tariq, S. K. Sahoo, H. Ahmad, A. A. Shaikh, B. Kodamasingh, and D. Khan, *Some integral inequalities via new family of preinvex functions*, Mathematical Modelling and Numerical Simulation With Applications, *2*(2) (2022), 117-126.

[24] J. Vanterler da C. Sousa, and E. Capelas de Oliveira, *M-fractional derivative with classical properties*, (2017), arXiv:1704.08187.

[25] J. Vanterler da C. Sousa, and E. Capelas de Oliveira, *A New Truncated M-Fractional Derivative Type Unifying Some Fractional Derivative Types with Classical Properties*, International Journal of Analysis and Applications, *16*(1) (2018), 83-96.

[26] A. M. Wazwaz, *Linear and nonlinear integral equations methods and applications*, Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg, 2011.

[27] Y. H. Youssri and W. M. Abd-Elhameed, *Numerical spectral LEGENDRE-GALERKIN algorithm for solving time fractional Telegraph equation*, Romanian Journal of Physics, *63*(107) (2018).

[28] Y. H. Youssri and A. G. Atta, *Petrov-Galerkin Lucas Polynomials Procedure for the Time-Fractional Diffusion Equation*, Contemporary Mathematics, (2023).

[29] L. Zada, R. Nawaz, K. S. Nisar, M. Tahir, M. Yavuz, M. K. A. Kaabar, and F. Martnez, *New approximate-analytical solutions to partial differential equations via auxiliary function method*, Partial Differential Equations in Applied Mathematics, *4*(100045) (2021).