Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. *, No. *, *, pp. 1-13
DOI:10.22034/cmde.2024.59080.2509

Unveiling traveling waves and solitons of Dirac integrable system via homogenous balance and singular manifolds methods

Samah M. Mabrouk[1], **Ahmed S. Rashed**[1,2,*], **and Rasha Saleh**[1]
[1] Department of Physics and Engineering Mathematics, Faculty of Engineering, Zagazig University, Egypt.
[2] Basic Science Department, Faculty of Engineering, Delta University for Science and Technology, Gamasa, 11152, Egypt.

Abstract

This study utilizes two robust methodologies to examine the precise solutions of the Dirac integrable system. The Homogeneous Balance Method (HB) is initially employed to generate an accurate solution. The system of equations for the quasi-solution is solved, where all the equations are of the same nature. The quasi-solution of the traveling wave results in the solitary wave solution of the system. The singular manifold method (SMM) is utilized following the Lie reduction of the Dirac system in order to search for the traveling wave solutions of the system. Both approaches demonstrate the existence of traveling wave solutions inside the system. The precise solutions of the Dirac system are shown in three-dimensional graphs. We have created solutions to the examined problem, including bright solutions, periodic soliton solutions, and complicated solutions.

Keywords. Dirac Integrable System, Homogeneous Balance Method, Singular Manifold Method.
2010 Mathematics Subject Classification. 58J47, 35C05.

1. Introduction

Nonlinear evolution equations represent a powerful mathematical framework for modeling dynamic processes that deviate significantly from linear behavior [1–4]. These equations have remarkable applications in fields as diverse as fluid dynamics, quantum mechanics, pattern formation, and information theory. Unlike their linear counterparts, nonlinear evolution equations exhibit intricate and often unexpected behaviors, such as soliton formation, wave breaking, chaos, and pattern formation, making them both intellectually challenging and practically valuable. The study of nonlinear evolution equations has garnered significant attention due to its ability to capture the complexities of real-world phenomena. These equations go beyond simple linear relationships and incorporate the interplay of various nonlinear effects, leading to a more accurate representation of natural processes. Consequently, understanding the properties and dynamics of nonlinear evolution equations is crucial for unraveling the underlying mechanisms governing complex systems [5].

Nonlinear evolution equations find applications in various scientific and engineering disciplines. Nonlinear evolution equations play a significant role in understanding the emergence of patterns in diverse systems, including chemical reactions, biological processes, and material science. Examples include the Fitz-Hugh-Nagumo model for neural dynamics and the Gray-Scott model for chemical pattern formation [6]. In soliton theory, solitons are stable, solitary waves that retain both speed and shape during propagation. Nonlinear Schrödinger and Korteweg-de Vries equations are examples of evolution equations used to study soliton behavior. These equations find applications in optical communications, fiber optics, and Bose-Einstein condensates [7]. For nonlinear optics, nonlinear evolution equations are essential for studying light-matter interactions in nonlinear optical materials. They help understand phenomena like self-focusing, self-phase modulation, and four-wave mixing, enabling applications in laser technology, optical communications, and photonic devices [8, 9] . In image processing field, Nonlinear evolution equations have been

Received: 05 November 2023 ; Accepted: 09 February 2024.
* Corresponding author. Email: ahmed.s.rashed@gmail.com.

utilized in image denoising, image inpainting, and image segmentation tasks. The famous Rudin-Osher-Fatemi (ROF) model, based on the nonlinear total variation equation, is widely used for image restoration and enhancement [10, 11]. Another important application is mathematical biology, where nonlinear evolution equations are employed to model biological phenomena, such as population dynamics, biochemical reactions, and the spread of epidemics. These models aid in understanding ecological systems, tumor growth, neural networks, and gene regulation [12, 13]. Moreover, nonlinear evolution equations are employed in control theory to analyze and design feedback control systems. They allow for the control of complex nonlinear dynamics and find applications in robotics, aerospace engineering, and autonomous systems [5, 14]. The approaches of solution are really significant. The approaches employed encompass a range of analytical, semi-analytical, and numerical methods [15–21]. Sine-Cosine [22–26], tanh-coth [24, 27–29], inverse scattering [30, 32], Hirota bilinear [33–37], extended homogenous method [38–42], Exp-function [43–46], Elliptic function [47–52], Bäcklund transformation [53–57], symmetry transformations and singular manifolds methods [58–75] are common methods for investigating the behavior of nonlinear PDEs. The concept of integrable couplings arose and subsequent research was initiated during the examination of the connections between Virasoro algebras and hereditary operators [76–80].

The main aim of the current work is to deeply investigate the Dirac integrable system [78], Which appear in a vast number of applications specially in quantum electrodynamics and the standard model. Dirac equation describes muons, tayons in electron studies. It is also used in quantum chromodynamics and in phenomenological hadron models, for example, in the relativistic model for quasi-independent quarks, which has been applied for the description of hadron properties. Two methods namely, homogeneous balance and Lie infinitesimals methods. Both methods result in new families of solutions in addition of comparable solutions with other methods in the literature. These solutions are crucial for comprehending the behavior of many physical phenomena, such as the propagation of waves in the presence of magnetic fields and the formation of optical solitons within waveguides.

The paper is structured in the following manner. In section 2, the homogeneous balance method is applied to Dirac integrable system to discover its closed form solutions. Section 3 is devoted to apply Lie infinitesimals method accompanied with singular manifolds method to construct new families of solutions. The conclusion remarks are described in section 4.

2. Description of the homogeneous balance method

The homogeneous balancing (HB) method has been widely recognized as a highly successful approach for determining explicit solitary wave solutions. The subsequent section provides a description of the methodology of homogenous balance [81–83].

Assume that a system of partial differential equations (PDEs) to be in the form:

$$\begin{cases} p_1\left(\varphi, \psi, \varphi_x, \psi_x, \varphi_t,\ \varphi_{xx}, \psi_{xx}\ , \dots\right) = 0, \\ p_2\left(\varphi, \psi, \varphi_x, \psi_x, \varphi_t,\ \varphi_{xx}, \psi_{xx}\ , \dots\right) = 0. \end{cases} \tag{2.1}$$

Where $\mathbf{p_1}$ and $\mathbf{p_2}$ are polynomials in the dependent variables, φ, ψ , and their partial derivatives.

A function $\xi = \xi(x, t)$ can be considered as a "quasi-solution" of (2.1), if there are functions $f = f(\xi)$ and $g = g(\xi)$, of only one argument so that, a nominated linear combination of;

$$1, f(\xi), f_x(\xi), f_t(\xi), f_x x(\xi), f_x t(\xi), \dots \tag{2.2}$$

And

$$1, g(\xi), g_x(\xi), g_t(\xi), ,\ g_{xx}(\xi), g_{xt}(\xi), \dots \tag{2.3}$$

are solutions to system (2.1). The HB technique is outlined in the following prosecutorial steps:

Step 1: The solution to system (2.1) is postulated as a linear combination of (2.2) and (2.3), ensuring a balance

between the strongly nonlinear and the highest order derivative components in the system.
Step 2: The selected combination from the previous step is introduced into the system (2.1). The terms with the largest degree of $\xi(x,t)$ are gathered, and the total of their coefficients is equated to zero. The outcome is a set of ordinary differential equations (ODEs) involving the functions $f(\xi)$ and $g(\xi)$.
Step 3: Replace the nonlinear terms, which include derivatives of $g(\xi)$ and $f(\xi)$, and gather all the terms that have the same order of $f,\ f^{'},\ f^{''},\ f^{'''},\ ...,\ g,\ g^{'},\ g^{''},\ g^{'''},\ \ldots$ and then, equate the coefficients of these terms to zero. Obtain a system of ODEs in terms of the variable $\xi\ (x,\ t)$ that is uniform or consistent throughout. Based on the homogeneous property of this system of equations, $\xi(x,t)$ may be forecasted as an exponential function.
Step 4: Replace the functions $f(\xi),\ g(\xi)$, and $\xi(x,t)$ in the linear combination from step 1. This will yield the solution of the system (2.1).

2.1. **Solitary wave solution of Dirac integrable system.** The Dirac integrable system is presented as;

$$p_t = q_{xx} + q^3 + \frac{1}{2}qp^2, \tag{2.4}$$

$$q_t = -\frac{1}{2}p_{xx} - \frac{1}{4}p^3 - \frac{1}{2}pq^2. \tag{2.5}$$

where p and q are functions of the spatial coordinate x and temporal coordinate t. In this section, the homogeneous balancing approach is utilized to discover the precise solution of the Dirac integrable system (2.4) and (2.5). Select the solution of this system as a linear combination in the specified form:

$$p(x,t) = \frac{\partial^{\beta_1} f(\xi)}{\partial x^{\beta_1}}, \tag{2.6}$$

$$q(x,t) = \frac{\partial^{\beta_2} g(\xi)}{\partial x^{\beta_2}}. \tag{2.7}$$

The equilibrium between the most non-linear and the most significant derivative components in the system (2.4) and (2.5) leads to $\beta_1 = \beta_2 = 2$. The linear combinations (2.6) and (2.7) can be expressed as:

$$p(x,t) = f^{''}\xi_x^2 + f^{'}\xi_{xx}, \tag{2.8}$$

$$q(x,t) = g^{''}\xi_x^2 + g^{'}\xi_{xx}. \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.4) results in:

$$\begin{aligned} &\left(\frac{1}{2}g''f^{''2} + g^{''3}\right)\xi_x^6 + \left(3\ g^{''2}g^{'}\xi_{xx} + g^{(4)} + g''f^{''}f^{'}\xi_{xx} + \frac{1}{2}g'f^{''2}\xi_{xx}\right)\xi_x^4 \\ +&\left(6\ g^{'''}\xi_{xx} - f^{'''}\xi_t + 3g^{''}g^{'2}\xi_{xx}^2 + g^{'}f^{''}f^{'}\xi_{xx}^2 + \frac{1}{2}g^{''}f^{'2}\xi_{xx}^2\right)\xi_x^2 + \left(4g^{''}\xi_{xxx} - 2f^{''}\xi_{xt}\right)\xi_x \\ &+g^{'}\xi_{4x} + \frac{1}{2}g^{'}f^{'2}\xi_{xx}^3 + 3g^{''}\xi_{xx}^2 - f^{''}\xi_t\xi_{xx} + g^{'3}\xi_{xx}^3 - f^{'}\xi_{xxt} = 0. \end{aligned} \tag{2.10}$$

Substituting (2.8) and (2.9) into (2.5) results in:

$$\begin{aligned} &\left(\frac{1}{2}f''g^{''2} + \frac{1}{4}f^{''3}\right)\xi_x^6 + \left(\frac{3}{4}f^{''2}f^{'}\xi_{xx} + \frac{1}{2}f^{(4)} + g''f^{''}g^{'}\xi_{xx} + \frac{1}{2}f'g^{''2}\xi_{xx}\right)\xi_x^4 \\ +&\left(3\ f^{'''}\xi_{xx} + g^{'''}\xi_t + \frac{3}{4}f^{''}f^{'2}\xi_{xx}^2 + g^{'}g^{''}f^{'}\xi_{xx}^2 + \frac{1}{2}f^{''}g^{'2}\xi_{xx}^2\right)\xi_x^2 + \left(2f^{''}\xi_{xxx} + 2g^{''}\xi_{xt}\right)\xi_x \\ &+\frac{1}{2}f^{'}\xi_{4x} + \frac{1}{2}f^{'}g^{'2}\xi_{xx}^3 + \frac{3}{2}f^{''}\xi_{xx}^2 + g^{''}\xi_t\xi_{xx} + \frac{1}{4}f^{'3}\xi_{xx}^3 + g^{'}\xi_{xxt} = 0. \end{aligned} \tag{2.11}$$

Setting the coefficients of $\xi_x^6 = 0$ yields a system of ODEs in the form:

$$\begin{cases} \frac{1}{2}g''f''^2 + g''^3 = 0, \\ \frac{1}{2}f''g''^2 + \frac{1}{4}f''^3 = 0. \end{cases} \tag{2.12}$$

The solutions of (2.12) can be formulated in the form:

$$f = C_1 \ln \xi, g = C_2 \ln \xi. \tag{2.13}$$

The relationship between the nonlinear derivatives of $g(\xi)$ and $f(\xi)$ may be described as follows:

$$\begin{cases} f'^2 = -C_1 f'' , \quad f''^2 = -\frac{1}{6}C_1 f^{(4)}, \quad f' f'' = -\frac{1}{2}C_1 f''' , \quad f'^3 = -\frac{1}{2}C_1^2 f''', \\ g'^2 = -C_2 g'' , \quad g''^2 = -\frac{1}{6}C_2 g^{(4)}, g' g'' = -\frac{1}{2}C_2 g''', \quad g'^3 = -\frac{1}{2}C_2^2 g''' , \\ f' g' = -C_1 g'' = -C_2 f'' , \quad g' f'' = f' g'' = -\frac{1}{2}C_1 g''' = -\frac{1}{2}C_2 f''', \\ g'' f'' g' = f''^2 g' = \frac{1}{24}C_1 g^{(5)}, \quad g' f'' f' = \frac{1}{6}C_2^2 g^{(4)} = \frac{1}{6}C_1 C_2 f^{(4)}. \end{cases} \tag{2.14}$$

Using (2.13) and (2.14), the Equations (2.10) and (2.11) are simplified to be:

$$\begin{aligned} &\left(\frac{1}{2}\xi_{4x} + \frac{C_2}{C_1}\xi_{xxt}\right) f' + \left(3\xi_x^2 + 3\xi\ \xi_{xx} + 2\xi_x\xi_{xxx} + 2\frac{C_2\xi_x\xi_{xt}}{C_1} + \frac{C_2\xi_t\xi_{xx}}{C_1}\right) f'' \\ &+ \left(\frac{3}{4}C_2^2\xi^2\xi_{xx} + 2\frac{C_2\xi\xi_x\xi_t}{C_1} + \frac{3}{2}C_2^2\xi\ \xi_x^2 + \frac{3}{4}C_1^2\xi\ \xi_x^2 + 6\xi\ \xi_x\xi_{xx} + \frac{3}{8}C_1^2\xi^2\xi_{xx}\right) f''' \\ &+ \left(\frac{1}{2}C_1^2\xi^2\xi_x\xi_{xx} + 2\xi^3\xi_x + C_2^2\xi^2\xi_x\xi_{xx} + \frac{1}{2}C_1^2\xi\ \xi_x^3\right) f^{(4)} + \left(\frac{1}{8}C_1^2\xi^3\xi_x\xi_{xx} + \frac{1}{4}C_2^2\xi^3\xi_x\xi_{xx}\right) f^{(5)} = 0, \end{aligned} \tag{2.15}$$

$$\begin{aligned} &\left(\xi_{4x} - \frac{C_1}{C_2}\xi_{xxt}\right) g' + \left(6\xi_x^2 + 6\xi\ \xi_{xx} + 4\xi_x\xi_{xxx} - 2\frac{C_1\xi_x\xi_{xt}}{C_2} - \frac{C_1\xi_t\xi_{xx}}{C_2}\right) g'' \\ &+ \left(\frac{3}{4}C_1^2\xi^2\xi_{xx} - 2\frac{C_1\xi\xi_x\xi_t}{C_2} + \frac{3}{2}C_1^2\xi\ \xi_x^2 + 3C_2^2\xi\ \xi_x^2 + 12\xi\ \xi_x\xi_{xx} + \frac{3}{2}C_2^2\xi^2\xi_{xx}\right) g''' \\ &+ \left(2C_2^2\xi^2\xi_x\xi_{xx} + 2C_2^2\xi\ \xi_x^3 + C_1^2\xi^2\xi_x\xi_{xx} + C_1^2\xi\ \xi_x^3 + 4\xi^3\xi_x\right) g^{(4)} + \left(\frac{1}{2}C_2^2\xi^3\xi_x\xi_{xx} + \frac{1}{4}C_1^2\xi^3\xi_x\xi_{xx}\right) g^{(5)} = 0. \end{aligned} \tag{2.16}$$

Setting the coefficients of $f^{(5)}, f^{(4)}, \ldots,\ f'$ in (2.15) and the coefficients of $g^{(5)}, g^{(4)}, \ldots,\ g'$in (2.6) to zero results in a system of PDEs for $\xi(x,\ t)$, as follows:

$$\frac{1}{8}C_1^2\xi^3\xi_x\xi_{xx} + \frac{1}{4}C_2^2\xi^3\xi_x\xi_{xx} = 0, \tag{2.17}$$

$$\frac{1}{2}C_1^2\xi^2\xi_x\xi_{xx} + 2\xi^3\xi_x + C_2^2\xi^2\xi_x\xi_{xx} + \frac{1}{2}C_1^2\xi\ \xi_x^3 = 0, \tag{2.18}$$

$$\left(\frac{3}{4}C_2^2\xi^2\xi_{xx} + 2\frac{C_2\xi\xi_x\xi_t}{C_1} + \frac{3}{2}C_2^2\xi\ \xi_x^2 + \frac{3}{4}C_1^2\xi\ \xi_x^2 + 6\xi\ \xi_x\xi_{xx} + \frac{3}{8}C_1^2\xi^2\xi_{xx}\right) f''' = 0, \tag{2.19}$$

$$3\xi_x^2 + 3\xi\ \xi_{xx} + 2\xi_x\xi_{xxx} + 2\frac{C_2\xi_x\xi_{xt}}{C_1} + \frac{C_2\xi_t\xi_{xx}}{C_1} = 0, \tag{2.20}$$

$$\frac{1}{2}\xi_{4x} - \frac{C_2}{C_1}\xi_{xxt} = 0, \tag{2.21}$$

$$\frac{1}{2}C_2^2\xi^3\xi_x\xi_{xx} + \frac{1}{4}C_1^2\xi^3\xi_x\xi_{xx} = 0, \tag{2.22}$$

$$2C_2^2\xi^2\xi_x\xi_{xx}+2C_2^2\xi\ \xi_x^3+C_1^2\xi^2\xi_x\xi_{xx}+C_1^2\xi\ \xi_x^3+4\xi^3\xi_x=0, \tag{2.23}$$

$$\frac{3}{4}C_1^2\xi^2\xi_{xx}-2\frac{C_1\xi\xi_x\xi_t}{C_2}-\frac{3}{2}C_1^2\xi\ \xi_x^2+3C_2^2\xi\ \xi_x^2+12\xi\ \xi_x\xi_{xx}+\frac{3}{2}C_2^2\xi^2\xi_{xx}=0, \tag{2.24}$$

$$6\xi_x^2+6\xi\ \xi_{xx}+4\xi_x\xi_{xxx}-2\frac{C_1\xi_x\xi_{xt}}{C_2}-\frac{C_1\xi_t\xi_{xx}}{C_2}=0, \tag{2.25}$$

$$\xi_{4x}-\frac{C_1}{C_2}\xi_{xxt}=0. \tag{2.26}$$

The system (2.17) - (2.26) is solved with the aid of Maple tool yields,

$$\xi(x,\ \ t)=1+e^{\alpha x+\beta t+\gamma}, \tag{2.27}$$

where α,β and γ are arbitrary constants and the solutions of the Dirac integrable system (2.4) - (2.5) are given by:

$$\begin{cases} p(x,\ t)=\frac{\sqrt{2}\alpha^2}{4}sech^2(\alpha x+\beta t+\gamma), \\ q(x,\ t)=\frac{i\alpha^2}{4}sech^2(\alpha x+\beta t+\gamma). \end{cases} \tag{2.28}$$

$$\begin{cases} p(x,\ t)=-\frac{\sqrt{2}\alpha^2}{4}\sin\left(\frac{1}{2}(\alpha x+\beta t+\gamma)\right)+\frac{\sqrt{2}\alpha^2}{4}\cos\left(\frac{1}{2}(\alpha x+\beta t+\gamma)\right), \\ q(x,\ t)=\frac{i\alpha^2}{4}\sin\left(\frac{1}{2}(\alpha x+\beta t+\gamma)\right)-\frac{i\alpha^2}{4}\cos\left(\frac{1}{2}(\alpha x+\beta t+\gamma)\right). \end{cases} \tag{2.29}$$

$$\begin{cases} p(x,\ t)=\frac{\sqrt{2}\alpha^2}{4}\coth^2(\alpha x+\beta t+\gamma), \\ q(x,\ t)=\frac{i\alpha^2}{4}\coth^2(\alpha x+\beta t+\gamma). \end{cases} \tag{2.30}$$

The solutions of the system (2.4) - (2.5) are depicted hereafter in Figures 1-6.

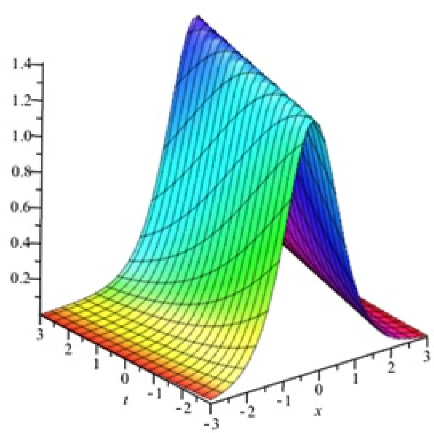

FIGURE 1. The Bright solution (2.28) of the Dirac integrable system $P(x,t)$ at $\alpha=2,\beta=-0.5,\gamma=-1$.

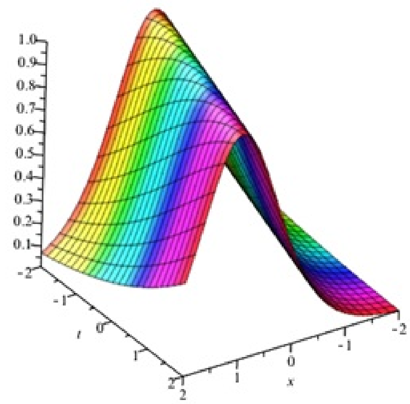

FIGURE 2. The Bright solution (2.28) of the Dirac integrable system $q(x,t)$ at $\alpha = 2, \beta = -0.5, \gamma = -1$.

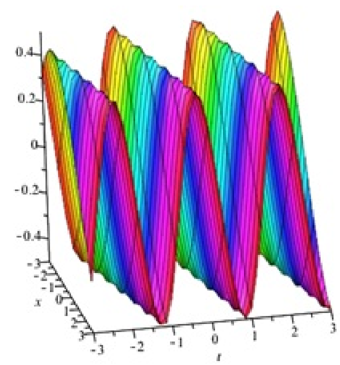

FIGURE 3. The Bright solution (2.29) of the Dirac integrable system $P(x,t)$ at $\alpha = 1, \beta = -6, \gamma = 10$.

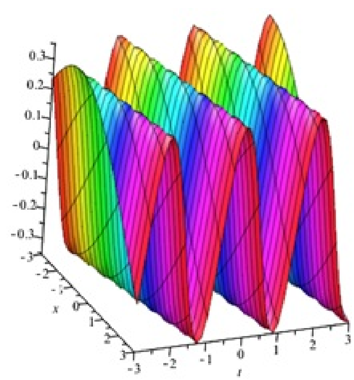

FIGURE 4. The Bright solution (2.29) of the Dirac integrable system $q(x,t)$ at $\alpha = 1, \beta = -6, \gamma = 10$.

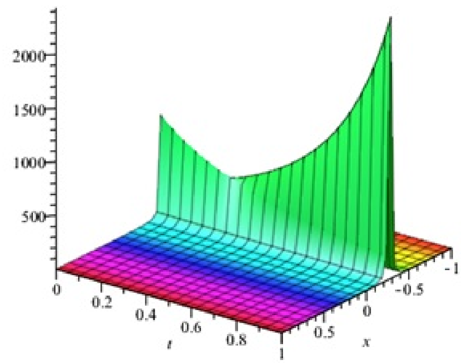

FIGURE 5. The Bright solution (2.30) of the Dirac integrable system $P(x,t)$ at $\alpha = 0.5, \beta = 0.01, \gamma = 0.125$.

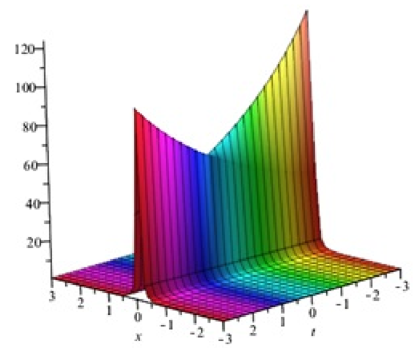

FIGURE 6. The Bright solution (2.30) of the Dirac integrable system $P(x,t)$ at $\alpha = 0.5, \beta = 0.01, \gamma = 0.125$.

3. Reduction of the Dirac integrable system by Lie transformations theory

The Lie transformation method is utilized in this section to convert the Dirac integrable system into a set of ordinary differential equations. The Dirac integrable system (2.4) - (2.5) possesses the following Lie infinitesimal vectors:

$$X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial t}, X_3 = q\frac{\partial}{\partial p} - \frac{p}{2}\frac{\partial}{\partial q}, X_4 = t\frac{\partial}{\partial x} + xq\frac{\partial}{\partial p} - \frac{xp}{2}\frac{\partial}{\partial q}, X_5 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{p}{2}\frac{\partial}{\partial p} - \frac{q}{2}\frac{\partial}{\partial q}. \tag{3.1}$$

Using the combined vector $X_1 + X_2 = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}$, the characteristic equation is described by:

$$\frac{dx}{\xi(x,t,u)} = \frac{dt}{\tau(x,t,u)} = \frac{dy}{Y(x,t,u)} = \frac{du}{\varphi(x,t,u)}. \tag{3.2}$$

The invariant transformations are given by:

$$\eta = x - t,\ q(\eta) = q(x,t), p(\eta) = p\ (x,\ t). \tag{3.3}$$

Then, the Dirac integrable system is reduced to the following nonlinear ODE system:

$$q^{''} + p^{'} +\ q^3 + \frac{1}{2}qp^2 = 0, \tag{3.4}$$

$$\frac{1}{2}p^{''} - q^{'} + \frac{1}{4}\ p^3 + \frac{1}{2}pq^2 = 0. \tag{3.5}$$

3.1. **Investigating of the solution using singular manifold method (SMM).** The singular manifold approach is utilized in this part to discover novel sets of solutions for the Dirac integrable system (2.4) - (2.5). According to the SMM [73, 74], the solutions of equations (3.4) and (3.5) are written in a series form:

$$\begin{cases} p(\eta) = \sum_{j=0}^{\infty} p_j \varphi(\eta)^{j-\alpha}, \\ q(\eta) = \sum_{j=0}^{\infty} q_j \varphi(\eta)^{j-\alpha}. \end{cases} \tag{3.6}$$

The dominating behavior yields an α value of 1. The Equation (3.6) can be reformulated as:

$$\begin{cases} p(\eta) = \frac{p_0}{\varphi} + p_1, \\ q(\eta) = \frac{q_0}{\varphi} + q_1. \end{cases} \tag{3.7}$$

By substituting the results in (3.7) into Equation (3.4), a polynomial expression is obtained that involves the variable φ and its derivatives:

$$\sum_{i=0}^{3} C_i \left(\frac{\varphi'}{\varphi}\right)^i = 0. \tag{3.8}$$

Equating all the coefficients C_i of $\left(\frac{\varphi'}{\varphi}\right)^i$ to zero implies the coefficients of $\left(\frac{\varphi'}{\varphi}\right)^3$:

$$\begin{cases} \frac{1}{2}\, {p_0}^3 + 2p_0{\varphi^{'}}^2 + {q_0}^2 p_0 = 0, \\ \ {q_0}^3 + 2q_0{\varphi^{'}}^2 + \frac{1}{2}{p_0}^2 q_0 = 0. \end{cases} \tag{3.9}$$

From (3.9), one can get:

$$p_0 = 2\ \varphi',\ q_0 = 2i\ \varphi^{'}. \tag{3.10}$$

Substituting (3.10) into (3.7) results in the Bäcklund transformation of (3.4) - (3.5) in the form:

$$p(\eta) = \frac{2\varphi'}{\varphi} + p_1, q(\eta) = i\frac{2\varphi'}{\varphi} + q_1. \tag{3.11}$$

Using (3.11) into (3.6) reduces the polynomial to:

$$\sum_{i=0}^{1} C_i \left(\frac{\varphi'}{\varphi}\right)^i = 0. \tag{3.12}$$

Equating the coefficient of $\left(\frac{\varphi'}{\varphi}\right)^i$ to zero results in:

$$\begin{cases} 2q_0\varphi^{'} - p_0\varphi^{''} + {q_0}^2 p_1 + \frac{3}{2}{p_0}^2 p_1 - 2p_0^{'}\varphi^{'} + 2p_0 q_0 q_1 = 0\ , \\ -p_0\varphi^{'} - q_0\varphi^{''} + 3{q_0}^2 q_1 + \frac{1}{2}{p_0}^2 q_1 - 2q_0^{'}\varphi^{'} + p_0 q_0 p_1 = 0, \\ p_0^{''} + {q_1}^2 p_0 + \frac{3}{2}{p_1}^2 p_0 - 2q_0^{'} + 2q_1 q_0 p_1 = 0, \\ q_0^{''} + 3{q_1}^2 q_0 + \frac{1}{2}{p_1}^2 q_0 + p_0^{'} + q_1 p_0 p_1 = 0. \end{cases} \tag{3.13}$$

The solution of the system (3.13) is:

$$p_1 = 2\ i - \frac{\varphi^{''}}{\varphi^{'}},\ q_1 = 1 - i\ \frac{\varphi^{''}}{\varphi^{'}}. \tag{3.14}$$

The Schwarzian derivatives of φ could be expressed in the form:

$$\left(\frac{\varphi^{''}}{\varphi^{'}}\right)^{'} - \frac{1}{2}\left(\frac{\varphi^{''}}{\varphi^{'}}\right)^2 = 1. \tag{3.15}$$

Solving (3.15) implies:

$$\varphi = \sqrt{2}c_2 \tan\left(\sqrt{\frac{1}{2}}\,(\eta + c_1)\right) + c_3. \tag{3.16}$$

Substituting (3.16) and (3.14) into (3.11), considering the relation between $p(\eta), q(\eta)$ *and* $p(x,t), q(x,t)$ described by (3.3) leads to:

$$\begin{cases} p\ (x\ t) = \frac{2C_2 - \sqrt{2}C_3 \tan\left(\sqrt{\frac{1}{2}}(x-t+c_1)\right)}{C_3 + \sqrt{2}C_2 \tan\left(\sqrt{\frac{1}{2}}(x-t+c_1)\right)} + 2i, \\ q\ (x\ t) = i\frac{2C_2 - \sqrt{2}C_3 \tan\left(\sqrt{\frac{1}{2}}(x-t+c_1)\right)}{C_3 + \sqrt{2}C_2 \tan\left(\sqrt{\frac{1}{2}}(x-t+c_1)\right)} - 1. \end{cases} \tag{3.17}$$

The solution $q(x,t)$ is depicted hereafter in Figure 7.

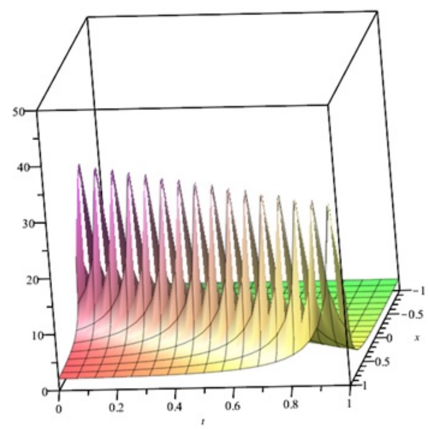

FIGURE 7. The Dirac integrable system solution of $q(x,t)$.

4. CONCLUSION

The Dirac integrable system has been investigated using two powerful methods, homogenous balance and Lie infinitesimals with singular manifolds methods. The motivation is to employ well known method to obtain comparable solutions with more simpler techniques. The solutions acquired are trigonometric and hyperbolic solutions. The solutions play a crucial role in comprehending the behavior of the Dirac system and its significance in various applications, particularly in quantum mechanics and fluid dynamics.

CONFLICT OF INTEREST

The authors declare that there is no conflict of interests regarding the publication of this article.

DATA AVAILABILITY

The authors confirm that the data supporting the findings of this study are available within the article.

REFERENCES

[1] G. Hashemi, *A novel analytical approximation approach for strongly nonlinear oscillation systems based on the energy balance method and He's frequency-amplitude formulation*, Computational Methods for Differential Equations, *11*(3) (2023), 464-477.
[2] J. Manafian, L. A. Dawood, and M. Lakestani, *New solutions to a generalized fifth-order kdv like equation with prime number p=3 via a generalized bilinear differential operator*, Partial Differential Equations in Applied Mathematics, *9* (2024), 100600.
[3] O. A. Ilhan, J. Manafian, H. M. Baskonus, and M. Lakestani, *Solitary wave solitons to one model in the shallow water waves*,The European Physical Journal Plus, *136*(3) (2021), 337.

[4] B. Kalegowda and R. Raghavachar, *Multi-soliton solutions to the generalized boussinesq equation of tenth order*, Computational Methods for Differential Equations, *11*(4) (2023), 727-737.

[5] J. J. E. Slotine and W. Li, *Applied nonlinear control*, Prentice Hall, 1991.

[6] S. Kondo and T. Miura,*Reaction-diffusion model as a framework for understanding biological pattern formation*, Science, *329* (2010), 1616-1620.

[7] B. A. Malomed, *Soliton management in periodic systems*, Springer New York, New York, 2006.

[8] R. W. Boyd, *Nonlinear optics*, Elsevier Science, 2003.

[9] G. P. Agrawal, *Nonlinear fiber optics*, Elsevier Science, 2012.

[10] T. F. Chan, E. l. Selim, and M. Nikolova, *Algorithms for finding global minimizers of image segmentation and denoising models*, SIAM Journal on Applied Mathematics, *66*(5) (2006), 1632-1648.

[11] G. Gilboa and S. Osher, *Nonlocal operators with applications to image processing*, Multiscale Modeling and Simulation, *7*(3) (2008), 1005-1028.

[12] J. D. Murray and S. S. Antman, *Mathematical biology. I, an introduction*, Springer New York, 2002.

[13] L. Edelstein-Keshet, *Mathematical models in biology*, Society for Industrial and Applied Mathematics Philadelphia, Philadelphia, 2005.

[14] H. K. Khalil, *Nonlinear systems*, Prentice Hall, 2002.

[15] M. Jafari, S. Mahdion, A. Akgül, and S. M. Eldin, *New conservation laws of the boussinesq and generalized kadomtsev–petviashvili equations via homotopy operator*, Results in Physics, *47* (2023), 106369.

[16] M. Jafari and R. Darvazebanzade, *Approximate symmetry group analysis and similarity reductions of the perturbed mKdV-KS equation*, Computational Methods for Differential Equations, *11*(1) (2023), 175-182.

[17] M. Jafari, A. Zaeim, and A. Tanhaeivash, *Symmetry group analysis and conservation laws of the potential modified KdV equation using the scaling method*, International Journal of Geometric Methods in Modern Physics, *19*(7) (2022), 2250098.

[18] M. Jafari, A. Zaeim, and S. Mahdion, *Scaling symmetry and a new conservation law of the harry dym equation*, Mathematics Interdisciplinary Research, *6*(2) (2021), 151-158.

[19] M. Jafari and R. Darvazebanzade, *Analyzing of approximate symmetry and new conservation laws of perturbed generalized Benjamin-Bona-Mahony equation*, AUT Journal of Mathematics and Computing, *5*(1) (2024), 61-69.

[20] M. Jafari and S. Mahdion, *Non-classical symmetry and new exact solutions of the kudryashov-sinelshchikov and modified KdV-ZK equations*, AUT Journal of Mathematics and Computing, *4*(2) (2023), 195-203.

[21] M. Mohamed, S. M. Mabrouk, and A. S. Rashed, *Mathematical investigation of the infection dynamics of covid-19 using the fractional differential quadrature method*, Computation, *11*(10) (2023), 198.

[22] E. M. Zayed and M. A. Abdelaziz, *Exact solutions for the nonlinear schrödinger equation with variable coefficients using the generalized extended tanh-function, the sine-cosine and the exp-function methods*, Applied Mathematics and Computation, *218*(5) (2011), 2259-2268.

[23] G. Betchewe, B. B. Thomas, K. K. Victor, and K. T. Crepin, *Explicit series solutions to nonlinear evolution equations: The sine-cosine method*, Applied Mathematics and Computation *215*(12) (2010), 4239-4247.

[24] M. Yaghobi Moghaddam, A. Asgari, and H. Yazdani, *Exact travelling wave solutions for the generalized nonlinear Schrödinger (GNLS) equation with a source by extended tanh-coth, sine-cosine and exp-function methods*, Applied Mathematics and Computation, *210*(2) (2009), 422-435.

[25] F. Taşcan and A. Bekir, *Analytic solutions of the (2+1)-dimensional nonlinear evolution equations using the sine-cosine method*, Applied Mathematics and Computation, *215*(8) (2009), 3134-3139.

[26] A.-M. Wazwaz, *he tanh–coth and the sine-cosine methods for kinks, solitons, and periodic solutions for the Pochhammer-Chree equations*, Applied Mathematics and Computation, *195*(1) (2008), 24-33.

[27] L. Wazzan, *A modified tanh-coth method for solving the KdV and the KdV-burgers' equations*, Communications in Nonlinear Science and Numerical Simulation, *14*(2) (2009), 443-450.

[28] L. Wazzan, *A modified tanh-coth method for solving the general Burgers-Fisher and the Kuramoto-Sivashinsky equations*, Communications in Nonlinear Science and Numerical Simulation, *14*(6) (2009), 2642-2652.

[29] A. Bekir and A. C. Cevikel, *Solitary wave solutions of two nonlinear physical models by tanh-coth method*, Communications in Nonlinear Science and Numerical Simulation, *14*(5) (2009), 1804-1809.

[30] H. Sasaki, *Inverse scattering problems for the Hartree equation whose interaction potential decays rapidly*, Journal of Differential Equations, *252*(2) (2012), 2004-2023.

[31] M. Lassas, J. L. Mueller, S. Siltanen, and A. Stahel, *The Novikov-Veselov equation and the inverse scattering method, part I: Analysis*, Physica D: Nonlinear Phenomena, *241*(16) (2012), 1322-1335.

[32] X. Ai and G. Gui, *On the inverse scattering problem and the low regularity solutions for the Dullin-Gottwald-Holm equation*, Nonlinear Analysis: Real World Applications, *11*(2) (2010), 888-894.

[33] A.-M. Wazwaz, *A study on two extensions of the Bogoyavlenskii-Schieff equation*, Communications in Nonlinear Science and Numerical Simulation, *17*(4) (2012), 1500-1505.

[34] A.-M. Wazwaz, *Structures of multiple soliton solutions of the generalized, asymmetric and modified Nizhnik-Novikov-Veselov equations*, Applied Mathematics and Computation, *218*(22) (2012), 11344-11349.

[35] W.-X. Ma, Y. Zhang, Y. Tang, and J. Tu, *Hirota bilinear equations with linear subspaces of solutions*, Applied Mathematics and Computation, *218*(13) (2012), 7174-7183.

[36] A.-M. Wazwaz, *N-soliton solutions for the integrable bidirectional sixth-order Sawada-Kotera equation*, Applied Mathematics and Computation, *216*(8) (2010), 2317-2320.

[37] Y. Wang and L. Wei, *New exact solutions to the (2+1)-dimensional Konopelchenko-Dubrovsky equation*, Communications in Nonlinear Science and Numerical Simulation, *15*(2) (2010), 216-224.

[38] M. Eslami, B. Fathi vajargah, and M. Mirzazadeh, *Exact solutions of modified Zakharov-Kuznetsov equation by the homogeneous balance method*, Ain Shams Engineering Journal, *5*(1) (2014), 221-225.

[39] Y. Zhou, F. Yang, and Q. Liu, *Reduction of the Sharma-Tasso-Olver equation and series solutions*, Communications in Nonlinear Science and Numerical Simulation, *16*(2) (2011), 641-646.

[40] Q. Pang, *Study on the behavior of oscillating solitons using the (2+1)-dimensional nonlinear system*, Applied Mathematics and Computation *217*(5) (2010), 2015-2023.

[41] A. S. Abdel Rady, E. S. Osman, and M. Khalfallah, *On soliton solutions for a generalized HHirota-Satsuma coupled KdV equation*, Communications in Nonlinear Science and Numerical Simulation, *15*(2) (2010), 264-274.

[42] M. Khalfallah, *Exact traveling wave solutions of the Boussinesq-Burgers equation*, Mathematical and Computer Modelling, *49*(3) (2009), 666-671.

[43] K. Khan and M. Ali Akbar, *Traveling wave solutions of the (2+1)-dimensional Zoomeron equation and the Burgers equations via the MSE method and the exp-function method*, Ain Shams Engineering Journal, *5*(1) (2014), 247-256.

[44] K. Parand and J. A. Rad, *Exp-function method for some nonlinear PDE's and a nonlinear ODE's*, Journal of King Saud University-Science, *24*(1) (2012), 1-10.

[45] A. Ebaid, *An improvement on the exp-function method when balancing the highest order linear and nonlinear terms*, Journal of Mathematical Analysis and Applications, *392*(1) (2012), 1-5.

[46] J. Biazar and Z. Ayati, *Exp and modified exp function methods for nonlinear Drinfeld-Sokolov system*, Journal of King Saud University-Science, *24*(4) (2012), 315-318.

[47] A. H. Bhrawy, M. A. Abdelkawy, and A. Biswas, *Cnoidal and snoidal wave solutions to coupled nonlinear wave equations by the extended Jacobi's elliptic function method*, Communications in Nonlinear Science and Numerical Simulation, *18*(4) (2013), 915-925.

[48] K. A. Gepreel and A. R. Shehata, *Rational Jacobi elliptic solutions for nonlinear differential-difference lattice equations*, Applied Mathematics Letters *25*(9) (2012), 1173-1178.

[49] M. K. Elboree, *The Jacobi elliptic function method and its application for two component BKP hierarchy equations*, Computers & Mathematics with Applications, *62*(12) (2011), 4402-4414.

[50] S. Z. Rida and M. Khalfallah, *New periodic wave and soliton solutions for a Kadomtsev-Petviashvili (KP) like equation coupled to a Schrödinger equation*, Communications in Nonlinear Science and Numerical Simulation, *15*(10) (2010), 2818-2827.

[51] X. Deng, J. Cao, and X. Li, *Travelling wave solutions for the nonlinear dispersion Drinfel'd-Sokolov (D(m,n)) system*, Communications in Nonlinear Science and Numerical Simulation, *15* (2) (2010), 281-290.

[52] A. S. Abdel Rady and M. Khalfallah, *On soliton solutions for Boussinesq-Burgers equations*, Communications in Nonlinear Science and Numerical Simulation, *15*(4) (2010), 886-894.

[53] Z. Lü, J. Su and F. Xie, *Construction of exact solutions to the Jimbo-Miwa equation through Bäcklund transformation and symbolic computation*, Computers & Mathematics with Applications, *65*(4) (2013), 648-656.

[54] A. D. Polyanin and A. I. Zhurov, *On RF-pairs, Bäcklund transformations and linearization of nonlinear equations*, Communications in Nonlinear Science and Numerical Simulation, *17*(2) (2012), 536-544.

[55] W.-X. Ma and A. Abdeljabbar, *A bilinear Bäcklund transformation of a (3+1)-dimensional generalized KP equation*, Applied Mathematics Letters *25* (10) (2012), 1500-1504.

[56] B. Lu, *Coupling Bäcklund trasnsformation of Riccati equation and division theorem method for traveling wave solutions of some class of NLPDEs*, Communications in Nonlinear Science and Numerical Simulation, *17*(12) (2012), 4626-4633.

[57] J. M. Conde, P. R. Gordoa and A. Pickering, *A new kind of Bäcklund transformation for partial differential equations*, Reports on Mathematical Physics, *70*(2) (2012), 149-161.

[58] A. S. Rashed, *Analysis of (3+1)-dimensional unsteady gas flow using optimal system of lie symmetries*, Mathematics and Computers in Simulation, *156* (2019), 327-346.

[59] M. M. Kassem and A. S. Rashed, *N-solitons and cuspon waves solutions of $(2+1)$-dimensional Broer-Kaup-Kupershmidt equations via hidden symmetries of lie optimal system*, Chinese Journal of Physics, *57* (2019), 90-104.

[60] S. M. Mabrouk and A. S. Rashed, *Analysis of (3 + 1)-dimensional Boiti- Leon-Manna-Pempinelli equation via Lax pair investigation and group transformation method*, Computers & Mathematics with Applications, *74*(10) (2017), 2546-2556.

[61] A. S. Rashed and M. M. Kassem, *Hidden symmetries and exact solutions of integro-differential Jaulent-Miodek evolution equation*, Applied Mathematics and Computation, *247* (2014), 1141-1155.

[62] A. S. Rashed and M. M. Kassem, *Group analysis for natural convection from a vertical plate*, Journal of Computational and Applied Mathematics, *222*(2) (2008), 392-403.

[63] M. M. Kassem and A. S. Rashed, *Group solution of a time dependent chemical convective process*, Applied Mathematics and Computation, *215*(5) (2009), 1671-1684.

[64] M.-X. Yu, B. Tian, Y.-Q. Yuan, Y. Sun, and X.-X. Du, *Conservation laws, solitons, breather and rogue waves for the (2+1)-dimensional variable-coefficient Nizhnik-Novikov-Veselov system in an inhomogeneous medium*, Chinese Journal of Physics, *56*(2) (2018), 645-658.

[65] Y.-H. Wang and H. Wang, *A coupled kdv system: Consistent tanh expansion, soliton-cnoidal wave solutions and nonlocal symmetries*, Chinese Journal of Physics *56* (2) (2018), 598-604.

[66] A. Patel and V. Kumar, *Dark and kink soliton solutions of the generalized ZK-BBM equation by iterative scheme*, Chinese Journal of Physics, *56*(3) (2018), 819-829.

[67] A. Isa Aliyu, M. Inc, A. Yusuf, and D. Baleanu, *Symmetry analysis, explicit solutions, and conservation laws of a sixth-order nonlinear Ramani equation*, Symmetry, *10* (2018), 341.

[68] M. Inc, A. Yusuf, A. I. Aliyu, and D. Baleanu, *Lie symmetry analysis and explicit solutions for the time fractional generalized Burgers-Huxley equation*, Optical and Quantum Electronics, *50*(2) (2018), 91-96.

[69] M. Inc, M. Hashemi, and A. Isa Aliyu, *Exact solutions and conservation laws of the Bogoyavlenskii equation*, Acta Physica Polonica *133* (2018), 1133-1137.

[70] M. Inc, A. I. Aliyu, A. Yusuf, and D. Baleanu, *On the classification of conservation laws and soliton solutions of the long short-wave interaction system*, Modern Physics Letters B, *32*(18) (2018), 1850202.

[71] D. Baleanu, M. Inc, A. Yusuf, and A. Isa Aliyu, *Optimal system, nonlinear self-adjointness and conservation laws for generalized shallow water wave equation*, Open Physics, *16* (2018), 364-370.

[72] D. Baleanu, M. Inc, A. Yusuf, and A. I. Aliyu, *Lie symmetry analysis and conservation laws for the time fractional simplified modified Kawahara equation*, Open Physics, *16*(1) (2018), 302-310.

[73] R. Saleh, M. Kassem, and S. Mabrouk, *Exact solutions of Calgero-Bogoyavlenskii-Schiff equation using the singular manifold method after Lie reductions*, Mathematical Methods in the Applied Sciences, *40*(16) (2017), 5851-5862.

[74] R. Saleh and A. S. Rashed, *New exact solutions of (3 + 1)-dimensional generalized Kadomtsev-Petviashvili equation using a combination of Lie symmetry and singular manifold methods*, Mathematical Methods in the Applied Sciences, *43*(4) (2020), 2045-2055.

[75] A. S. Rashed, S. M. Mabrouk, and A.-M. Wazwaz, *Forward scattering for non-linear wave propagation in (3 + 1)-dimensional Kimbo-Miwa equation using singular manifold and group transformation methods*, Waves in Random and Complex Media, *32*(2) (2022), 663-675.

[76] Y. Ye, Z. Li, C. Li, S. Shen, and W.-X. Ma, *A generalized Dirac soliton hierarchy and its bi-hamiltonian structure*, Applied Mathematics Letters, *60* (2016), 67-72.

[77] J. Zhang, Y. Zhao, F. You, and Z. Popowicz, *A new super extension of Dirac hierarchy*, Abstract and Applied Analysis, *2014* (2014), 472101.

[78] B. Wei, C. Hu, X. Guan, Z. Luan, M. Yao, and W. Liu, *Analytic study on soliton solutions for a dirac integrable equation*, Optik - International Journal for Light and Electron Optics, *183* (2019), 869-874.

[79] X.-X. Xu and Y.-P. Sun, *An integrable coupling hierarchy of Dirac integrable hierarchy, its liouville integrability and Darboux transformation*, The Journal of Nonlinear Sciences and Applications, *10*(6) (2017), 3328-3343.

[80] W.-X. Ma, *Multi-component bi-hamiltonian Dirac integrable equations*, Chaos, Solitons and Fractals, *39*(1) (2009), 282-287.

[81] E. Fan, *Two new applications of the homogeneous balance method*, Physics Letters A, *265*(5) (2000), 353-357.

[82] J.-L. Zhang, Y.-M. Wang, M.-L. Wang, and Z.-D. Fang, *New applications of the homogeneous balance principle*, Chinese Physics *12*(3) (2003), 245-250.

[83] S. M. Mabrouk, *Chase-repulsion analysis for (2+1)-dimensional Lotka-Volterra system*, International Journal of Engineering Research and Technology, *8*(6) (2019), 875-879