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# Unveiling traveling waves and solitons of dirac integrable system via homogenous balance and singular manifolds methods

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**Abstract**

This study utilizes two robust methodologies to examine the precise solutions of the Dirac integrable system. The Homogeneous Balance Method (HB) is initially employed to generate an accurate solution. The system of equations for the quasi-solution is solved, where all the equations are of the same nature. The quasi-solution of the traveling wave results in the solitary wave solution of the system. The singular manifold method (SMM) is utilized following the Lie reduction of the Dirac system in order to search for the traveling wave solutions of the system. Both approaches demonstrate the existence of traveling wave solutions inside the system. The precise solutions of the Dirac system are shown in three-dimensional graphs. We have created solutions to the examined problem, including bright solutions, periodic soliton solutions, and complicated solutions.

**Keywords.** Dirac integrable system, Homogeneous balance method, Singular manifold method.
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## 1. Introduction

Nonlinear evolution equations represent a powerful mathematical framework for modeling dynamic processes that deviate significantly from linear behavior [22, 23, 34, 50]. These equations have remarkable applications in fields as diverse as fluid dynamics, quantum mechanics, pattern formation, and information theory. Unlike their linear counterparts, nonlinear evolution equations exhibit intricate and often unexpected behaviors, such as soliton formation, wave breaking, chaos, and pattern formation, making them both intellectually challenging and practically valuable. The study of nonlinear evolution equations has garnered significant attention due to its ability to capture the complexities of real-world phenomena. These equations go beyond simple linear relationships and incorporate the interplay of various nonlinear effects, leading to a more accurate representation of natural processes. Consequently, understanding the properties and dynamics of nonlinear evolution equations is crucial for unraveling the underlying mechanisms governing complex systems [65].

Nonlinear evolution equations find applications in various scientific and engineering disciplines. Nonlinear evolution equations play a significant role in understanding the emergence of patterns in diverse systems, including chemical reactions, biological processes, and material science. Examples include the Fitz-Hugh-Nagumo model for neural dynamics and the Gray-Scott model for chemical pattern formation [40]. In soliton theory, solitons are stable, solitary waves that retain both speed and shape during propagation. Nonlinear Schrödinger and Korteweg-de Vries equations are examples of evolution equations used to study soliton behavior. These equations find applications in optical communications, fiber optics, and Bose-Einstein condensates [49]. For nonlinear optics, nonlinear evolution equations are essential for studying light-matter interactions in nonlinear optical materials. They help understand phenomena like self-focusing, self-phase modulation, and four-wave mixing, enabling applications in laser technology, optical communications, and photonic devices [3, 11] . In image processing field, Nonlinear evolution equations have been

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utilized in image denoising, image inpainting, and image segmentation tasks. The famous Rudin-Osher-Fatemi (ROF) model, based on the nonlinear total variation equation, is widely used for image restoration and enhancement [12, 21]. Another important application is mathematical biology, where nonlinear evolution equations are employed to model biological phenomena, such as population dynamics, biochemical reactions, and the spread of epidemics. These models aid in understanding ecological systems, tumor growth, neural networks, and gene regulation [16, 52]. Moreover, nonlinear evolution equations are employed in control theory to analyze and design feedback control systems. They allow for the control of complex nonlinear dynamics and find applications in robotics, aerospace engineering, and autonomous systems [38, 65]. The approaches of solution are really significant. The approaches employed encompass a range of analytical, semi-analytical, and numerical methods [28–33, 51]. Sine-Cosine [8, 66, 69, 77, 80], tanh-coth [7, 73, 74, 77], inverse scattering [4, 64], Hirota bilinear [46, 67, 70–72], extended homogenous method [1, 18, 37, 53, 83], Exp-function [10, 15, 39, 54], Elliptic function [2, 9, 14, 17, 20, 61], Bäcklund transformation [13, 42, 43, 45, 56], symmetry transformations and singular manifolds methods [5, 6, 24–27, 35, 36, 47, 55, 57–60, 62, 63, 68, 79] are common methods for investigating the behavior of nonlinear PDEs. The concept of integrable couplings arose and subsequent research was initiated during the examination of the connections between Virasoro algebras and hereditary operators [44, 75, 76, 78, 82].

The main aim of the current work is to deeply investigate the Dirac integrable system [75], which appears in a vast number of applications specially in quantum electrodynamics and the standard model. Dirac equation describes muons and tayons in electron studies. It is also used in quantum chromodynamics and in phenomenological hadron models, for example, in the relativistic model for quasi-independent quarks, which has been applied for the description of hadron properties. Two methods namely, homogeneous balance and Lie infinitesimals methods. Both methods result in new families of solutions in addition of comparable solutions with other methods in the literature. These solutions are crucial for comprehending the behavior of many physical phenomena, such as the propagation of waves in the presence of magnetic fields and the formation of optical solitons within waveguides.

The paper is structured in the following manner. In section 2, the homogeneous balance method is applied to Dirac integrable system to discover its closed form solutions. Section 3 is devoted to apply Lie infinitesimals method accompanied with singular manifolds method to construct new families of solutions. The conclusion remarks are described in section 4.

## 2. Description of the homogeneous balance method

The homogeneous balancing (HB) method has been widely recognized as a highly successful approach for determining explicit solitary wave solutions. The subsequent section provides a description of the methodology of homogenous balance [19, 48, 81].

Assume that a system of partial differential equations (PDEs) to be in the form:

$$\begin{cases} p_1\left(\varphi, \psi, \varphi_x, \psi_x, \varphi_t,\ \varphi_{xx}, \psi_{xx}\ ,\ \ \ldots\right) = 0, \\ p_2\left(\varphi, \psi, \varphi_x, \psi_x, \varphi_t,\ \varphi_{xx}, \psi_{xx}\ , \ldots\right) = 0, \end{cases} \tag{2.1}$$

where $\mathbf{p_1}$ and $\mathbf{p_2}$ are polynomials in the dependent variables, $\varphi, \psi$, and their partial derivatives.

A function $\xi = \xi(x,t)$ can be considered as a "quasi-solution" of (2.1), if there are functions $f = f(\xi)$ and $g = g(\xi)$, of only one argument so that, a nominated linear combination of;

$$1, f(\xi), f_x(\xi), f_t(\xi), f_x x(\xi), f_x t(\xi), \ldots. \tag{2.2}$$

and

$$1, g(\xi), g_x(\xi), g_t(\xi), ,\ g_{xx}(\xi), g_{xt}(\xi), \ldots. \tag{2.3}$$

are solutions to system (2.1). The HB technique is outlined in the following prosecutorial steps:

*Step 1:* The solution to system (2.1) is postulated as a linear combination of (2.2) and (2.3), ensuring a balance

between the strongly nonlinear and the highest order derivative components in the system.
*Step 2:* The selected combination from the previous step is introduced into the system (2.1). The terms with the largest degree of $\xi(x,t)$ are gathered, and the total of their coefficients is equated to zero. The outcome is a set of ordinary differential equations (ODEs) involving the functions $f(\xi)$ and $g(\xi)$.
*Step 3:* Replace the nonlinear terms, which include derivatives of $g(\xi)$ and $f(\xi)$, and gather all the terms that have the same order of $f,\ f^{'},\ f^{''},\ f^{'''},\ ...,\ g,\ g^{'},\ g^{''},\ g^{'''},\ \ldots$ and then, equate the coefficients of these terms to zero. Obtain a system of ODEs in terms of the variable $\xi\ (x,\ t)$ that is uniform or consistent throughout. Based on the homogeneous property of this system of equations, $\xi(x,t)$ may be forecasted as an exponential function.
*Step 4:* Replace the functions $f(\xi),\ g(\xi)$, and $\xi(x,t)$ in the linear combination from step 1. This will yield the solution of the system (2.1).

2.1. **Solitary wave solution of Dirac integrable system.** The Dirac integrable system is presented as;

$$p_t = q_{xx} + q^3 + \frac{1}{2}qp^2, \tag{2.4}$$

$$q_t = -\frac{1}{2}p_{xx} - \frac{1}{4}p^3 - \frac{1}{2}pq^2, \tag{2.5}$$

where $p$ and $q$ are functions of the spatial coordinate $x$ and temporal coordinate $t$. In this section, the homogeneous balancing approach is utilized to discover the precise solution of the Dirac integrable system (2.4) and (2.5). Select the solution of this system as a linear combination in the specified form:

$$p(x,t) = \frac{\partial^{\beta_1} f(\xi)}{\partial x^{\beta_1}}, \tag{2.6}$$

$$q(x,t) = \frac{\partial^{\beta_2} g(\xi)}{\partial x^{\beta_2}}. \tag{2.7}$$

The equilibrium between the most non-linear and the most significant derivative components in the system (2.4) and (2.5) leads to $\beta_1 = \beta_2 = 2$. The linear combinations (2.6) and (2.7) can be expressed as:

$$p(x,t) = f^{''}\xi_x^2 + f^{'}\xi_{xx}, \tag{2.8}$$

$$q(x,t) = g^{''}\xi_x^2 + g^{'}\xi_{xx}. \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.4) results in:

$$\begin{aligned}
&\left(\frac{1}{2}g''f^{''2} + g^{''3}\right)\xi_x^6 + \left(3\ g^{''2}g^{'}\xi_{xx} + g^{(4)} + g''f^{''}f^{'}\xi_{xx} + \frac{1}{2}g'f^{''2}\xi_{xx}\right)\xi_x^4 \\
&+\left(6\ g^{'''}\xi_{xx} - f^{'''}\xi_t + 3g^{''}g^{'2}\xi_{xx}^2 + g^{'}f^{''}f^{'}\xi_{xx}^2 + \frac{1}{2}g^{''}f^{'2}\xi_{xx}^2\right)\xi_x^2 + \left(4g^{''}\xi_{xxx} - 2f^{''}\xi_{xt}\right)\xi_x \\
&+g^{'}\xi_{4x} + \frac{1}{2}g^{'}f^{'2}\xi_{xx}^3 + 3g^{''}\xi_{xx}^2 - f^{''}\xi_t\xi_{xx} + g^{'3}\xi_{xx}^3 - f^{'}\xi_{xxt} = 0.
\end{aligned} \tag{2.10}$$

Substituting (2.8) and (2.9) into (2.5) results in:

$$\begin{aligned}
&\left(\frac{1}{2}f''g^{''2} + \frac{1}{4}f^{''3}\right)\xi_x^6 + \left(\frac{3}{4}f^{''2}f^{'}\xi_{xx} + \frac{1}{2}f^{(4)} + g''f^{''}g^{'}\xi_{xx} + \frac{1}{2}f'g^{''2}\xi_{xx}\right)\xi_x^4 \\
&+\left(3\ f^{'''}\xi_{xx} + g^{'''}\xi_t + \frac{3}{4}f^{''}f^{'2}\xi_{xx}^2 + g^{'}g^{''}f^{'}\xi_{xx}^2 + \frac{1}{2}f^{''}g^{'2}\xi_{xx}^2\right)\xi_x^2 + \left(2f^{''}\xi_{xxx} + 2g^{''}\xi_{xt}\right)\xi_x \\
&+\frac{1}{2}f^{'}\xi_{4x} + \frac{1}{2}f^{'}g^{'2}\xi_{xx}^3 + \frac{3}{2}f^{''}\xi_{xx}^2 + g^{''}\xi_t\xi_{xx} + \frac{1}{4}f^{'3}\xi_{xx}^3 + g^{'}\xi_{xxt} = 0.
\end{aligned} \tag{2.11}$$

Setting the coefficients of $\xi_x^6 = 0$ yields a system of ODEs in the form:

$$\begin{cases} \frac{1}{2}g''f''^2 + g''^3 = 0, \\ \frac{1}{2}f''g''^2 + \frac{1}{4}f''^3 = 0. \end{cases} \tag{2.12}$$

The solutions of (2.12) can be formulated in the form:

$$f = C_1 \ln \xi, g = C_2 \ln \xi. \tag{2.13}$$

The relationship between the nonlinear derivatives of $g(\xi)$ and $f(\xi)$ may be described as follows:

$$\begin{cases} f'^2 = -C_1 f'' , \quad f''^2 = -\frac{1}{6}C_1 f^{(4)}, \quad f'f'' = -\frac{1}{2}C_1 f''' , \quad f'^3 = -\frac{1}{2}C_1^2 f''', \\ g'^2 = -C_2 g'' , \quad g''^2 = -\frac{1}{6}C_2 g^{(4)}, g'g'' = -\frac{1}{2}C_2 g''', \quad g'^3 = -\frac{1}{2}C_2^2 g''' , \\ f'g' = -C_1 g'' = -C_2 f'' , \quad g'f'' = f'g'' = -\frac{1}{2}C_1 g''' = -\frac{1}{2}C_2 f''', \\ g''f''g' = f''^2 g' = \frac{1}{24}C_1 g^{(5)}, \quad g'f''f' = \frac{1}{6}C_2^2 g^{(4)} = \frac{1}{6}C_1 C_2 f^{(4)}. \end{cases} \tag{2.14}$$

Using (2.13) and (2.14), the Equations (2.10) and (2.11) are simplified to be:

$$\begin{aligned} &\left(\frac{1}{2}\xi_{4x} + \frac{C_2}{C_1}\xi_{xxt}\right) f' + \left(3\xi_x^2 + 3\xi\ \xi_{xx} + 2\xi_x\xi_{xxx} + 2\frac{C_2\xi_x\xi_{xt}}{C_1} + \frac{C_2\xi_t\xi_{xx}}{C_1}\right) f'' \\ &+ \left(\frac{3}{4}C_2^2\xi^2\xi_{xx} + 2\frac{C_2\xi\xi_x\xi_t}{C_1} + \frac{3}{2}C_2^2\xi\ \xi_x^2 + \frac{3}{4}C_1^2\xi\ \xi_x^2 + 6\xi\ \xi_x\xi_{xx} + \frac{3}{8}C_1^2\xi^2\xi_{xx}\right) f''' \\ &+ \left(\frac{1}{2}C_1^2\xi^2\xi_x\xi_{xx} + 2\xi^3\xi_x + C_2^2\xi^2\xi_x\xi_{xx} + \frac{1}{2}C_1^2\xi\ \xi_x^3\right) f^{(4)} + \left(\frac{1}{8}C_1^2\xi^3\xi_x\xi_{xx} + \frac{1}{4}C_2^2\xi^3\xi_x\xi_{xx}\right) f^{(5)} = 0, \end{aligned} \tag{2.15}$$

$$\begin{aligned} &\left(\xi_{4x} - \frac{C_1}{C_2}\xi_{xxt}\right) g' + \left(6\xi_x^2 + 6\xi\ \xi_{xx} + 4\xi_x\xi_{xxx} - 2\frac{C_1\xi_x\xi_{xt}}{C_2} - \frac{C_1\xi_t\xi_{xx}}{C_2}\right) g'' \\ &+ \left(\frac{3}{4}C_1^2\xi^2\xi_{xx} - 2\frac{C_1\xi\xi_x\xi_t}{C_2} + \frac{3}{2}C_1^2\xi\ \xi_x^2 + 3C_2^2\xi\ \xi_x^2 + 12\xi\ \xi_x\xi_{xx} + \frac{3}{2}C_2^2\xi^2\xi_{xx}\right) g''' \\ &+ \left(2C_2^2\xi^2\xi_x\xi_{xx} + 2C_2^2\xi\ \xi_x^3 + C_1^2\xi^2\xi_x\xi_{xx} + C_1^2\xi\ \xi_x^3 + 4\xi^3\xi_x\right) g^{(4)} \\ &+ \left(\frac{1}{2}C_2^2\xi^3\xi_x\xi_{xx} + \frac{1}{4}C_1^2\xi^3\xi_x\xi_{xx}\right) g^{(5)} = 0. \end{aligned} \tag{2.16}$$

Setting the coefficients of $f^{(5)}, f^{(4)}, \ldots,\ f'$ in (2.15) and the coefficients of $g^{(5)}, g^{(4)}, \ldots,\ g'$in (2.6) to zero results in a system of PDEs for $\xi(x,\ t)$, as follows:

$$\frac{1}{8}C_1^2\xi^3\xi_x\xi_{xx} + \frac{1}{4}C_2^2\xi^3\xi_x\xi_{xx} = 0, \tag{2.17}$$

$$\frac{1}{2}C_1^2\xi^2\xi_x\xi_{xx} + 2\xi^3\xi_x + C_2^2\xi^2\xi_x\xi_{xx} + \frac{1}{2}C_1^2\xi\ \xi_x^3 = 0, \tag{2.18}$$

$$\left(\frac{3}{4}C_2^2\xi^2\xi_{xx} + 2\frac{C_2\xi\xi_x\xi_t}{C_1} + \frac{3}{2}C_2^2\xi\ \xi_x^2 + \frac{3}{4}C_1^2\xi\ \xi_x^2 + 6\xi\ \xi_x\xi_{xx} + \frac{3}{8}C_1^2\xi^2\xi_{xx}\right) f''' = 0, \tag{2.19}$$

$$3\xi_x^2 + 3\xi\ \xi_{xx} + 2\xi_x\xi_{xxx} + 2\frac{C_2\xi_x\xi_{xt}}{C_1} + \frac{C_2\xi_t\xi_{xx}}{C_1} = 0, \tag{2.20}$$

$$\frac{1}{2}\xi_{4x} - \frac{C_2}{C_1}\xi_{xxt} = 0, \tag{2.21}$$

$$\frac{1}{2}C_2^2\xi^3\xi_x\xi_{xx} + \frac{1}{4}C_1^2\xi^3\xi_x\xi_{xx} = 0, \tag{2.22}$$

$$2C_2^2\xi^2\xi_x\xi_{xx} + 2C_2^2\xi\ \xi_x^3 + C_1^2\xi^2\xi_x\xi_{xx} + C_1^2\xi\ \xi_x^3 + 4\xi^3\xi_x = 0, \tag{2.23}$$

$$\frac{3}{4}C_1^2\xi^2\xi_{xx} - 2\frac{C_1\xi\xi_x\xi_t}{C_2} - \frac{3}{2}C_1^2\xi\ \xi_x^2 + 3C_2^2\xi\ \xi_x^2 + 12\xi\ \xi_x\xi_{xx} + \frac{3}{2}C_2^2\xi^2\xi_{xx} = 0, \tag{2.24}$$

$$6\xi_x^2 + 6\xi\ \xi_{xx} + 4\xi_x\xi_{xxx} - 2\frac{C_1\xi_x\xi_{xt}}{C_2} - \frac{C_1\xi_t\xi_{xx}}{C_2} = 0, \tag{2.25}$$

$$\xi_{4x} - \frac{C_1}{C_2}\xi_{xxt} = 0. \tag{2.26}$$

The system (2.17)-(2.26) is solved with the aid of Maple tool yields,

$$\xi(x,\ \ t) = 1 + e^{\alpha x+\beta t+\gamma}, \tag{2.27}$$

where $\alpha, \beta$ and $\gamma$ are arbitrary constants and the solutions of the Dirac integrable system (2.4)-(2.5) are given by:

$$\begin{cases} p(x,\ t) = \frac{\sqrt{2}\alpha^2}{4} sech^2(\alpha x + \beta t + \gamma), \\ q(x,\ t) = \frac{i\alpha^2}{4} sech^2(\alpha x + \beta t + \gamma). \end{cases} \tag{2.28}$$

$$\begin{cases} p(x,\ t) = -\frac{\sqrt{2}\alpha^2}{4}\sin\left(\frac{1}{2}(\alpha x + \beta t + \gamma)\right) + \frac{\sqrt{2}\alpha^2}{4}\cos\left(\frac{1}{2}(\alpha x + \beta t + \gamma)\right), \\ q(x,\ t) = \frac{i\alpha^2}{4}\sin\left(\frac{1}{2}(\alpha x + \beta t + \gamma)\right) - \frac{i\alpha^2}{4}\cos\left(\frac{1}{2}(\alpha x + \beta t + \gamma)\right). \end{cases} \tag{2.29}$$

$$\begin{cases} p(x,\ t) = \frac{\sqrt{2}\alpha^2}{4}\coth^2(\alpha x + \beta t + \gamma), \\ q(x,\ t) = \frac{i\alpha^2}{4}\coth^2(\alpha x + \beta t + \gamma). \end{cases} \tag{2.30}$$

The solutions of the system (2.4)-(2.5) are depicted hereafter in Figures 1-6.

## 3. Reduction of the Dirac integrable system by Lie transformations theory

The Lie transformation method is utilized in this section to convert the Dirac integrable system into a set of ordinary differential equations. The Dirac integrable system (2.4)-(2.5) possesses the following Lie infinitesimal vectors:

$$X_1 = \frac{\partial}{\partial x},\ X_2 = \frac{\partial}{\partial t}, X_3 = q\frac{\partial}{\partial p} - \frac{p}{2}\frac{\partial}{\partial q}, X_4 = t\frac{\partial}{\partial x} + xq\frac{\partial}{\partial p} - \frac{xp}{2}\frac{\partial}{\partial q}, X_5 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{p}{2}\frac{\partial}{\partial p} - \frac{q}{2}\frac{\partial}{\partial q}. \tag{3.1}$$

Using the combined vector $X_1 + X_2 = \frac{\partial}{\partial x} + \frac{\partial}{\partial t}$, the characteristic equation is described by:

$$\frac{dx}{\xi(x,t,u)} = \frac{dt}{\tau(x,t,u)} = \frac{dy}{Y(x,t,u)} = \frac{du}{\varphi(x,t,u)}. \tag{3.2}$$

The invariant transformations are given by:

$$\eta = x - t,\ q(\eta) = q(x,t), p(\eta) = p\ (x,\ t). \tag{3.3}$$

Then, the Dirac integrable system is reduced to the following nonlinear ODE system:

$$q^{''} + p^{'} +\ q^3 + \frac{1}{2}qp^2 = 0, \tag{3.4}$$

$$\frac{1}{2}p^{''} - q^{'} + \frac{1}{4}\ p^3 + \frac{1}{2}pq^2 = 0. \tag{3.5}$$

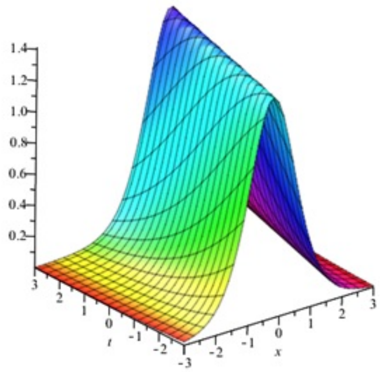


FIGURE 1. The Bright solution (2.28) of the Dirac integrable system $P(x,t)$ at $\alpha = 2, \beta = -0.5, \gamma = -1$.

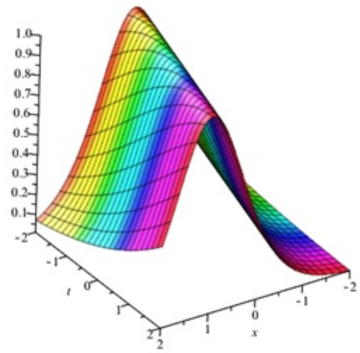


FIGURE 2. The Bright solution (2.28) of the Dirac integrable system $q(x,t)$ at $\alpha = 2, \beta = -0.5, \gamma = -1$.

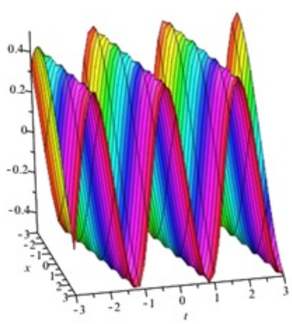


FIGURE 3. The Bright solution (2.29) of the Dirac integrable system $P(x,t)$ at $\alpha = 1, \beta = -6, \gamma = 10$.

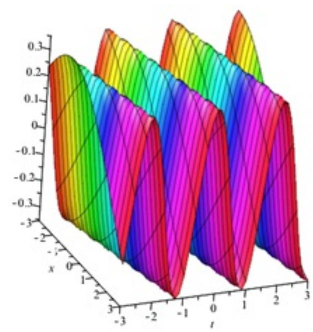


FIGURE 4. The Bright solution (2.29) of the Dirac integrable system $q(x,t)$ at $\alpha = 1, \beta = -6, \gamma = 10$.

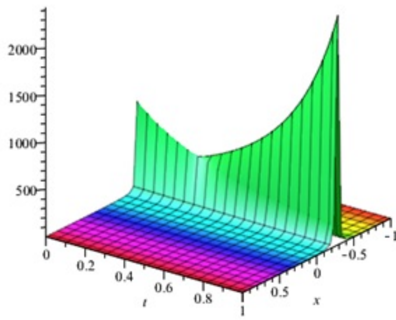


FIGURE 5. The Bright solution (2.30) of the Dirac integrable system $P(x,t)$ at $\alpha = 0.5, \beta = 0.01, \gamma = 0.125$.

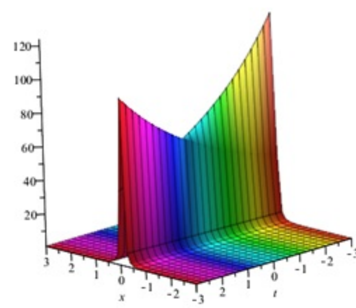


FIGURE 6. The Bright solution (2.30) of the Dirac integrable system $P(x,t)$ at $\alpha = 0.5, \beta = 0.01, \gamma = 0.125$.

3.1. **Investigating of the solution using singular manifold method (SMM).** The singular manifold approach is utilized in this part to discover novel sets of solutions for the Dirac integrable system (2.4)-(2.5). According to the SMM [62, 63], the solutions of Equations (3.4) and (3.5) are written in a series form:

$$\begin{cases} p(\eta) = \sum_{j=0}^{\infty} p_j \varphi(\eta)^{j-\alpha}, \\ q(\eta) = \sum_{j=0}^{\infty} q_j \varphi(\eta)^{j-\alpha}. \end{cases} \tag{3.6}$$

The dominating behavior yields an $\alpha$ value of 1. The Equation (3.6) can be reformulated as:

$$\begin{cases} p(\eta) = \frac{p_0}{\varphi} + p_1, \\ q(\eta) = \frac{q_0}{\varphi} + q_1. \end{cases} \tag{3.7}$$

By substituting the results in (3.7) into Equation (3.4), a polynomial expression is obtained that involves the variable $\varphi$ and its derivatives:

$$\sum_{i=0}^{3} C_i \left(\frac{\varphi'}{\varphi}\right)^i = 0. \tag{3.8}$$

Equating all the coefficients $C_i$ of $\left(\frac{\varphi'}{\varphi}\right)^i$ to zero implies the coefficients of $\left(\frac{\varphi'}{\varphi}\right)^3$ :

$$\begin{cases} \frac{1}{2}\ {p_0}^3 + 2p_0{\varphi'}^2 + {q_0}^2 p_0 = 0, \\ \ {q_0}^3 + 2q_0{\varphi'}^2 + \frac{1}{2}{p_0}^2 q_0 = 0. \end{cases} \tag{3.9}$$

From (3.9), one can get:

$$p_0 = 2\ \varphi',\ q_0 = 2i\ \varphi'. \tag{3.10}$$

Substituting (3.10) into (3.7) results in the Bäcklund transformation of (3.4)-(3.5) in the form:

$$p(\eta) = \frac{2\varphi'}{\varphi} + p_1, q(\eta) = i\frac{2\varphi'}{\varphi} + q_1. \tag{3.11}$$

Using (3.11) into (3.6) reduces the polynomial to:

$$\sum_{i=0}^{1} C_i \left(\frac{\varphi'}{\varphi}\right)^i = 0. \tag{3.12}$$

Equating the coefficient of $\left(\frac{\varphi'}{\varphi}\right)^i$ to zero results in:

$$\begin{cases} 2q_0\varphi' - p_0\varphi'' + {q_0}^2 p_1 + \frac{3}{2}{p_0}^2 p_1 - 2p_0'\varphi' + 2p_0q_0q_1 = 0\ , \\ -p_0\varphi' - q_0\varphi'' + 3{q_0}^2 q_1 + \frac{1}{2}{p_0}^2 q_1 - 2q_0'\varphi' + p_0q_0p_1 = 0, \\ p_0'' + {q_1}^2 p_0 + \frac{3}{2}{p_1}^2 p_0 - 2q_0' + 2q_1q_0p_1 = 0, \\ q_0'' + 3{q_1}^2 q_0 + \frac{1}{2}{p_1}^2 q_0 + p_0' + q_1p_0p_1 = 0. \end{cases} \tag{3.13}$$

The solution of the system (3.13) is:

$$p_1 = 2\ i - \frac{\varphi''}{\varphi'}, \quad q_1 = 1 - i\ \frac{\varphi''}{\varphi'}. \tag{3.14}$$

The Schwarzian derivatives of $\varphi$ could be expressed in the form:

$$\left(\frac{\varphi''}{\varphi'}\right)' - \frac{1}{2}\left(\frac{\varphi''}{\varphi'}\right)^2 = 1. \tag{3.15}$$

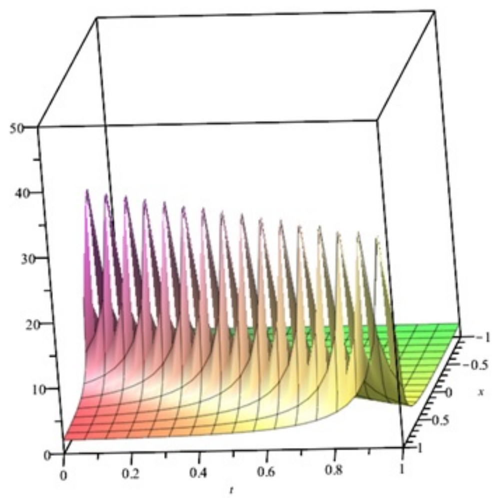


FIGURE 7. The Dirac integrable system solution of $q(x,t)$.

Solving (3.15) implies:

$$\varphi = \sqrt{2}c_2 \tan\left(\sqrt{\frac{1}{2}}\,(\eta + c_1)\right) + c_3. \tag{3.16}$$

Substituting (3.16) and (3.14) into (3.11), considering the relation between $p(\eta), q(\eta)$ *and* $p(x,t), q(x,t)$ described by (3.3) leads to:

$$\begin{cases} p\ (x\ t) = \dfrac{2C_2 - \sqrt{2}C_3 \tan\left(\sqrt{\frac{1}{2}}(x-t+c_1)\right)}{C_3 + \sqrt{2}C_2 \tan\left(\sqrt{\frac{1}{2}}(x-t+c_1)\right)} + 2i, \\ q\ (x\ t) = i\dfrac{2C_2 - \sqrt{2}C_3 \tan\left(\sqrt{\frac{1}{2}}(x-t+c_1)\right)}{C_3 + \sqrt{2}C_2 \tan\left(\sqrt{\frac{1}{2}}(x-t+c_1)\right)} - 1. \end{cases} \tag{3.17}$$

The solution $q(x,t)$ is depicted hereafter in Figure 7.

## 4. Conclusion

The Dirac integrable system has been investigated using two powerful methods, homogenous balance and Lie infinitesimals with singular manifolds methods. The motivation is to employ well-known method to obtain comparable solutions with simpler techniques. The solutions acquired are trigonometric and hyperbolic solutions. The solutions play a crucial role in comprehending the behavior of the Dirac system and its significance in various applications, particularly in quantum mechanics and fluid dynamics.

## Conflict of Interest

The authors declare that there is no conflict of interests regarding the publication of this article.

## Data Availability

The authors confirm that the data supporting the findings of this study are available within the article.

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