



## q-Exponential fixed point theorem for mixed monotone operator with q-fractional problem

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### Abstract

In this work, we examine the existence and uniqueness (EU) of q-Exponential positive solution (q-EPS) of the hybrid q-fractional boundary value problem (q-FBVP). We prove the q-Exponential fixed point theorem (q-EFPT) with a new set  $\rho_{h,e_1}$  in the Banach space  $\mathbf{E}$  to check the EU of q-EPS of the q-FBVP. In the long run, an exemplum is given to show the correctness of our results.

**Keywords.** q-Derivative, Positive solution, Mixed monotone operator, q-Integral, q-Exponential fixed point theorem.

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### 1. INTRODUCTION

Fractional differential equations (FDE) include fractional derivatives. Nowadays, there are several different concepts and definitions of fractional derivatives, fractional integrals and their applications in different fields of mathematics. In [3, 11], the existence and uniqueness of the positive solutions of the fractional boundary value problem have been investigated by using fixed point theorems in a set including  $\varphi(h, e)$ -concave operators. In [6], unique solutions of the fractional boundary value problem in ordered Banach spaces where the operators are mixed monotone were examined. Two important types of these derivatives and integrals that have been of our interest are q-derivative, q-integral, Riemann-Liouville fractional derivative (RLFD) and Riemann-Liouville fractional integral (RLFI). In this work, we investigate the EU of the q-EPS of the problem below:

$$\left\{ \begin{array}{l} -D_q D_{0+}^\gamma \left( \frac{w(\tau) + 3e_1(\tau)}{E_q(\lambda\tau)} \right) - [\chi(\tau, w(\tau), \vartheta(\tau)) + \varrho(\tau, w(\tau)) + \phi(\tau, \vartheta(\tau))] = 0, \\ 0 < \tau < 1, \quad 2 \leq \gamma < 3, \quad 0 < q < 1, \\ w(0) = w'(0) = e_1(0) = e_1'(0) = 0, \\ \left[ I_{0+}^\alpha \left( \frac{w(\tau) + 3e_1(\tau)}{E_q(\lambda\tau)} \right) \right]_{\tau=1} = 0, \quad 1 \leq \alpha \leq 2, \end{array} \right. \quad (1.1)$$

where  $\lambda$  is a positive real number,  $e_1(\tau)$  is a differentiable function and  $0 < e_1(\tau)$ ,  $D_{0+}^\gamma$  is the RLFD of order  $\gamma$ ,  $D_q$  is the q-derivative and  $I_{0+}^\alpha$  is the RLFI of order  $\alpha$ . Let's assume that  $O = [0, 1]$  and  $\Upsilon = [0, +\infty)$ . The functions  $\chi : O \times \Upsilon \times \Upsilon \rightarrow \Upsilon$  and  $\varrho, \phi : O \times \Upsilon \rightarrow \Upsilon$  are continuous.

**Remark 1.1.** In this work, the  $\vartheta$  variable in the  $\chi$  function equal to  $\vartheta(\tau) = 1 - \frac{I_{q,0} w(\tau)}{\tau}$ ,  $q, \tau \in (0, 1)$ .

We have extracted 1.2 from reference [9].

**Definition 1.2.** The q-Exponential function  $E_q(\tau)$  is expressed as  $E_q(\tau) = \sum_{k=0}^{\infty} \frac{1}{\{k\}_q!} \tau^k$  if  $|q| > 1$  or  $0 < |q| < 1$ ,  $|\tau| < |1 - q|^{-1}$ . Where  $\{k\}_q! = \prod_{k=1}^n \frac{1-q^k}{1-q}$  and  $\{0\}_q = 0$ .

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In [1], A. Boutiara and M. Benbachir concerned the EU of solution of the  $q$ -system of nonlinear FDE on four points. Their analysis relies on two issues, one of them is topological degree and the other is the fixed point theorem (FPT) of Banach contraction principle.

$$\begin{aligned} D_q^{\alpha_1} w_1(\mu) &= \chi_1(\mu, w_1(\mu), w_2(\mu)), \quad \mu \in O := [0, 1], \\ D_q^{\alpha_2} w_2(\mu) &= \chi_2(\mu, w_1(\mu), w_2(\mu)), \end{aligned}$$

with boundary conditions:

$$\begin{aligned} w_1(0) &= a_1 I_q^{\beta_1} w(\eta_1), \quad 0 < \eta_1 < 1, \quad \beta_1 > 0, \\ w_1(1) &= b_1 I_q^{\alpha_1} w(\sigma_1), \quad 0 < \sigma_1 < 1, \quad \alpha_1 > 0, \\ w_2(0) &= a_2 I_q^{\beta_2} w(\eta_2), \quad 0 < \eta_2 < 1, \quad \beta_2 > 0, \\ w_2(1) &= b_2 I_q^{\alpha_2} w(\sigma_2), \quad 0 < \sigma_2 < 1, \quad \alpha_2 > 0, \end{aligned}$$

for  $i = 1, 2$ ,  $D_q^{\alpha_i}$  are the  $q$ -derivative from the Caputo type of order  $1 < \alpha_i \leq 2$ .

In [4], C. Zhai and L. Wang proved the EU of positive solutions (PS) of the following FDE by using FPT:

$$\begin{aligned} D_{0+}^{\alpha} w(\tau) + \chi(\tau, w(\tau)) &= b, \quad 0 < \tau < 1, \\ w(0) = w'(0) = 0, \quad w(1) &= \beta \int_0^1 w(s) ds, \end{aligned}$$

where  $2 < \alpha \leq 3$ ,  $0 < \beta < \alpha$  and  $b > 0$  is a constants.

In [7], M. Gholami and A. Neamaty have proved the EU of solution of the problem below by using the  $\lambda$ -FPT of mixed monotone operator (MMO):

$$\begin{aligned} -D_{0+}^{\nu} w(\tau) &= \lambda^{-1}(\chi(\tau, w(\tau), \vartheta(\tau)) + \varrho(\tau, w(\tau)) + \phi(\tau, \vartheta(\tau))), \quad 0 < \tau < 1, \quad 3 \leq \nu \leq 4, \\ w(0) = w'(0) = w''(0) &= 0, \\ [D_{0+}^{\rho} w(\tau)]_{\tau=1} &= 0, \quad 1 \leq \rho \leq 2. \end{aligned}$$

This problem includes the RLFD and the positive number of  $\lambda$ . In [10], Y. Liu, Ch. Yan, and W. Jiang checked the existence of solution to the below FBVP:

$$\begin{aligned} {}^C D_{1-}^{\alpha} D_{0+}^{\beta} w(\tau) + \chi(\tau, w(\tau)) &= b, \quad 0 < \tau < 1, \\ w(0) = 0, \quad w'(1) = D_{0+}^{\beta} w(1) &= 0, \end{aligned}$$

where  ${}^C D_{1-}^{\alpha}$  is the left-sided Capato derivative of order  $\alpha$ ,  $0 < \alpha \leq 1$  and  $D_{0+}^{\beta}$  is the right RLFD of order  $\beta$ ,  $1 < \beta \leq 2$  and  $\alpha + \beta \geq 2$ .  $b$  is a constant.

In [12], Y. sang, and Y. Ren checked the EU of the PS to the following BVP using the method of the MMO:

$$\begin{aligned} w^{(4)}(\tau) &= \chi(\tau, w(\tau), (Hw)(\tau)), \quad \tau \in [0, 1], \\ w(0) = w'(0) &= 0, \\ w''(1) = 0, \quad w'''(1) &= \varrho(w(1)). \end{aligned}$$

Our purpose is to survey the EU of the  $q$ -EPS of the hybrid  $q$ -FBVP (1.1) using the  $q$ -FPT with the new set  $\rho_{h, e_1}$ . We work in a real Banach space  $(\mathbf{E}, \|\cdot\|)$ , with  $\rho$  cone. (Read [7] for more information).

## 2. PRELIMINARIES

In the second part, we state the preliminaries that we need to prove the main results. From the references [2] and [8], we have extracted Definitions below.



**Definition 2.1.** RLFI  $I_{0+}^\alpha w$  of order  $\alpha > 0$  on finite interval  $\Omega = [0, b](0 < b < \infty)$  on the real axis  $\mathfrak{R}$ , is defined below:

$$I_{0+}^\alpha w(u) = \frac{1}{\Gamma(\alpha)} \int_0^u \frac{w(\tau)}{(u - \tau)^{1-\alpha}} d\tau, \quad u > 0, \alpha > 0,$$

here  $\Gamma(\alpha)$  is the Gamma function.

**Definition 2.2.** Let  $\gamma > 0$ .  $D_{0+}^\gamma w$  is called RLFD of order  $\gamma > 0$  and is expressed as below:

$$(D_{0+}^\gamma w)(u) := \left(\frac{d}{du}\right)^{\mathfrak{N}} (I_{0+}^{\mathfrak{N}-\gamma} w)(u) = \frac{1}{\Gamma(\mathfrak{N} - \gamma)} \left(\frac{d}{du}\right)^{\mathfrak{N}} \int_0^u \frac{w(\tau)}{(u - \tau)^{\gamma - \mathfrak{N} + 1}} d\tau, \quad (\mathfrak{N} = [\gamma] + 1, u > 0),$$

where  $[\gamma]$  means the integer part of  $\gamma$ .

**Definition 2.3.** Let  $\varphi : [0, \infty) \rightarrow \mathfrak{R}$  be a function.  $D_q \varphi$  is called  $q$ -derivative, and it is defined as follows:

$$(D_q \varphi)(u) = \frac{\varphi(u) - \varphi(qu)}{u - qu}, \quad u \neq 0, q \in \mathfrak{R}^+ \setminus 1.$$

**Definition 2.4.** Let  $\varphi : [0, \infty) \rightarrow \mathfrak{R}$  be a function.  $I_{q,0} \varphi$  is called  $q$ -integral, and it is defined below:

$$(I_{q,0} \varphi)(u) = \int_0^u \varphi(\tau) d_q(\tau) = u(1 - q) \sum_{k=0}^{\infty} \varphi(uq^k) q^k, \quad 0 < |q| < 1, u \in \mathfrak{R}^+.$$

Also, we have:

$$\begin{aligned} (I_{q,a} \varphi)(u) &= \int_a^u \varphi(\tau) d_q(\tau) = \int_0^u \varphi(\tau) d_q(\tau) - \int_0^a \varphi(\tau) d_q \tau, \quad a \in \mathfrak{R}^+, \\ (D_q I_{q,a} \varphi)(u) &= \varphi(u), \quad a \in \mathfrak{R}^+, \\ (I_{q,a} D_q \varphi)(u) &= \varphi(u) - \varphi(a), \quad a \in \mathfrak{R}^+. \end{aligned} \tag{2.1}$$

The following Lemmas and Conclusion are extracted from references [2], [8] and [5].

**Lemma 2.5.** For  $\alpha, \beta \in \mathfrak{R}^+$ , relation below is established:

$$(I_{q,a}^\beta I_{q,a}^\alpha \varphi)(u) = (I_{q,a}^{\alpha+\beta} \varphi)(u),$$

**Lemma 2.6.** For  $\alpha \in \mathfrak{R}$  and  $\lambda \in (-1, \infty)$ , the following property is valid:

$$I_{q,a}^\alpha ((u - a)^\lambda) = \frac{\Gamma_q(\lambda + 1)}{\Gamma_q(\alpha + \lambda + 1)} (u - a)^{\alpha+\lambda}, \quad (0 < a < u < b).$$

**Lemma 2.7.** If  $v \in L^1[0, 1]$ ,  $\eta > 0$  and  $\kappa > 0$ , then

$$\begin{aligned} I_{0+}^\kappa I_{0+}^\eta v(\tau) &= I_{0+}^{\kappa+\eta} v(\tau), \\ D_{0+}^\eta I_{0+}^\eta v(\tau) &= v(\tau). \end{aligned}$$

**Lemma 2.8.** If  $\nu > -1$ ,  $\alpha > 0$  and  $\tau > 0$  thus

$$D_{0+}^{-\nu} \tau^\alpha = \frac{\Gamma(1 + \nu)}{\Gamma(1 + \nu + \alpha)} \tau^{\nu+\alpha}.$$

**Remark 2.9.** If  $v$  has a RLFD of order  $\alpha > 0$  and  $v \in \mathfrak{C}(0, 1) \cap \mathfrak{L}(0, 1)$ , so

$$I_{0+}^\alpha D_{0+}^\alpha v(\tau) = v(\tau) - \sum_{j=1}^n \mathfrak{C}_j (\tau - a)^{\alpha-j},$$

for any  $\mathfrak{C}_j \in \mathfrak{R}$  where  $\mathfrak{N} = [\alpha] + 1$ .



**Lemma 2.10.** *Let  $w, v \in \mathfrak{C}(0, 1) \cap \mathfrak{L}(0, 1)$ ,  $D_q D^\gamma w \in \mathfrak{L}^1[0, 1]$ ,  $I_{q,0}v \in \mathfrak{L}^1[0, 1]$ ,  $0 < q < 1$  and  $w(0) = 0$  be given. Thus, the problem  $-D_q D_{0+}^\gamma \left( \frac{w(\tau) + 3e_1(\tau)}{E_q(\lambda\tau)} \right) = v(\tau)$  has a general solution as follows:*

$$w(\tau) = E_q(\lambda\tau) \left( \sum_{j=1}^{\mathfrak{N}} \mathfrak{C}_j \tau^{\gamma-j} - I_{0+}^\gamma I_{q,0}v(\tau) \right) - 3e_1(\tau),$$

where  $\mathfrak{C}_j \in \mathfrak{R}(j = 1, 2, \dots, n)$ ,  $\mathfrak{N} = [\gamma]$ .

*Proof.* First, we take the  $q$ -integral from the sides of the equation and we get

$$-I_{q,0} D_q D_{0+}^\gamma \left( \frac{w(\tau) + 3e_1(\tau)}{E_q(\lambda\tau)} \right) = I_{q,0}v(\tau).$$

By using of (2.1), we attain  $-D_{0+}^\gamma \left( \frac{w(\tau) + 3e_1(\tau)}{E_q(\lambda\tau)} \right) = I_{q,0}v(\tau)$ . Now we take the  $I_{0+}^\gamma$  from the sides, therefore,

$$-I_{0+}^\gamma D_{0+}^\gamma \left( \frac{w(\tau) + 3e_1(\tau)}{E_q(\lambda\tau)} \right) = I_{0+}^\gamma I_{q,0}v(\tau).$$

By using 2.9, we obtain

$$\sum_{j=1}^{\mathfrak{N}} \mathfrak{C}_j \tau^{\gamma-j} - \left( \frac{w(\tau) + 3e_1(\tau)}{E_q(\lambda\tau)} \right) = I_{0+}^\gamma I_{q,0}v(\tau).$$

So, we have:

$$w(\tau) = E_q(\lambda\tau) \left( \sum_{j=1}^{\mathfrak{N}} \mathfrak{C}_j \tau^{\gamma-j} - I_{0+}^\gamma I_{q,0}v(\tau) \right) - 3e_1(\tau).$$

□

Now we state a theorem in which we earn the Green function of the solution to the problem.

**Theorem 2.11.** *Assume  $v \in \mathfrak{C}[0, 1] \cap \mathfrak{L}^1[0, 1]$ ,  $D_q D^\gamma w \in \mathfrak{L}^1[0, 1]$ ,  $I_{q,0}v \in \mathfrak{L}^1[0, 1]$  and  $0 < q < 1$  be given. Thus, the hybrid  $q$ -FBVP*

$$\begin{aligned} -D_q D_{0+}^\gamma \left( \frac{w(\tau) + 3e_1(\tau)}{E_q(\lambda\tau)} \right) &= v(\tau), \quad 2 \leq \gamma < 3, \quad 0 < q < 1, \\ w(0) = w'(0) = e_1(0) = e_1'(0) &= 0, \\ \left[ I_{0+}^\alpha \left( \frac{w(\tau) + 3e_1(\tau)}{E_q(\lambda\tau)} \right) \right]_{\tau=1} &= 0, \quad 1 \leq \alpha \leq 2, \end{aligned}$$

has a  $q$ -EPS as follows

$$w(\tau) = E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) Y(\varsigma) d\varsigma - 3e_1(\tau) = E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0}v(\varsigma) d\varsigma - 3e_1(\tau), \quad (2.2)$$

where  $G(\tau, \varsigma)$  a Green function is as below

$$\mathfrak{G}(\tau, \varsigma) = \frac{1}{\Gamma(\gamma)} \begin{cases} \tau^{\gamma-1}(1-\varsigma)^{\gamma+\alpha-1} + (\tau-\varsigma)^{\gamma-1}, & 0 \leq \varsigma \leq \tau \leq 1, \\ \tau^{\gamma-1}(1-\varsigma)^{\gamma+\alpha-1}, & 0 \leq \tau \leq \varsigma \leq 1. \end{cases} \quad (2.3)$$

*Proof.* According to 2.10, the problem has a general solution as follows:

$$w(\tau) = E_q(\lambda\tau) (\mathfrak{C}_1 \tau^{\gamma-1} + \mathfrak{C}_2 \tau^{\gamma-2} + \mathfrak{C}_3 \tau^{\gamma-3} - I_{0+}^\gamma I_{q,0}v(\tau)) - 3e_1(\tau).$$

Using the first boundary condition, we have that  $\mathfrak{C}_2 = \mathfrak{C}_3 = 0$ . Hence,

$$w(\tau) = E_q(\lambda\tau) (\mathfrak{C}_1 \tau^{\gamma-1} - I_{0+}^\gamma I_{q,0}v(\tau)) - 3e_1(\tau).$$



Also, using the second boundary condition, we get the constant  $\mathfrak{C}_1$ :

$$I_{0+}^{\alpha} \left( \frac{w(\tau) + 3e_1(\tau)}{E_q(\lambda\tau)} \right) = \mathfrak{C}_1 I_{0+}^{\alpha} \tau^{\gamma-1} - I_{0+}^{\alpha} I_{0+}^{\gamma} I_{q,0} v(\tau).$$

By using of 2.5, 2.8, and 2.2 we obtain:

$$I_{0+}^{\alpha} \left( \frac{w(\tau) + 3e_1(\tau)}{E_q(\lambda\tau)} \right) = \mathfrak{C}_1 \frac{\Gamma(\gamma)}{\Gamma(\alpha + \gamma)} t^{\alpha+\gamma-1} - \frac{1}{\Gamma(\alpha + \gamma)} \int_0^{\tau} (\tau - \varsigma)^{\gamma+\alpha-1} I_{q,0} v(\varsigma) d\varsigma.$$

By putting  $\tau = 1$ , we get:

$$0 = \mathfrak{C}_1 \frac{\Gamma(\gamma)}{\Gamma(\alpha + \gamma)} - \frac{1}{\Gamma(\alpha + \gamma)} \int_0^1 (1 - \varsigma)^{\gamma+\alpha-1} I_{q,0} v(\varsigma) d\varsigma,$$

we assume that  $I_{q,0} v(\varsigma) = Y(\varsigma)$ . Therefore,  $\mathfrak{C}_1 = \frac{1}{\Gamma(\gamma)} \int_0^1 (1 - \varsigma)^{\gamma+\alpha-1} Y(\varsigma) d\varsigma$ . By placing  $\mathfrak{C}_1$  and  $\mathfrak{C}_2 = \mathfrak{C}_3 = 0$  in  $w(\tau)$ , we have

$$\begin{aligned} w(\tau) &= E_q(\lambda\tau) \left( \frac{\tau^{\gamma-1}}{\Gamma(\gamma)} \int_0^1 (1 - \varsigma)^{\gamma+\alpha-1} Y(\varsigma) d\varsigma - \frac{1}{\Gamma(\gamma)} \int_0^{\tau} (\tau - \varsigma)^{\gamma-1} Y(\varsigma) d\varsigma \right) - 3e_1(\tau) \\ &= E_q(\lambda\tau) \left( \frac{1}{\Gamma(\gamma)} \int_0^{\tau} (\tau^{\gamma-1} (1 - \varsigma)^{\gamma+\alpha-1} - (\tau - \varsigma)^{\gamma-1}) Y(\varsigma) d\varsigma \right. \\ &\quad \left. + \frac{1}{\Gamma(\gamma)} \int_{\tau}^1 \tau^{\gamma-1} (1 - \varsigma)^{\gamma+\alpha-1} Y(\varsigma) d\varsigma \right) - 3e_1(\tau). \end{aligned}$$

Therefore, the Green function is as below:

$$\mathfrak{G}(\tau, \varsigma) = \frac{1}{\Gamma(\gamma)} \begin{cases} \tau^{\gamma-1} (1 - \varsigma)^{\gamma+\alpha-1} + (\tau - \varsigma)^{\gamma-1}, & 0 \leq \varsigma \leq \tau \leq 1, \\ \tau^{\gamma-1} (1 - \varsigma)^{\gamma+\alpha-1}, & 0 \leq \tau \leq \varsigma \leq 1. \end{cases}$$

So, we obtain that

$$w(\tau) = E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) Y(\varsigma) d\varsigma - 3e_1(\tau).$$

□

In the following Lemma, the characteristics of the Green function in (2.3) were stated and proved that we need in the main part.

**Lemma 2.12.** *The Green function defined in Theorem 2.11 has the following properties:*

- (1) for every  $\tau, \varsigma \in [0, 1]$  then,  $\mathfrak{G}(\tau, \varsigma) \geq 0$ ,
- (2) for any  $\tau, \varsigma \in [0, 1]$  so,

$$\tau^{\gamma-1} [(1 - \varsigma)^{\gamma+\alpha-1} - (1 - \varsigma)^{\gamma-1}] \leq \Gamma(\gamma) \mathfrak{G}(\tau, \varsigma) \leq \tau^{\gamma-1} (1 - \varsigma)^{\gamma+\alpha-1}.$$

*Proof.* (1). It can be concluded  $\mathfrak{G}(0, 0) = 0$ . Also, for any constant  $\varsigma$ , the Green function  $\mathfrak{G}$  is increasing with respect to  $\tau$ . That is, for  $0 \leq \tau \leq \varsigma \leq 1$

$$\frac{\partial \mathfrak{G}(\tau, \varsigma)}{\partial \tau} = \frac{(\gamma - 1) \tau^{\gamma-2} (1 - \varsigma)^{\gamma+\alpha-1}}{\Gamma(\gamma)} \geq 0,$$

and for  $0 \leq \varsigma \leq \tau \leq 1$ , we have:

$$\frac{\partial \mathfrak{G}(\tau, \varsigma)}{\partial \tau} = \frac{(\gamma - 1) \tau^{\gamma-2} (1 - \varsigma)^{\gamma+\alpha-1} - (\gamma - 1) \tau^{\gamma-2} (1 - \frac{\varsigma}{\tau})^{\gamma-2}}{\Gamma(\gamma)} \geq 0.$$



That's mean,  $\mathfrak{G}(\tau, \varsigma) \geq 0$ . The verdict of the first part has been proven.

(2). It is obvious that  $\Gamma(\gamma)\mathfrak{G}(\tau, \varsigma) \leq \tau^{\gamma-1}(1-\varsigma)^{\gamma+\alpha-1}$ . First, we prove the left side of the inequality, then

$$0 \leq \tau - \varsigma \leq \tau - \tau\varsigma = \tau(1 - \varsigma),$$

so, we get  $(\tau - \varsigma)^{\gamma-1} \leq \tau^{\gamma-1}(1 - \varsigma)^{\gamma-1}$ . Therefore,

$$\begin{aligned} \Gamma(\gamma)\mathfrak{G}(\tau, \varsigma) &= \tau^{\gamma-1}(1 - \varsigma)^{\gamma+\alpha-1} - (\tau - \varsigma)^{\gamma-1} \\ &\geq \tau^{\gamma-1}(1 - \varsigma)^{\gamma+\alpha-1} - \tau^{\gamma-1}(1 - \varsigma)^{\gamma-1} \\ &= \tau^{\gamma-1}[(1 - \varsigma)^{\gamma+\alpha-1} - (1 - \varsigma)^{\gamma-1}], \end{aligned}$$

if  $0 \leq \tau \leq \varsigma \leq 1$  thus, we obtain:

$$\begin{aligned} \Gamma(\gamma)\mathfrak{G}(\tau, \varsigma) &= \tau^{\gamma-1}(1 - \varsigma)^{\gamma+\alpha-1} \\ &\geq \tau^{\gamma-1}[(1 - \varsigma)^{\gamma+\alpha-1} - (1 - \varsigma)^{\gamma-1}]. \end{aligned}$$

□

### 3. MAIN RESULTS

In this section, we investigate EU of the q-EPS for the hybrid q-FBVP (1.1) using of the q-EFPT of MMO in cone  $\rho_{h, e_1}$ . Banach space  $\mathbf{E}$  can be kit out with a partially order arranged with  $u, v \in \mathfrak{C}[0, 1]$ ,  $u \leq v \Leftrightarrow u(\tau) \leq v(\tau)$  for  $\tau \in [0, 1]$ .  $\forall u, v \in \mathbf{E}$ , the marking an equivalence relation  $u \sim v$  that is, there exists  $\mu_1 > 0$  and  $\mu_2 > 0$ , therefore  $\mu_1 u \leq v \leq \mu_2 u$ . If  $h \in \mathbf{E}$  and  $h > 0$  are given, we define the  $\rho_h = \{u \in \mathbf{E} \mid \exists \mu_1 \leq \mu_2 \text{ and } \mu_1 h \leq u \leq \mu_2 h\}$ . It is obvious that  $\rho_h \subseteq \rho$  for any  $h \in \rho$ . We consider  $\rho$  as a normal cone with a normality constant of 1. Let  $e_1 \in \rho$ ,  $0 \leq e_1 \leq h$  be given and  $e_1$  is differentiable. Then set  $\rho_{h, e_1}$  is expressed as below:

$$\begin{aligned} \rho_{h, e_1} &= \{u \in \mathbf{E} \mid \exists \mu_3 \text{ and } \mu_4, \mu_3 h \leq u + e_1 \leq \mu_4 h\} \\ &= \{u \in \mathbf{E} \mid u + e_1 \in \mathbb{P}_h\} \\ &= \{u \in \mathbf{E} \mid \exists \mu_3 \text{ and } \mu_4, \mu_3 h + (\mu_3 - 1)e_1 \leq u \leq \mu_4 h + (\mu_4 - 1)e_1\} \end{aligned}$$

It is clear that  $\rho_h \subseteq \rho_{h, e_1}$  for any  $h \in \rho_{h, e_1}$ .

Definitions 3.1 and 3.2 are extracted from [4] and [12] respectively.

**Definition 3.1.** An operator  $\mathcal{A} : \rho_{h, e_1} \times \rho_{h, e_1} \rightarrow \rho_{h, e_1}$  is called a MMO if  $\mathcal{A}(w, \vartheta)$  is decreasing relative to  $\vartheta$  and increasing relative to  $w$ , means that,  $w_i, \vartheta_i \in \rho_{h, e_1}$  ( $i = 1, 2$ ),  $w_1 \leq w_2$ ,  $\vartheta_1 \geq \vartheta_2$  imply

$$\mathcal{A}(w_1, \vartheta_1) \leq \mathcal{A}(w_2, \vartheta_2),$$

the element  $u \in \rho_{h, e_1}$  is called a fixed point of  $\mathcal{A}$  if  $\mathcal{A}(u, u) = u$ .

**Definition 3.2.** If  $\mathcal{B} : \rho_{h, e_1} \rightarrow \mathbf{E}$  is an operator. For each  $u \in \rho_{h, e_1}$ , there is a function  $\varphi : [0, 1] \rightarrow [0, 1]$  such that for any  $\lambda \in (0, 1)$ ,  $\varphi(\lambda) > \lambda$ ,

$$\mathcal{B}(\lambda u + (\lambda - 1)e_1) \geq \varphi(\lambda)\mathcal{B}(u) + (\varphi(\lambda) - 1)e_1.$$

Therefore  $\mathcal{B}$  is named a generalized  $\varphi$ -( $h, e_1$ )-concave.

**Definition 3.3.** If  $T$  is MMO in cone  $\rho_{h, e_1}$ , the element  $u \in \rho_{h, e_1}$  is called a q-Exponential fixed point (q-EFP) of  $T$  if  $E_q(\lambda\tau)T(u, u) = u$ .

The following Lemmas are derived from [4].

**Lemma 3.4.** If  $u \in \rho_{h, e_1}$ , then for  $\lambda > 0$ ,

$$\lambda u + (\lambda - 1)e_1 \in \rho_{h, e_1}.$$



**Lemma 3.5.** *If  $u, v \in \rho_{h, e_1}$ , thus for  $0 < \mu < 1, \nu > 1$  such that (S.T):*

$$\mu v + (\mu - 1)e_1 \leq v \leq \nu v + (\nu - 1)e_1.$$

*So, we can elect a tiny  $\epsilon \in (0, 1)$  S.T*

$$\epsilon v + (\epsilon - 1)e_1 \leq u \leq \epsilon^{-1}v + (\epsilon^{-1} - 1)e_1.$$

Now, we state and prove the main Lemma and Theorems. This Lemma and Theorems are specific to conical space of  $\rho_{h, e_1}$ , and this theorem is called the q-EFPT of MMO on cone  $\rho_{h, e_1}$ .

**Lemma 3.6.** *We assume that  $T : \rho \times \rho \rightarrow \rho$  is a MMO, which we consider as a normal cone  $\rho$  in the real Banach space  $\mathbf{E}$ . There is a function  $\iota : [0, 1] \rightarrow [0, 1]$  S.T for any  $\tau, \iota(\tau) \geq \tau$ :*

$$T(\tau w + (\tau - 1)e_1, \tau^{-1}\vartheta + (\tau^{-1} - 1)e_1) \geq \iota(\tau)T(w, \vartheta) + (\iota(\tau) - 1)e_1, \quad w, \vartheta \in \mathbb{P}_{h, e_1}. \tag{3.1}$$

(J) *For any  $h_0 \in \rho_{h, e_1}$  that  $h_0 \neq 0, T(h_0, h_0) \in \rho_{h, e_1}$ .*

*Thus,*

- (1)  $T : \rho_{h, e_1} \times \rho_{h, e_1} \rightarrow \rho_{h, e_1}$ ;
- (2) *There is  $w_0, \vartheta_0 \in \rho_{h, e_1}$  and  $\epsilon \in (0, 1)$  S.T  $\epsilon\vartheta_0 \leq w_0 < E_q(\lambda\tau)\vartheta_0$ ,*  
 $w_0 \leq E_q(\lambda\tau)T(w_0, \vartheta_0) \leq E_q(\lambda\tau)T(\vartheta_0, w_0) \leq E_q(\lambda\tau)\vartheta_0$
- (3)  $T$  has a q-EFP  $w^*$  in  $\rho_{h, e_1}$ ;
- (4) *For each initial values  $w_0, \vartheta_0 \in \rho_{h, e_1}$ , constructing successively the sequences*

$$w_{\mathfrak{N}} = E_q(\lambda\tau)T(w_{\mathfrak{N}-1}, \vartheta_{\mathfrak{N}-1}), \quad \mathfrak{N} = 1, 2, \dots,$$

$$\vartheta_{\mathfrak{N}} = E_q(\lambda\tau)T(\vartheta_{\mathfrak{N}-1}, w_{\mathfrak{N}-1}), \quad \mathfrak{N} = 1, 2, \dots$$

*in result  $w_{\mathfrak{N}} \rightarrow w^*$  and  $\vartheta_{\mathfrak{N}} \rightarrow \vartheta^*$  as  $\mathfrak{N} \rightarrow \infty$ .*

*Proof.* According to the (3.1), we achieve:

$$T(\tau^{-1}h_0 + (\tau^{-1} - 1)e_1, \tau h_0 + (\tau - 1)e_1) \geq \iota^{-1}(\tau)T(h_0, h_0) + (\iota^{-1}(\tau) - 1)e_1. \tag{3.2}$$

Because  $h_0 \in \rho_{h, e_1}$ , there is a  $\eta \in (0, 1)$  S.T,

$$\eta h + (\eta - 1)e_1 \leq h_0 \leq \eta^{-1}h + (\eta^{-1} - 1)e_1. \tag{3.3}$$

According to the property of mixed monotone  $T$  and relation (3.1), (3.2) and (3.3) we get:

$$\begin{aligned} T(h, h) &\geq T(\eta h_0 + (\eta - 1)e_1, \eta^{-1}h_0 + (\eta^{-1} - 1)e_1) \\ &\geq \iota(\eta)T(h_0, h_0) + (\iota(\eta) - 1)e_1, \end{aligned}$$

$$\begin{aligned} T(h, h) &\leq T(\eta^{-1}h_0 + (\eta^{-1} - 1)e_1, \eta h_0 + (\eta - 1)e_1) \\ &\leq \iota^{-1}(\eta)T(h_0, h_0) + (\iota^{-1}(\eta) - 1)e_1. \end{aligned}$$

Since  $T(h_0, h_0) \in \rho_{h, e_1}$ , we conclude that  $T(h, h) \in \rho_{h, e_1}$ .

Therefore,  $T : \rho_{h, e_1} \times \rho_{h, e_1} \rightarrow \rho_{h, e_1}$ . Because  $T(h, h) \in \rho_{h, e_1}$ , we consider a number like  $\tau_0 \in (0, 1)$  S.T

$$\tau_0 h + (\tau_0 - 1)e_1 \leq T(h, h) \leq \tau_0^{-1}h + (\tau_0^{-1} - 1)e_1. \tag{3.4}$$

Since  $\tau_0 < \iota(\tau_0) \leq 1$  therefore,  $\frac{\iota(\tau_0)}{\tau_0} \leq \frac{1}{\tau_0}$ . We can find positive number like  $m$  that  $(\frac{\iota(\tau_0)}{\tau_0})^m \geq \frac{1}{\tau_0}$  then,

$$\iota^m(\tau_0) \geq \tau_0^{m-1}. \tag{3.5}$$

We put

$$w_0 = \tau_0^m h + (\tau_0^m - 1)e_1,$$

$$\vartheta_0 = \tau_0^{-m} h + (\tau_0^{-m} - 1)e_1.$$



It is clear that,  $w_0, \vartheta_0 \in \rho_{h, e_1}$  and

$$\begin{aligned} w_0 &= \tau_0^m h + (\tau_0^m - 1)e_1 < \tau_0^{-m} h + (\tau_0^{-m} - 1)e_1 \\ &= \vartheta_0 < E_q(\lambda\tau)\vartheta_0, \end{aligned}$$

as a result,  $w_0 < E_q(\lambda\tau)\vartheta_0$ . It is enough that  $\mathbf{E} = \max E_q(\lambda\tau)$ ,  $\tau \in (0, 1)$  and for large natural values  $\mathfrak{N}$ , we can be considered  $\mathfrak{N}$  that  $\epsilon = \frac{\mathbf{E}}{\mathfrak{N}} < 1$ , consequently,  $\epsilon\vartheta_0 \leq w_0$ . Since  $\mathbf{T}$  is a MMO, we get that  $\mathbf{T}(w_0, \vartheta_0) \leq \mathbf{T}(\vartheta_0, w_0)$ , by multiplying  $E_q(\lambda\tau)$  on the sides inequality, we obtain

$$E_q(\lambda\tau)\mathbf{T}(w_0, \vartheta_0) \leq E_q(\lambda\tau)\mathbf{T}(\vartheta_0, w_0).$$

Using the relation (3.1), (3.4) and (3.5) we have:

$$\begin{aligned} E_q(\lambda\tau)\mathbf{T}(w_0, \vartheta_0) &= E_q(\lambda\tau)\mathbf{T}(\tau_0^m h + (\tau_0^m - 1)e_1, \tau_0^{-m} h + (\tau_0^{-m} - 1)e_1) \\ &= E_q(\lambda\tau)\mathbf{T}\{\tau_0[\tau_0^{m-1} h + (\tau_0^{m-1} - 1)e_1] + (\tau_0 - 1)e_1, \tau_0^{-1}[\tau_0^{1-m} h + (\tau_0^{1-m} - 1)e_1] \\ &\quad + (\tau_0^{-1} - 1)e_1\} \\ &\geq \iota(\tau_0)\mathbf{T}(\tau_0^{m-1} h + (\tau_0^{m-1} - 1)e_1, \tau_0^{1-m} h + (\tau_0^{1-m} - 1)e_1) + (\iota(\tau_0) - 1)e_1 \\ &= \iota(\tau_0)\mathbf{T}\{\tau_0[\tau_0^{m-2} h + (\tau_0^{m-2} - 1)e_1] + (\tau_0 - 1)e_1, \tau_0^{-1}[\tau_0^{2-m} h + (\tau_0^{2-m} - 1)e_1] \\ &\quad + (\tau_0^{-1} - 1)e_1\} + (\iota(\tau_0) - 1)e_1 \\ &\geq \iota^2(\tau_0)\mathbf{T}(\tau_0^{m-2} h + (\tau_0^{m-2} - 1)e_1, \tau_0^{2-m} h + (\tau_0^{2-m} - 1)e_1) + (\iota^2(\tau_0) - 1)e_1 \\ &\vdots \\ &\geq \iota^{m-1}(\tau_0)\mathbf{T}(\tau_0 h + [\tau_0 - 1]e_1, \tau_0^{-1} h + [\tau_0^{-1} - 1]e_1) + (\iota^{m-1}(\tau_0) - 1)e_1 \\ &\geq \iota^m(\tau_0)\mathbf{T}(h, h) + (\iota^m(\tau_0) - 1)e_1 \\ &\geq \iota^m(\tau_0)[\tau_0 h + (\tau_0 - 1)e_1] + (\iota^m(\tau_0) - 1)e_1 \\ &\geq \tau_0^{m-1}[\tau_0 h + (\tau_0 - 1)e_1] + (\tau_0^{m-1} - 1)e_1 \\ &= \tau_0^m h + \tau_0^m e_1 - \tau_0^{m-1} e_1 + \tau_0^{m-1} e_1 - e_1 \\ &= \tau_0^m h + (\tau_0^m - 1)e_1 = w_0. \end{aligned}$$

That is  $E_q(\lambda\tau)\mathbf{T}(w_0, \vartheta_0) \geq w_0$ . By using the relation (3.2), (3.4) and (3.5) we obtain:

$$\begin{aligned} E_q(\lambda\tau)\mathbf{T}(\vartheta_0, w_0) &= E_q(\lambda\tau)\mathbf{T}(\tau_0^{-m} h + [\tau_0^{-m} - 1]e_1, \tau_0^m h + [\tau_0^m - 1]e_1) \\ &= E_q(\lambda\tau)\mathbf{T}\{\tau_0^{-1}[\tau_0^{1-m} h + (\tau_0^{1-m} - 1)e_1] + (\tau_0^{-1} - 1)e_1, \tau_0[\tau_0^{m-1} h + (\tau_0^{m-1} - 1)e_1] \\ &\quad + (\tau_0 - 1)e_1\} \\ &\leq E_q(\lambda\tau)[\iota^{-1}(\tau_0)\mathbf{T}(\tau_0^{1-m} h + (\tau_0^{1-m} - 1)e_1, \tau_0^{m-1} h + (\tau_0^{m-1} - 1)e_1) + (\iota^{-1}(\tau_0) - 1)e_1] \\ &= E_q(\lambda\tau)[\iota^{-1}(\tau_0)\mathbf{T}(\tau_0^{-1}\{\tau_0^{2-m} h + (\tau_0^{2-m} - 1)e_1\} + (\tau_0^{-1} - 1)e_1, \tau_0\{\tau_0^{m-2} h + (\tau_0^{m-2} - 1)e_1\} \\ &\quad + (\tau_0 - 1)e_1) + (\iota^{-1}(\tau_0) - 1)e_1] \\ &\leq E_q(\lambda\tau)[\iota^{-2}(\tau_0)\mathbf{T}(\tau_0^{2-m} h + (\tau_0^{2-m} - 1)e_1, \tau_0^{m-2} h + \{\tau_0^{m-2} - 1\}e_1) + (\iota^{-2}(\tau_0) - 1)e_1] \\ &\vdots \\ &\leq E_q(\lambda\tau)[\iota^{-m+1}(\tau_0)\mathbf{T}(\tau_0^{-1} h + (\tau_0^{-1} - 1)e_1, \tau_0 h + (\tau_0 - 1)e_1) + (\iota^{-m+1}(\tau_0) - 1)e_1] \end{aligned}$$





$$\begin{aligned}
 &\leq E_q(\lambda\tau) [\iota^{-m}(\tau_0)\mathbf{T}(h, h) + (\iota^{-m}(\tau_0) - 1)e_1] \\
 &\leq E_q(\lambda\tau) [\iota^{-m}(\tau_0)(\tau_0^{-1}h + (\tau_0^{-1} - 1)e_1) + (\iota^{-m}(\tau_0) - 1)e_1] \\
 &\leq E_q(\lambda\tau) [\tau_0^{-m+1}(\tau_0^{-1}h + (\tau_0^{-1} - 1)e_1) + (\tau_0^{-m+1} - 1)e_1] \\
 &= E_q(\lambda\tau) [\tau_0^{-m}h + (\tau_0^{-m} - 1)e_1] \\
 &= E_q(\lambda\tau)\vartheta_0.
 \end{aligned}$$

That is  $E_q(\lambda\tau)\mathbf{T}(w_0, \vartheta_0) \leq E_q(\lambda\tau)\vartheta_0$ . Then, we prove that

$$w_0 \leq E_q(\lambda\tau)\mathbf{T}(w_0, \vartheta_0) \leq E_q(\lambda\tau)\mathbf{T}(\vartheta_0, w_0) \leq E_q(\lambda\tau)\vartheta_0. \tag{3.6}$$

By constructing recursive sequences

$$\begin{aligned}
 w_{\mathfrak{N}} &= E_q(\lambda\tau)\mathbf{T}(w_{\mathfrak{N}-1}, \vartheta_{\mathfrak{N}-1}), \\
 \vartheta_{\mathfrak{N}} &= E_q(\lambda\tau)\mathbf{T}(\vartheta_{\mathfrak{N}-1}, w_{\mathfrak{N}-1}), \quad \mathfrak{N} = 1, 2, \dots
 \end{aligned}$$

By using the relation (3.6),  $w_1 = E_q(\lambda\tau)\mathbf{T}(w_0, \vartheta_0) \leq E_q(\lambda\tau)\mathbf{T}(\vartheta_0, w_0) = \vartheta_1$ , in general, we can take  $w_{\mathfrak{N}} < \vartheta_{\mathfrak{N}}$ , ( $\mathfrak{N} = 1, 2, \dots$ ), therefore, we get:

$$w_0 \leq w_1 \leq \dots \leq w_{\mathfrak{N}} \leq \dots \leq \vartheta_{\mathfrak{N}} \leq \dots \leq \vartheta_1 \leq E_q(\lambda\tau)\vartheta_0. \tag{3.7}$$

Since  $w_0, \vartheta_0 \in \rho_{h, e_1}$ , using 3.5, we choose a  $\mu > 0$ , S.T  $w_0 \geq \mu(E_q(\lambda\tau)\vartheta_0) + (\mu - 1)e_1$ , by using the relation (3.7), we obtain:

$$w_{\mathfrak{N}} \geq w_0 \geq \mu(E_q(\lambda\tau)\vartheta_0) + (\mu - 1)e_1 \geq \mu\vartheta_1 + (\mu - 1)e_1, \quad \mathfrak{N} = 1, 2, \dots$$

Suppose  $\tau_{\mathfrak{N}} = \sup\{\tau > 0 | w_{\mathfrak{N}} \geq \tau\vartheta_{\mathfrak{N}} + (\tau - 1)e_1\}$ . We take from (3.7)

$$\tau_{\mathfrak{N}} \in (0, 1), \quad w_{\mathfrak{N}} \geq \tau_{\mathfrak{N}}\vartheta_{\mathfrak{N}} + (\tau_{\mathfrak{N}} - 1)e_1, \tag{3.8}$$

so, by using (3.7) and (3.8), we have:

$$w_{\mathfrak{N}+1} \geq w_{\mathfrak{N}} \geq \tau_{\mathfrak{N}}\vartheta_{\mathfrak{N}} + (\tau_{\mathfrak{N}} - 1)e_1 \geq \tau_{\mathfrak{N}}\vartheta_{\mathfrak{N}+1} + (\tau_{\mathfrak{N}} - 1)e_1, \quad \mathfrak{N} = 1, 2, \dots$$

In result of  $\tau_{\mathfrak{N}+1} \geq \tau_{\mathfrak{N}}$  that is the sequence  $\{\tau_{\mathfrak{N}}\}$  is an increasing. Suppose  $\lim_{\mathfrak{N} \rightarrow \infty} \tau_{\mathfrak{N}} = \tau^*$ . We show that  $\tau^* = 1$ .

Differently,  $0 < \tau^* < 1$ . We examine two cases.

Case 1:  $\exists N_0 \in N$ , S.T  $\tau_N = \tau^*$ . For all  $\mathfrak{N} > N_0$  we get  $\tau_{\mathfrak{N}} = \tau^*$ . Thus, for each  $\mathfrak{N} \geq N_0$

$$\begin{aligned}
 w_{\mathfrak{N}+1} &= E_q(\lambda\tau)\mathbf{T}(w_{\mathfrak{N}}, \vartheta_{\mathfrak{N}}) \geq E_q(\lambda\tau)\mathbf{T}[\tau_{\mathfrak{N}}\vartheta_{\mathfrak{N}} + (\tau_{\mathfrak{N}} - 1)e_1, \tau_{\mathfrak{N}}^{-1}w_{\mathfrak{N}} + (\tau_{\mathfrak{N}}^{-1} - 1)e_1] \\
 &= E_q(\lambda\tau)\mathbf{T}[\tau^*\vartheta_{\mathfrak{N}} + (\tau^* - 1)e_1, (\tau^*)^{-1}w_{\mathfrak{N}} + ((\tau^*)^{-1} - 1)e_1] \\
 &\geq E_q(\lambda\tau) [\iota(\tau^*)\mathbf{T}(\vartheta_{\mathfrak{N}}, w_{\mathfrak{N}}) + (\iota(\tau^*) - 1)e_1], \\
 &= E_q(\lambda\tau)\iota(\tau^*)\vartheta_{\mathfrak{N}+1} + E_q(\lambda\tau)(\iota(\tau^*) - 1)e_1 \\
 &\geq \iota(\tau^*)\vartheta_{\mathfrak{N}+1} + (\iota(\tau^*) - 1)e_1.
 \end{aligned}$$

According to the increasing sequence  $\{\tau_{\mathfrak{N}}\}$ , we have  $\tau_{\mathfrak{N}+1} \geq \iota(\tau^*) > \tau^*$  by We take limits from both sides as  $\mathfrak{N} \rightarrow \infty$ , we obtain  $\tau^* \geq \iota(\tau^*) > \tau^*$ , which is contrary to our assumption.



Case 2:  $\forall \mathfrak{N} \in N$ ,  $\tau_{\mathfrak{N}} < \tau^*$ . So, we get:

$$\begin{aligned}
w_{\mathfrak{N}+1} &= E_q(\lambda\tau)\mathbf{T}(w_{\mathfrak{N}}, \vartheta_{\mathfrak{N}}) \geq E_q(\lambda\tau)\mathbf{T}(\tau_{\mathfrak{N}}\vartheta_{\mathfrak{N}} + (\tau_{\mathfrak{N}} - 1)e_1, \tau_{\mathfrak{N}}^{-1}w_{\mathfrak{N}} + (\tau_{\mathfrak{N}}^{-1} - 1)e_1) \\
&= E_q(\lambda\tau)\mathbf{T}\left[\frac{\tau_{\mathfrak{N}}}{\tau^*}\{\tau^*\vartheta_{\mathfrak{N}} + (\tau^* - 1)e_1\} + \left(\frac{\tau_{\mathfrak{N}}}{\tau^*} - 1\right)e_1, \left(\frac{\tau_{\mathfrak{N}}}{\tau^*}\right)^{-1}\{(\tau^*)^{-1}w_{\mathfrak{N}} + (\tau^*)^{-1} - 1\}e_1\right] + \left\{\left(\frac{\tau_{\mathfrak{N}}}{\tau^*}\right)^{-1} - 1\right\}e_1 \\
&\geq E_q(\lambda\tau)\left[\left(\frac{\tau_{\mathfrak{N}}}{\tau^*}\right)\mathbf{T}\{\tau^*\vartheta_{\mathfrak{N}} + (\tau^* - 1)e_1, (\tau^*)^{-1}w_{\mathfrak{N}} + ((\tau^*)^{-1} - 1)e_1\} + \left(\frac{\tau_{\mathfrak{N}}}{\tau^*} - 1\right)e_1\right] \\
&\geq E_q(\lambda\tau)\left[\left(\frac{\tau_{\mathfrak{N}}}{\tau^*}\right)(\iota(\tau^*)\mathbf{T}(w_{\mathfrak{N}}, \vartheta_{\mathfrak{N}}) + (\iota(\tau^*) - 1)e_1) + \left(\frac{\tau_{\mathfrak{N}}}{\tau^*} - 1\right)e_1\right] \\
&= E_q(\lambda\tau)\left[\left(\frac{\tau_{\mathfrak{N}}}{\tau^*}\right)\iota(\tau^*)\vartheta_{\mathfrak{N}+1} + E_q(\lambda\tau)(\iota(\tau^*) - 1)e_1 + E_q(\lambda\tau)\left(\frac{\tau_{\mathfrak{N}}}{\tau^*} - 1\right)e_1\right] \\
&= E_q(\lambda\tau)\left[\left(\frac{\tau_{\mathfrak{N}}}{\tau^*}\right)\iota(\tau^*)\vartheta_{\mathfrak{N}+1} + E_q(\lambda\tau)\left(\frac{\tau_{\mathfrak{N}}}{\tau^*}\iota(\tau^*) - 1\right)e_1\right].
\end{aligned}$$

By the sequence  $\{\tau_{\mathfrak{N}}\}$ , we have:

$$\tau_{\mathfrak{N}+1} \geq \iota\left(\frac{\tau_{\mathfrak{N}}}{\tau^*}\right)\iota(\tau^*) \geq \frac{\tau_{\mathfrak{N}}}{\tau^*}\iota(\tau^*).$$

We take limits from both sides as  $\mathfrak{N} \rightarrow \infty$ , we get:

$$\tau^* \geq \frac{\tau^*}{\tau^*}\iota(\tau^*) \geq \frac{\tau_{\mathfrak{N}}}{\tau^*}\iota(\tau^*),$$

which is against our hypothesis, in result of  $\tau^* = 1$ . For every  $\rho$  that is a natural number, we obtain:

$$\begin{aligned}
0 &\leq w_{\mathfrak{N}+\rho} - w_{\mathfrak{N}} \leq \vartheta_{\mathfrak{N}} - w_{\mathfrak{N}} \leq \vartheta_{\mathfrak{N}} - \tau_{\mathfrak{N}}\vartheta_{\mathfrak{N}} - (\tau_{\mathfrak{N}} - 1)e_1 \\
&= (1 - \tau_{\mathfrak{N}})\vartheta_{\mathfrak{N}} + (1 - \tau_{\mathfrak{N}})e_1 \leq (1 - \tau_{\mathfrak{N}})E_q(\lambda\tau)\vartheta_0 + (1 - \tau_{\mathfrak{N}})e_1.
\end{aligned}$$

Therefore

$$\begin{aligned}
0 &\leq \vartheta_{\mathfrak{N}} - \vartheta_{\mathfrak{N}+\rho} \leq \vartheta_{\mathfrak{N}} - w_{\mathfrak{N}} \leq \vartheta_{\mathfrak{N}} - \tau_{\mathfrak{N}}\vartheta_{\mathfrak{N}} - (\tau_{\mathfrak{N}} - 1)e_1 \\
&= (1 - \tau_{\mathfrak{N}})\vartheta_{\mathfrak{N}} + (1 - \tau_{\mathfrak{N}})e_1 \leq (1 - \tau_{\mathfrak{N}})E_q(\lambda\tau)\vartheta_0 + (1 - \tau_{\mathfrak{N}})e_1.
\end{aligned}$$

For constant normal  $N$  and  $\rho$  normal cone we get:

$$\begin{aligned}
\|w_{\mathfrak{N}+\rho} - w_{\mathfrak{N}}\| &\leq N(1 - \tau_{\mathfrak{N}})\|E_q(\lambda\tau)\vartheta_0 + e_1\| \rightarrow 0, \quad (as \mathfrak{N} \rightarrow \infty), \\
\|\vartheta_{\mathfrak{N}} - \vartheta_{\mathfrak{N}+\rho}\| &\leq N(1 - \tau_{\mathfrak{N}})\|E_q(\lambda\tau)\vartheta_0 + e_1\| \rightarrow 0, \quad (as \mathfrak{N} \rightarrow \infty),
\end{aligned}$$

then,  $\{w_{\mathfrak{N}}\}$  and  $\{\vartheta_{\mathfrak{N}}\}$  are Cauchy sequences. Since  $\mathbf{E}$  is complete, there exists  $w^*$  and  $\vartheta^*$  S.T  $w_{\mathfrak{N}} \rightarrow w^*$  and  $\vartheta_{\mathfrak{N}} \rightarrow \vartheta^*$  as  $\mathfrak{N} \rightarrow \infty$ . Using the relation (3.8) We know that:

$$w_0 \leq w_{\mathfrak{N}} \leq w^* \leq \vartheta^* \leq \vartheta_{\mathfrak{N}} \leq E_q(\lambda\tau)\vartheta_0.$$

Then,  $w^*, \vartheta^* \in \rho_{h, e_1}$  and

$$0 \leq \vartheta^* - w^* \leq \vartheta_{\mathfrak{N}} - w_{\mathfrak{N}} \leq (1 - \tau_{\mathfrak{N}})E_q(\lambda\tau)\vartheta_0 + e_1.$$

We conclude that:

$$\|\vartheta^* - w^*\| \leq N(1 - \tau_{\mathfrak{N}})\|E_q(\lambda\tau)\vartheta_0 + e_1\| \rightarrow 0, \quad (as \mathfrak{N} \rightarrow \infty),$$

that is  $w^* = \vartheta^*$ , if  $u^* := w^* = \vartheta^*$ , we have:

$$w_{\mathfrak{N}+1} = E_q(\lambda\tau)\mathbf{T}(w_{\mathfrak{N}}, \vartheta_{\mathfrak{N}}) \leq E_q(\lambda\tau)\mathbf{T}(u^*, u^*) \leq E_q(\lambda\tau)\mathbf{T}(\vartheta_{\mathfrak{N}}, w_{\mathfrak{N}}) = \vartheta_{\mathfrak{N}+1}.$$

As  $\mathfrak{N} \rightarrow \infty$  we get that  $E_q(\lambda\tau)\mathbf{T}(u^*, u^*) = u^*$ . That is  $u^*$  is a  $\mathbf{q}$ -EFP of  $\mathbf{T}$  in  $\rho_{h, e_1}$ . Now we display that  $u^*$  is a uniquely  $\mathbf{q}$ -EFP of  $\mathbf{T}$  in  $\rho_{h, e_1}$ . Assume that  $\bar{u}$  is a  $\mathbf{q}$ -EFP of  $\mathbf{T}$  in  $\rho_{h, e_1}$ . By using of 3.5, there are  $s_1, s_1^{-1} > 0$  S.T

$$s_1\bar{u} + (s_1 - 1)e_1 \leq u^* \leq s_1^{-1}\bar{u} + (s_1^{-1} - 1)e_1,$$

we put  $s_2 = \sup\{s_1 > 0 | s_1\bar{u} + (s_1 - 1)e_1 \leq u^* \leq s_1^{-1}\bar{u} + (s_1^{-1} - 1)e_1\}$ . We show that  $s_2 \geq 1$ . If  $0 < s_2 < 1$  thus,

$$s_2\bar{u} + (s_2 - 1)e_1 \leq u^* \leq s_2^{-1}\bar{u} + (s_2^{-1} - 1)e_1,$$



therefore,

$$\begin{aligned} u^* &= E_q(\lambda\tau)\mathbf{T}(u^*, u^*) \geq E_q(\lambda\tau)\mathbf{T}(s_2\bar{u} + [s_2 - 1]e_1, s_2^{-1}\bar{u} + [s_2^{-1} - 1]e_1) \\ &\geq \iota(s_2)E_q(\lambda\tau)\mathbf{T}(\bar{u}, \bar{u}) + [\iota(s_2) - 1]e_1 = \iota(s_2)\bar{u} + [\iota(s_2) - 1]e_1, \\ u^* &= E_q(\lambda\tau)\mathbf{T}(u^*, u^*) \leq E_q(\lambda\tau)\mathbf{T}(s_2^{-1}\bar{u} + [s_2^{-1} - 1]e_1, s_2\bar{u} + [s_2 - 1]e_1) \\ &\leq \iota^{-1}(s_2)E_q(\lambda\tau)\mathbf{T}(\bar{u}, \bar{u}) + [\iota^{-1}(s_2) - 1]e_1 = \iota^{-1}(s_2)\bar{u} + (\iota^{-1}(s_2) - 1)e_1. \end{aligned}$$

By using of definition  $s_2$ , we can take that  $s_2 > \iota(s_2) > s_2$  and  $s_2^{-1} < \iota^{-1}(s_2) < s_2^{-1}$ , which is against our hypothesis. Therefore,  $s_2 \geq 1$  and

$$\begin{aligned} u^* &\geq \iota(s_2)\bar{u} + (\iota(s_2) - 1)e_1 \geq s_2\bar{u} + (s_2 - 1)e_1 \geq s_2\bar{u} \geq \bar{u}, \\ u^* &\leq \iota^{-1}(s_2)\bar{u} + (\iota^{-1}(s_2) - 1)e_1 \leq s_2^{-1}\bar{u} + (s_2^{-1} - 1)e_1 \leq s_2^{-1}\bar{u} \leq \bar{u}. \end{aligned}$$

That's mean,  $\bar{u} \leq u^* \leq \bar{u}$ , thus,  $u^* = \bar{u}$ . Therefore,  $u^*$  is an uniquely  $q$ -EFP of  $\mathbf{T}$  in  $\rho_{h,e_1}$ . For every  $u_0, v_0 \in \rho_{h,e_1}$ , we choose tiny number  $s_3, s_4 \in (0, 1)$  S.T:

$$\begin{aligned} s_2h + (s_2 - 1)e_1 &\leq u_0 \leq s_2^{-1}h + (s_2^{-1} - 1)e_1, \\ s_3h + (s_3 - 1)e_1 &\leq v_0 \leq s_3^{-1}h + (s_3^{-1} - 1)e_1. \end{aligned}$$

If  $s^* = \min\{s_3, s_4\}$ ,  $s^* \in (0, 1)$  and

$$s^*h + (s^* - 1)e_1 \leq u_0, v_0 \geq (s^*)^{-1}h + ((s^*)^{-1} - 1)e_1.$$

Assume that

$$\bar{w}_0 = (s^*)^m h + ((s^*)^m - 1)e_1, \bar{v}_0 = (s^*)^{-m} h + ((s^*)^{-m} - 1)e_1.$$

We see that, according 3.4,  $\bar{w}_0, \bar{v}_0 \in \rho_{h,e_1}$ ,  $\bar{w}_0 < u_0$  and  $\bar{v}_0 > v_0$ . If

$$\begin{aligned} \bar{w}_0 &= E_q(\lambda\tau)\mathbf{T}(\bar{w}_{\mathfrak{N}-1}, \bar{v}_{\mathfrak{N}-1}), \mathfrak{N} - 1, 2, \dots, \\ \bar{v}_0 &= E_q(\lambda\tau)\mathbf{T}(\bar{v}_{\mathfrak{N}-1}, \bar{w}_{\mathfrak{N}-1}), \mathfrak{N} - 1, 2, \dots \end{aligned}$$

Similarly to, there is  $v^* \in \rho_{h,e_1}$ , S.T

$$E_q(\lambda\tau)\mathbf{T}(v^*, v^*) = v^*, \lim_{\mathfrak{N} \rightarrow \infty} \bar{w}_{\mathfrak{N}} = \lim_{\mathfrak{N} \rightarrow \infty} \bar{v}_{\mathfrak{N}} = v^*.$$

Since fixed point  $\mathbf{T}$  is unique in  $\rho_{h,e_1}$ , we have  $u^* = v^*$ , and we get by analysis

$$\bar{w}_{\mathfrak{N}} \leq u_{\mathfrak{N}} \leq \bar{v}_{\mathfrak{N}}, \bar{w}_{\mathfrak{N}} \leq v_{\mathfrak{N}} \leq \bar{v}_{\mathfrak{N}}.$$

Because  $\rho$  is normal, we obtain:

$$\lim_{\mathfrak{N} \rightarrow \infty} \bar{u}_{\mathfrak{N}} = u^*, \lim_{\mathfrak{N} \rightarrow \infty} \bar{v}_{\mathfrak{N}} = u^*.$$

□

**Theorem 3.7.** Suppose that  $0 < \alpha(\tau) < 1$ ,  $0 < \sigma(\tau) < 1$ ,  $0 < \beta(\tau) < 1$ ,  $0 < \alpha(\tau) + \beta(\tau) < 1$ . Also, if  $\rho$  is a normal cone in  $\mathbf{E}$  and  $\mathcal{A} : \rho \times \rho \rightarrow \rho$  is a MMO,  $\mathcal{B}, \mathcal{C} : \rho \rightarrow \rho$ .  $\mathcal{B}$  is a generalized  $\psi$ - $(h, e_1)$ -concave operator and increasing, and  $\mathcal{C}$  is a decreasing and they apply in the following conditions:

(J<sub>1</sub>) There is function  $\varphi : [0, 1] \rightarrow [0, 1]$  S.T for any  $\tau \in (0, 1)$ ,  $\varphi(\tau) = \tau^{\alpha(\tau)+\beta(\tau)} > \tau$ :

$$\mathcal{A}(\tau u + (\tau - 1)e_1, \tau^{-1}v + (\tau^{-1} - 1)e_1) \geq \tau^{\alpha(\tau)+\beta(\tau)}\mathcal{A}(u, v) + (\tau^{\alpha(\tau)+\beta(\tau)} - 1)e_1, \forall u, v \in \rho_{h,e_1}; \tag{3.9}$$

(J<sub>2</sub>) There is function  $\psi : [0, 1] \rightarrow [0, 1]$  S.T for any  $\tau \in (0, 1)$ ,  $\psi(\tau) = \tau^{\sigma(\tau)} > \tau$ , operator  $\mathcal{B}$  applies in the following relation:

$$\mathcal{B}(\tau u + (\tau - 1)e_1) \geq \tau^{\sigma(\tau)}\mathcal{B}(u) + (\tau^{\sigma(\tau)} - 1)e_1, \forall u \in \rho_{h,e_1}, \tag{3.10}$$

operator  $\mathcal{C}$  is decreasing and it applies in the following relation:

$$\mathcal{C}(\tau^{-1}v + (\tau^{-1} - 1)e_1) \geq \tau^{\sigma(\tau)}\mathcal{C}(v) + (\tau^{\sigma(\tau)} - 1)e_1, \forall v \in \rho_{h,e_1}; \tag{3.11}$$



- (J<sub>3</sub>) *Exists*  $h_0 \in \rho_h \subseteq \rho_{h,e_1}$  S.T  $\mathcal{A}(h_0, h_0) \in \rho_{h,e_1}$ ,  $\mathcal{B}(h_0) \in \rho_{h,e_1}$  and  $\mathcal{C}(h_0) \in \rho_{h,e_1}$ ;  
(J<sub>4</sub>) *Exists* a constant  $\delta > 0$ , S.T for each  $u, v \in \rho_{h,e_1}$

$$\mathcal{A}(u, v) + e_1 \geq \delta(\mathcal{B}(u) + \mathcal{C}(v) + 2e_1) + (\delta - 1)e_1.$$

Thus,

- (1)  $\mathcal{A} : \rho_{h,e_1} \times \rho_{h,e_1} \rightarrow \rho_{h,e_1}$  and  $\mathcal{B}, \mathcal{C} : \rho_{h,e_1} \rightarrow \rho_{h,e_1}$ ;  
(2) *There is*  $w_0, \vartheta_0 \in \rho_{h,e_1}$  and  $\epsilon \in (0, 1)$  S.T  $\epsilon \vartheta_0 \leq w_0 < E_q(\lambda\tau)\vartheta_0$ ,

$$\begin{aligned} w_0 &\leq E_q(\lambda\tau)\mathbb{T}(w_0, \vartheta_0) = \mathcal{A}(w_0, \vartheta_0) + \mathcal{B}(w_0) + \mathcal{C}(\vartheta_0) + 3e_1 \\ &\leq E_q(\lambda\tau)\mathbb{T}(\vartheta_0, w_0) = \mathcal{A}(\vartheta_0, w_0) + \mathcal{B}(\vartheta_0) + \mathcal{C}(w_0) + 3e_1 \leq E_q(\lambda\tau)\vartheta_0; \end{aligned}$$

- (3)  $\mathbb{T}$  has an unique  $q$ -EFP  $u^*$  in  $\rho_{h,e_1}$ ;  
(4) For any initial value  $u_0, v_0 \in \rho_{h,e_1}$ , there are recursive sequences

$$u_{\mathfrak{N}+1} = E_q(\lambda\tau)\mathbb{T}(u_{\mathfrak{N}}, v_{\mathfrak{N}}) = \mathcal{A}(u_{\mathfrak{N}}, v_{\mathfrak{N}}) + \mathcal{B}(u_{\mathfrak{N}}) + \mathcal{C}(v_{\mathfrak{N}}) + 3e_1, \quad \mathfrak{N} = 1, 2, \dots$$

$$v_{\mathfrak{N}+1} = E_q(\lambda\tau)\mathbb{T}(v_{\mathfrak{N}}, u_{\mathfrak{N}}) = \mathcal{A}(v_{\mathfrak{N}}, u_{\mathfrak{N}}) + \mathcal{B}(v_{\mathfrak{N}}) + \mathcal{C}(u_{\mathfrak{N}}) + 3e_1, \quad \mathfrak{N} = 1, 2, \dots$$

we obtain  $u_{\mathfrak{N}} \rightarrow u^*$  and  $v_{\mathfrak{N}} \rightarrow u^*$  as  $\mathfrak{N} \rightarrow \infty$ .

*Proof.* Using the (J<sub>1</sub>), (J<sub>2</sub>) and relations (3.9), (3.10), and (3.11), for any  $\tau \in (0, 1)$  and  $u, v \in \rho_{h,e_1}$  we get:

$$\mathcal{A}(\tau^{-1}u + (\tau^{-1} - 1)e_1, \tau v + (\tau - 1)e_1) \leq \frac{1}{\tau^{\alpha(\tau)+\beta(\tau)}}\mathcal{A}(u, v) + \left(\frac{1}{\tau^{\alpha(\tau)+\beta(\tau)}} - 1\right)e_1, \quad (3.12)$$

$$\mathcal{B}(\tau^{-1}u + (\tau^{-1} - 1)e_1) \leq \frac{1}{\tau^{\sigma(\tau)}}\mathcal{B}(u) + \left(\frac{1}{\tau^{\sigma(\tau)}} - 1\right)e_1, \quad (3.13)$$

$$\mathcal{C}(\tau v + (\tau - 1)e_1) \leq \frac{1}{\tau^{\sigma(\tau)}}\mathcal{C}(v) + \left(\frac{1}{\tau^{\sigma(\tau)}} - 1\right)e_1. \quad (3.14)$$

Now, we display that  $\mathcal{A} : \rho_{h,e_1} \times \rho_{h,e_1} \rightarrow \rho_{h,e_1}$ ,  $\mathcal{B}, \mathcal{C} : \rho_{h,e_1} \rightarrow \rho_{h,e_1}$ . Since  $\mathcal{A}(h_0, h_0) \in \rho_h$ ,  $\mathcal{B}(h_0) \in \rho_h$  and  $\mathcal{C}(h_0) \in \rho_h$ , exists constant  $\gamma_i > 0$  and  $\mu_i > 0$ , ( $i = 1, 2, \dots$ ) S.T:

$$E_q(\lambda\tau)\gamma_1 h \leq \mathcal{A}(h_0, h_0) \leq E_q(\lambda\tau)\mu_1 h, \quad (3.15)$$

$$E_q(\lambda\tau)\gamma_2 h \leq \mathcal{B}(h_0) \leq E_q(\lambda\tau)\mu_2 h, \quad (3.16)$$

$$E_q(\lambda\tau)\gamma_3 h \leq \mathcal{C}(h_0) \leq E_q(\lambda\tau)\mu_3 h. \quad (3.17)$$

Because  $h_0 \in \rho_h \subseteq \rho_{h,e_1}$ , there exist constant  $\tau_0 \in (0, 1)$  S.T:

$$\tau_0 h + (\tau_0 - 1)e_1 \leq h_0 \leq \frac{1}{\tau_0} h + \left(\frac{1}{\tau_0} - 1\right)e_1.$$

By using of relation (3.9) - (3.17), we have:

$$\begin{aligned} \mathcal{A}(h, h) &\leq \mathcal{A}\left(\frac{1}{\tau_0}h_0 + \left(\frac{1}{\tau_0} - 1\right)e_1, \tau_0 h_0 + (\tau_0 - 1)e_1\right) \\ &\leq \frac{1}{\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}}\mathcal{A}(h_0, h_0) + \left(\frac{1}{\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}} - 1\right)e_1 \\ &\leq \frac{E_q(\lambda\tau)\mu_1 h}{\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}} + \left(\frac{1}{\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}} - 1\right)e_1, \end{aligned} \quad (3.18)$$



$$\begin{aligned}
 \mathcal{A}(h, h) &\geq \mathcal{A}\left(\tau_0 h_0 + (\tau_0 - 1)e_1, \frac{1}{\tau_0} h_0 + \left(\frac{1}{\tau_0} - 1\right)e_1\right) \\
 &\geq \tau_0^{\alpha(\tau_0)+\beta(\tau_0)} \mathcal{A}(h_0, h_0) + (\tau_0^{\alpha(\tau_0)+\beta(\tau_0)} - 1)e_1 \\
 &\geq E_q(\lambda\tau)\gamma_1 h \tau_0^{\alpha(\tau_0)+\beta(\tau_0)} + (\tau_0^{\alpha(\tau_0)+\beta(\tau_0)} - 1)e_1,
 \end{aligned} \tag{3.19}$$

$$\begin{aligned}
 \mathcal{B}(h) &\leq \mathcal{B}\left(\frac{1}{\tau_0} h_0 + \left(\frac{1}{\tau_0} - 1\right)e_1\right) \leq \frac{1}{\tau_0^{\sigma(\tau_0)}} \mathcal{B}(h_0) + \left(\frac{1}{\tau_0^{\sigma(\tau_0)}} - 1\right)e_1 \\
 &\leq \frac{E_q(\lambda\tau)\mu_2 h}{\tau_0^{\sigma(\tau_0)}} + \left(\frac{1}{\tau_0^{\sigma(\tau_0)}} - 1\right)e_1,
 \end{aligned} \tag{3.20}$$

$$\begin{aligned}
 \mathcal{B}(h) &\geq \mathcal{B}(\tau_0 h_0 + (\tau_0 - 1)e_1) \geq \tau_0^{\sigma(\tau_0)} \mathcal{B}(h_0) + (\tau_0^{\sigma(\tau_0)} - 1)e_1 \\
 &\geq E_q(\lambda\tau)\gamma_2 h \tau_0^{\sigma(\tau_0)} + (\tau_0^{\sigma(\tau_0)} - 1)e_1.
 \end{aligned} \tag{3.21}$$

$$\begin{aligned}
 \mathcal{C}(h) &\leq \mathcal{C}(\tau_0 h_0 + (\tau_0 - 1)e_1) \\
 &\leq \frac{1}{\tau_0^{\sigma(\tau_0)}} \mathcal{C}(h_0) + \left(\frac{1}{\tau_0^{\sigma(\tau_0)}} - 1\right)e_1 \\
 &\leq \frac{E_q(\lambda\tau)\mu_3 h}{\tau_0^{\sigma(\tau_0)}} + \left(\frac{1}{\tau_0^{\sigma(\tau_0)}} - 1\right)e_1,
 \end{aligned} \tag{3.22}$$

$$\begin{aligned}
 \mathcal{C}(h) &\geq \mathcal{C}\left(\frac{1}{\tau_0^{\sigma(\tau_0)}} h_0 + \left(\frac{1}{\tau_0^{\sigma(\tau_0)}} - 1\right)e_1\right) \\
 &\geq \tau_0^{\sigma(\tau_0)} \mathcal{C}(h_0) + (\tau_0^{\sigma(\tau_0)} - 1)e_1 \\
 &\geq E_q(\lambda\tau)\gamma_3 h \tau_0^{\sigma(\tau_0)} + (\tau_0^{\sigma(\tau_0)} - 1)e_1.
 \end{aligned} \tag{3.23}$$

It is obvious that  $\frac{\mu_1}{\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}}$ ,  $\gamma_1 \tau_0^{\alpha(\tau_0)+\beta(\tau_0)}$ ,  $\frac{\mu_2}{\tau_0^{\sigma(\tau_0)}}$ ,  $\gamma_2 \tau_0^{\sigma(\tau_0)}$ ,  $\frac{\mu_3}{\tau_0^{\sigma(\tau_0)}}$  and  $\gamma_3 \tau_0^{\sigma(\tau_0)}$  are greater than zero. As a result, we proved that  $\mathcal{A}(h, h) \in \rho_{h, e_1}$ ,  $\mathcal{B}(h) \in \rho_{h, e_1}$  and  $\mathcal{C}(h) \in \rho_{h, e_1}$ . For every  $u, v \in \rho_{h, e_1}$ , we select a sufficiently tiny number like  $\xi \in (0, 1)$  S.T:

$$\xi h + (\xi - 1)e_1 \leq u \leq \frac{1}{\xi} h + \left(\frac{1}{\xi} - 1\right)e_1, \tag{3.24}$$

$$\xi h + (\xi - 1)e_1 \leq v \leq \frac{1}{\xi} h + \left(\frac{1}{\xi} - 1\right)e_1. \tag{3.25}$$

By using of relations (3.9)-(3.14) and (3.18)-(3.25), we have:

$$\begin{aligned}
 \mathcal{A}(u, v) &\geq \mathcal{A}\left(\xi h + (\xi - 1)e_1, \frac{1}{\xi} h + \left(\frac{1}{\xi} - 1\right)e_1\right) \\
 &\geq \xi^{\alpha(\xi)+\beta(\xi)} \mathcal{A}(h, h) + (\xi^{\alpha(\xi)+\beta(\xi)} - 1)e_1 \\
 &\geq \xi^{\alpha(\xi)+\beta(\xi)} \left(E_q(\lambda\tau)\tau_0^{\alpha(\tau_0)+\beta(\tau_0)} \gamma_1 h + (\tau_0^{\alpha(\tau_0)+\beta(\tau_0)} - 1)e_1\right) + (\xi^{\alpha(\xi)+\beta(\xi)} - 1)e_1 \\
 &= E_q(\lambda\tau)\tau_0^{\alpha(\tau_0)+\beta(\tau_0)} \gamma_1 \xi^{\alpha(\xi)+\beta(\xi)} h + (\xi^{\alpha(\xi)+\beta(\xi)} \tau_0^{\alpha(\tau_0)+\beta(\tau_0)} - 1)e_1,
 \end{aligned}$$



$$\begin{aligned}
\mathcal{A}(u, v) &\leq \mathcal{A}\left(\frac{1}{\xi}h + \left(\frac{1}{\xi} - 1\right)e_1, \xi h + (\xi - 1)e_1\right) \\
&\leq \frac{1}{\xi^{\alpha(\xi)+\beta(\xi)}}\mathcal{A}(h, h) + \left(\frac{1}{\xi^{\alpha(\xi)+\beta(\xi)}} - 1\right)e_1 \\
&\leq \frac{1}{\xi^{\alpha(\xi)+\beta(\xi)}}\left(\frac{E_q(\lambda\tau)\mu_1 h}{\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}} + \left(\frac{1}{\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}} - 1\right)e_1\right) + \left(\frac{1}{\xi^{\alpha(\xi)+\beta(\xi)}} - 1\right)e_1 \\
&= \frac{E_q(\lambda\tau)\mu_1 h}{\xi^{\alpha(\xi)+\beta(\xi)}\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}} + \left(\frac{1}{\xi^{\alpha(\xi)+\beta(\xi)}\tau_0^{\alpha(\tau_0)+\beta(\tau_0)}} - 1\right)e_1.
\end{aligned}$$

Thus,  $\mathcal{A}(u, v) \in \rho_{h, e_1}$ .

$$\begin{aligned}
\mathcal{B}(u) &\geq \mathcal{B}(\xi h + (\xi - 1)e_1) \geq \xi^{\sigma(\xi)}\mathcal{B}(h) + (\xi^{\sigma(\xi)-1})e_1 \\
&\geq \xi^{\sigma(\xi)}\left(E_q(\lambda\tau)\gamma_2 h \tau_0^{\sigma(\tau_0)} + (\tau_0^{\sigma(\tau_0)} - 1)e_1\right) + (\xi^{\sigma(\xi)} - 1)e_1 \\
&= E_q(\lambda\tau)\xi^{\sigma(\xi)}\tau_0^{\sigma(\tau_0)}\gamma_2 h + (\xi^{\sigma(\xi)}\tau_0^{\sigma(\tau_0)} - 1)e_1, \\
\mathcal{B}(u) &\leq \mathcal{B}\left(\frac{1}{\xi}h + \left(\frac{1}{\xi} - 1\right)e_1\right) \leq \frac{1}{\xi^{\sigma(\xi)}}\mathcal{B}(h) + \left(\frac{1}{\xi^{\sigma(\xi)}} - 1\right)e_1 \\
&\leq \frac{1}{\xi^{\sigma(\xi)}}\left(\frac{E_q(\lambda\tau)\mu_2 h}{\tau_0^{\sigma(\tau_0)}} + \left(\frac{1}{\tau_0^{\sigma(\tau_0)}} - 1\right)e_1\right) + \left(\frac{1}{\xi^{\sigma(\xi)}} - 1\right)e_1 \\
&= \frac{E_q(\lambda\tau)\mu_2 h}{\xi^{\sigma(\xi)}\tau_0^{\sigma(\tau_0)}} + \left(\frac{1}{\xi^{\sigma(\xi)}\tau_0^{\sigma(\tau_0)}} - 1\right)e_1.
\end{aligned}$$

Then,  $\mathcal{B}(u) \in \rho_{h, e_1}$ .

$$\begin{aligned}
\mathcal{C}(v) &\geq \mathcal{C}\left(\frac{1}{\xi}h + \left(\frac{1}{\xi} - 1\right)e_1\right) \geq \xi^{\sigma(\xi)}\mathcal{C}(h) + (\xi^{\sigma(\xi)-1})e_1 \\
&\geq \xi^{\sigma(\xi)}\left(E_q(\lambda\tau)\gamma_3 h \tau_0^{\sigma(\tau_0)} + (\tau_0^{\sigma(\tau_0)} - 1)e_1\right) + (\xi^{\sigma(\xi)} - 1)e_1 \\
&= E_q(\lambda\tau)\xi^{\sigma(\xi)}\tau_0^{\sigma(\tau_0)}\gamma_3 h + (\xi^{\sigma(\xi)}\tau_0^{\sigma(\tau_0)} - 1)e_1, \\
\mathcal{C}(v) &\leq \mathcal{C}(\xi h + (\xi - 1)e_1) \leq \frac{1}{\xi^{\sigma(\xi)}}\mathcal{C}(h) + \left(\frac{1}{\xi^{\sigma(\xi)}} - 1\right)e_1 \\
&\leq \frac{1}{\xi^{\sigma(\xi)}}\left(\frac{E_q(\lambda\tau)\mu_3 h}{\tau_0^{\sigma(\tau_0)}} + \left(\frac{1}{\tau_0^{\sigma(\tau_0)}} - 1\right)e_1\right) + \left(\frac{1}{\xi^{\sigma(\xi)}} - 1\right)e_1 \\
&= \frac{E_q(\lambda\tau)\mu_3 h}{\xi^{\sigma(\xi)}\tau_0^{\sigma(\tau_0)}} + \left(\frac{1}{\xi^{\sigma(\xi)}\tau_0^{\sigma(\tau_0)}} - 1\right)e_1.
\end{aligned}$$

Therefore,  $\mathcal{C}(v) \in \rho_{h, e_1}$ . That is  $\mathcal{A} : \rho_{h, e_1} \times \rho_{h, e_1} \longrightarrow \rho_{h, e_1}$ ,  $\mathcal{B}, \mathcal{C} : \rho_{h, e_1} \longrightarrow \rho_{h, e_1}$ . We define operator

$$E_q(\lambda\tau)\mathbb{T}(u, v) = \mathcal{A}(u, v) + \mathcal{B}(u) + \mathcal{C}(v) + 3e_1, \quad \forall u, v \in \rho_{h, e_1}.$$

According to  $\mathcal{A}(h, h) \in \rho_{h, e_1}$ ,  $\mathcal{B}(h) \in \rho_{h, e_1}$  and  $\mathcal{C}(h) \in \rho_{h, e_1}$  we have,

$$E_q(\lambda\tau)\mathbb{T}(h, h) = \mathcal{A}(h, h) + \mathcal{B}(h) + \mathcal{C}(h) + 3e_1 \in \rho_{h, e_1}.$$

Now we demonstrate that for any  $\tau \in (0, 1)$  there exist  $\iota(\tau) \in (\tau, 1]$ , S.T for each  $u, v \in \rho_{h, e_1}$ ,

$$\mathbb{T}(u + (\tau - 1)e_1, \tau^{-1}v + (\tau^{-1} - 1)e_1) \geq \iota(\tau)\mathbb{T}(u, v) + (\iota(\tau) - 1)e_1. \quad (3.26)$$



For any  $u, v \in \rho_{h, e_1}$  and by using of  $(J_4)$  there is constant  $\delta > 0$  S.T:

$$(\mathcal{A} + e_1) + \delta(\mathcal{A} + e_1) \geq \delta(\mathcal{B} + \mathcal{C} + 2e_1) + \delta(\mathcal{A} + e_1) + (\delta - 1)e_1.$$

Thus, we obtain:

$$(\mathcal{A} + e_1)(1 + \delta) \geq \delta[\mathcal{A} + \mathcal{B} + \mathcal{C} + 3e_1] + (\delta - 1)e_1, \quad \forall u, v \in \rho_{h, e_1}.$$

We divide the sides by the  $(1 + \delta)$ :

$$\begin{aligned} (\mathcal{A} + e_1) &\geq \frac{\delta}{1 + \delta} E_q(\lambda\tau)\mathbf{T}(u, v) + \frac{\delta - 1}{1 + \delta} e_1 \\ &\geq \frac{\delta}{1 + \delta} E_q(\lambda\tau)\mathbf{T}(u, v) - \frac{e_1}{1 + \delta}, \quad \forall u, v \in \rho_{h, e_1}. \end{aligned}$$

Using relations(3.9), (3.10), (3.11) and (3.26), we get:

$$\begin{aligned} &E_q(\lambda\tau)\mathbf{T}(\tau u + (\tau - 1)e_1, \tau^{-1}v + (\tau^{-1} - 1)e_1) - \tau^{\sigma(\tau)} E_q(\lambda\tau)\mathbf{T}(u, v) \\ &= \mathcal{A}(\tau u + (\tau - 1)e_1, \tau^{-1}v + (\tau^{-1} - 1)e_1) + \mathcal{B}(\tau u + (\tau - 1)e_1) + \mathcal{C}(\tau^{-1}v + (\tau^{-1} - 1)e_1) \\ &+ 3e_1 - \tau^{\sigma(\tau)}[\mathcal{A}(u, v) + \mathcal{B}(u) + \mathcal{C}(v) - 3e_1] \\ &\geq \tau^{\alpha(\tau)+\beta(\tau)} \mathcal{A}(u, v) + (\tau^{\alpha(\tau)+\beta(\tau)} - 1)e_1 + \tau^{\sigma(\tau)} \mathcal{B}(u) + (\tau^{\sigma(\tau)} - 1)e_1 \\ &+ \tau^{\sigma(\tau)} \mathcal{C}(v) + (\tau^{\sigma(\tau)} - 1)e_1 + 3e_1 - \tau^{\sigma(\tau)} \mathcal{A}(u, v) - \tau^{\sigma(\tau)} \mathcal{B}(u) - \tau^{\sigma(\tau)} \mathcal{C}(v) - 3\tau^{\sigma(\tau)} e_1 \\ &= (\tau^{\alpha(\tau)+\beta(\tau)} - \tau^{\sigma(\tau)})(\mathcal{A} + e_1) + 3e_1(1 - \tau^{\sigma(\tau)}) \\ &\geq (\tau^{\alpha(\tau)+\beta(\tau)} - \tau^{\sigma(\tau)})(\mathcal{A} + e_1) \\ &\geq (\tau^{\alpha(\tau)+\beta(\tau)} - \tau^{\sigma(\tau)})(\mathcal{A} + e_1) + (\tau^{\alpha(\tau)+\beta(\tau)} - 1)e_1 \\ &\geq (\tau^{\alpha(\tau)+\beta(\tau)} - \tau^{\sigma(\tau)})\left[\frac{\delta}{1 + \delta} E_q(\lambda\tau)\mathbf{T}(u, v) - \frac{e_1}{1 + \delta}\right] + (\tau^{\alpha(\tau)+\beta(\tau)} - 1)e_1. \end{aligned}$$

So, we have:

$$\begin{aligned} &E_q(\lambda\tau)\mathbf{T}(\tau u + (\tau - 1)e_1, \tau^{-1}v + (\tau^{-1} - 1)e_1) \geq (\tau^{\alpha(\tau)+\beta(\tau)} - \tau^{\sigma(\tau)})\left[\frac{\delta}{1 + \delta} E_q(\lambda\tau)\mathbf{T}(u, v) \right. \\ &- \left. \frac{e_1}{1 + \delta}\right] + (\tau^{\alpha(\tau)+\beta(\tau)} - 1)e_1 + \tau^{\sigma(\tau)} E_q(\lambda\tau)\mathbf{T}(u, v) \\ &= \frac{\delta\tau^{\alpha(\tau)+\beta(\tau)} - \delta\tau^{\sigma(\tau)}}{1 + \delta} E_q(\lambda\tau)\mathbf{T}(u, v) + \frac{\delta\tau^{\sigma(\tau)} + \tau^{\sigma(\tau)}}{1 + \delta} E_q(\lambda\tau)\mathbf{T}(u, v) \\ &- \frac{\tau^{\alpha(\tau)+\beta(\tau)} + \tau^{\sigma(\tau)}}{1 + \delta} e_1 + \frac{\delta\tau^{\alpha(\tau)+\beta(\tau)} + \tau^{\alpha(\tau)+\beta(\tau)} - (1 + \delta)}{1 + \delta} e_1 \\ &= \frac{\tau^{\alpha(\tau)+\beta(\tau)} + \tau^{\sigma(\tau)}}{1 + \delta} E_q(\lambda\tau)\mathbf{T}(u, v) + \left(\frac{\tau^{\alpha(\tau)+\beta(\tau)} + \tau^{\sigma(\tau)}}{1 + \delta} - 1\right)e_1, \quad \forall u, v \in \rho_{h, e_1}. \end{aligned} \tag{3.27}$$

If  $\iota(\tau) = \frac{\tau^{\alpha(\tau)+\beta(\tau)} + \tau^{\sigma(\tau)}}{1 + \delta}$  thus,  $\iota(\tau) \in (\tau, 1], \tau \in (0, 1)$ . By using (3.27), We draw conclusions

$$\mathbf{T}(\tau u + (\tau - 1)e_1, \tau^{-1}v + (\tau^{-1} - 1)e_1) \geq \iota(\tau)\mathbf{T}(u, v) + (\iota(\tau) - 1)e_1, \quad \forall u, v \in \rho_{h, e_1}.$$

Therefore, condition  $(J)$  of 3.6 is true. Therefore, using 3.6, we draw conclusions that

- (1) There are  $w_0, \vartheta_0 \in \rho_{h, e_1}$  and  $r \in (0, 1)$  S.T  $r\vartheta_0 \leq w_0 < E_q(\lambda\tau)\vartheta_0$ ;  
 $w_0 \leq E_q(\lambda\tau)\mathbf{T}(w_0, \vartheta_0) \leq E_q(\lambda\tau)\mathbf{T}(\vartheta_0, w_0) \leq E_q(\lambda\tau)\vartheta_0$ ;
- (2)  $\mathbf{T}$  has an unique q-EFP  $u^*$  in  $\rho_{h, e_1}$ ;



(3) For any primitive value  $u_0, v_0 \in \rho_{h, e_1}$ , by making recursive sequences

$$\begin{aligned} u_{\mathfrak{N}} &= E_q(\lambda\tau)\mathbf{T}(u_{\mathfrak{N}-1}, v_{\mathfrak{N}-1}), \\ v_{\mathfrak{N}} &= E_q(\lambda\tau)\mathbf{T}(v_{\mathfrak{N}-1}, u_{\mathfrak{N}-1}), \quad \mathfrak{N} = 1, 2, \dots, \end{aligned}$$

we get  $u_{\mathfrak{N}} \rightarrow u^*$  and  $v_{\mathfrak{N}} \rightarrow v^*$  as  $\mathfrak{N} \rightarrow \infty$ . Means that, conditions 1-3 are fulfilled.  $\square$

**Theorem 3.8.** Assume that the conditions  $(J'_1) - (J'_5)$  in (1.1) are satisfied :

- $(J'_1)$   $\chi : O \times \Upsilon \times \Upsilon \rightarrow \Upsilon$  and  $\varrho, \phi : O \times \Upsilon \rightarrow \Upsilon$  are continuous.  $\chi(\tau, 0, 1) \neq 0$ ,  $\varrho(\tau, 0) \neq 0$  and  $\phi(\tau, 1) \neq 0$ ;
- $(J'_2)$  For each constant  $\tau \in O$  and  $\vartheta \in \Upsilon$  in  $w \in \Upsilon$ , function  $\chi(\tau, w(\tau), \vartheta(\tau))$  is increasing and for every constant  $\tau \in (0, 1)$  and  $w \in \Upsilon$  in  $\vartheta \in \Upsilon$  function  $\chi(\tau, w(\tau), \vartheta(\tau))$  is decreasing. For constant  $\tau \in O$  in  $w \in \Upsilon$ , the function  $\varrho(\tau, w)$  is increasing and for constant  $\tau \in O$  in  $\vartheta \in \Upsilon$ , the function  $\phi(\tau, \vartheta)$  is decreasing;
- $(J'_3)$  For any  $\tau, \eta \in (0, 1)$ ,  $w, \vartheta \in \Upsilon$  and the continuous functions  $0 < \alpha(\eta) < 1$ ,  $0 < \beta(\eta) < 1$  and  $0 < \alpha(\eta) + \beta(\eta) < 1$ . There is a function  $\eta^{\alpha(\eta) + \beta(\eta)} > \eta$  S.T:

$$\chi(\tau, \eta w + (\eta - 1)e_1, \eta^{-1}\vartheta + (\eta^{-1} - 1)e_1) \geq \eta^{\alpha(\eta) + \beta(\eta)} \chi(\tau, w, \vartheta) + (\eta^{\alpha(\eta) + \beta(\eta)} - 1)e_1;$$

- $(J'_4)$  For any  $\tau, \eta \in (0, 1)$ ,  $w, \vartheta \in \Upsilon$  and continuous function  $0 < \sigma(\eta) < 1$ , there is a function  $\eta^{\sigma(\eta)} > \eta$ , S.T

$$\begin{aligned} \varrho(\tau, \eta w + (\eta - 1)e_1) &\geq \eta^{\sigma(\eta)} \varrho(\tau, w) + (\eta^{\sigma(\eta)} - 1)e_1, \\ \phi(\tau, \eta^{-1}\vartheta + (\eta^{-1} - 1)e_1) &\geq \eta^{\sigma(\eta)} \phi(\tau, \vartheta) + (\eta^{\sigma(\eta)} - 1)e_1; \end{aligned}$$

- $(J'_5)$  For every  $\tau, \eta \in (0, 1)$ ,  $w, \vartheta \in \Upsilon$  exists a constant  $\delta > 0$ , S.T:

$$\chi(\tau, w(\tau), \vartheta(\tau)) \geq \delta[\varrho(\tau, w(\tau)) + \phi(\tau, \vartheta(\tau))] + (\delta - 1)e_1.$$

Thus,

- (1) There are  $\tau \in O$ ,  $w_0, \vartheta_0 \in \rho_{h, e_1}$  and  $\epsilon \in (0, 1)$ , S.T  $\epsilon\vartheta_0 \leq w_0 < E_q(\lambda\tau)\vartheta_0$  and

$$w_0(\tau) \leq E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0}[\chi(\varsigma, w_0(\varsigma), \vartheta_0(\varsigma)) + \varrho(\varsigma, w_0(\varsigma)) + \phi(\varsigma, \vartheta_0(\varsigma))] d\varsigma - 3e_1,$$

$$E_q(\lambda\tau)\vartheta_0 \geq E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0}[\chi(\varsigma, \vartheta_0(\varsigma), w_0(\varsigma)) + \varrho(\varsigma, w_0(\varsigma)) + \phi(\varsigma, w_0(\varsigma))] d\varsigma - 3e_1,$$

where  $h(\tau) = \tau^{\gamma-1}$ ,  $\tau \in [0, 1]$ ;

- (2) The hybrid q-FBVP (1.1) has an unique q-EFP  $w^*$  in  $\rho_{h, e_1}$ ;
- (3) For any  $u_0, v_0 \in \rho_{h, e_1}$ , by making recursive sequences

$$u_{\mathfrak{N}+1} = E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0}[\chi(\varsigma, u_{\mathfrak{N}}(\varsigma), v_{\mathfrak{N}}(\varsigma)) + \varrho(\varsigma, u_{\mathfrak{N}}(\varsigma)) + \phi(\varsigma, v_{\mathfrak{N}}(\varsigma))] d\varsigma - 3e_1,$$

$$v_{\mathfrak{N}+1} = E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0}[\chi(\varsigma, v_{\mathfrak{N}}(\varsigma), u_{\mathfrak{N}}(\varsigma)) + \varrho(\varsigma, v_{\mathfrak{N}}(\varsigma)) + \phi(\varsigma, u_{\mathfrak{N}}(\varsigma))] d\varsigma - 3e_1,$$

$$\mathfrak{N} = 0, 1, 2, \dots$$

we have  $\|u_{\mathfrak{N}} - w^*\| \rightarrow 0$  and  $\|v_{\mathfrak{N}} - w^*\| \rightarrow 0$  as  $\mathfrak{N} \rightarrow \infty$ .

*Proof.* The hybrid q-FBVP (1.1) has a solution as below according to 2.11:

$$w(\tau) = E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0}[\chi(\varsigma, w(\varsigma), \vartheta(\varsigma)) + \varrho(\varsigma, w(\varsigma)) + \phi(\varsigma, \vartheta(\varsigma))] d\varsigma - 3e_1,$$

where  $\mathfrak{G}(\tau, \varsigma)$  is the same as (2.3).

We display that  $w$  is a q-EPS of problem (1.1)  $\leftrightarrow w = \mathcal{A}(w, \vartheta) + \mathcal{B}(w) + \mathcal{C}(\vartheta) + 3e_1 = E_q(\lambda\tau)\mathbf{T}(w, \vartheta)$ . We express





operators  $\mathcal{A} : \rho \times \rho \rightarrow \mathbf{E}$  and  $\mathcal{B}, \mathcal{C} : \rho \rightarrow \mathbf{E}$  as below:

$$\mathcal{A}(w, \vartheta)(\tau) = E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \chi(\varsigma, w(\varsigma), \vartheta(\varsigma)) d\varsigma - e_1,$$

$$\mathcal{B}(w)(\tau) = E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \varrho(\varsigma, w(\varsigma)) d\varsigma - e_1,$$

$$\mathcal{C}(\vartheta)(\tau) = E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \phi(\varsigma, \vartheta(\varsigma)) d\varsigma - e_1.$$

For  $w_i, \vartheta_i \in \rho_{h, e_1}$ , ( $i = 1, 2$ ) with  $w_1 \leq w_2$  and  $\vartheta_1 \geq \vartheta_2$ , we know that  $\vartheta_1(\tau) \geq \vartheta_2(\tau)$ ,  $w_1(\tau) \leq w_2(\tau)$ . We show that operators  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$  apply to the following relations, respectively.

$$\mathcal{A}(w_1, \vartheta_1)(\tau) \leq \mathcal{A}(w_2, \vartheta_2)(\tau).$$

$$\mathcal{B}(w_1)(\tau) \leq \mathcal{B}(w_2)(\tau),$$

$$\mathcal{C}(\vartheta_1)(\tau) \leq \mathcal{C}(\vartheta_2)(\tau).$$

Using of  $(J'_2)$ , we have:

$$\begin{aligned} \mathcal{A}(w_1, \vartheta_1)(\tau) &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \chi(\varsigma, w_1(\varsigma), \vartheta_1(\varsigma)) d\varsigma - e_1 \\ &\leq E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \chi(\varsigma, w_2(\varsigma), \vartheta_2(\varsigma)) d\varsigma - e_1 \\ &= \mathcal{A}(w_2, \vartheta_2)(\tau). \end{aligned}$$

That is,  $\mathcal{A}(w_1, \vartheta_1)(\tau) \leq \mathcal{A}(w_2, \vartheta_2)(\tau)$ . Similarly, for  $\mathcal{B}$  and  $\mathcal{C}$  we obtain.

$$\begin{aligned} \mathcal{B}(w_1)(\tau) &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \varrho(\varsigma, w_1(\varsigma)) d\varsigma - e_1 \\ &\leq E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \varrho(\varsigma, w_2(\varsigma)) d\varsigma - e_1 \\ &= \mathcal{B}(w_2)(\tau). \end{aligned}$$

Means that,  $\mathcal{B}(w_1)(\tau) \leq \mathcal{B}(w_2)(\tau)$ .

$$\begin{aligned} \mathcal{C}(\vartheta_1)(\tau) &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \phi(\varsigma, \vartheta_1(\varsigma)) d\varsigma - e_1 \\ &\leq E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \phi(\varsigma, \vartheta_2(\varsigma)) d\varsigma - e_1 \\ &= \mathcal{C}(\vartheta_2)(\tau). \end{aligned}$$

That is,  $\mathcal{C}(\vartheta_1)(\tau) \leq \mathcal{C}(\vartheta_2)(\tau)$ . Now, for each  $w, \vartheta \in \rho_{h, e_1}$ ,  $\eta \in (0, 1)$  and  $\tau \in O$  using of  $(J'_3)$ , we show that operator  $\mathcal{A}$  satisfy in condition (3.9).

$$\begin{aligned} &\mathcal{A}(\eta w + (\eta - 1)e_1, \eta^{-1}\vartheta + (\eta^{-1} - 1)e_1)(\tau) \\ &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \chi(\varsigma, \eta w(\varsigma) + (\eta - 1)e_1, \eta^{-1}\vartheta(\varsigma) + (\eta^{-1} - 1)e_1) d\varsigma - e_1 \\ &\geq \eta^{\alpha(\eta) + \beta(\eta)} E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \chi(\varsigma, w(\varsigma), \vartheta(\varsigma)) d\varsigma + (\eta^{\alpha(\eta) + \beta(\eta)} - 1)e_1 - e_1 \\ &= \eta^{\alpha(\eta) + \beta(\eta)} \mathcal{A}(w, \vartheta)(\tau) + (\eta^{\alpha(\eta) + \beta(\eta)} - 1)e_1. \end{aligned}$$



That's mean, the operator  $\mathcal{A}$  satisfies in condition (3.9). Now, for any  $w, \vartheta \in \rho_{h, e_1}$ ,  $\eta \in (0, 1)$  and  $\tau \in [0, 1]$ , by using of  $(J'_4)$ , we show that  $\mathcal{B}$  and  $\mathcal{C}$  apply in conditions (3.10) and (3.11).

$$\begin{aligned} \mathcal{B}(\eta w + (\eta - 1)e_1) &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \varrho(\varsigma, \eta w(\varsigma) + (\eta - 1)e_1) d\varsigma - e_1 \\ &\geq E_q(\lambda\tau) [\eta^{\sigma(\eta)} E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \varrho(\varsigma, w(\varsigma)) d\varsigma + (\eta^{\sigma(\eta)} - 1)e_1] - e_1 \\ &= \eta^{\sigma(\eta)} \mathcal{B}(w)(\tau) + (\eta^{\sigma(\eta)} - 1)e_1. \end{aligned}$$

That is, the operator  $\mathcal{B}$  applies in condition (3.10).

$$\begin{aligned} \mathcal{C}(\eta^{-1}\vartheta + (\eta^{-1} - 1)e_1) &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \phi(\varsigma, \eta^{-1}\vartheta(\varsigma) + (\eta^{-1} - 1)e_1) d\varsigma - e_1 \\ &\geq E_q(\lambda\tau) [\eta^{\sigma(\eta)} \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \phi(\varsigma, \vartheta(\varsigma)) d\varsigma + (\eta^{\sigma(\eta)} - 1)e_1] - e_1 \\ &= \eta^{\sigma(\eta)} \mathcal{C}(\vartheta)(\tau) + (\eta^{\sigma(\eta)} - 1)e_1. \end{aligned}$$

That's mean, the operator  $\mathcal{C}$  satisfies in condition (3.11). Now we need to demonstrate that  $\mathcal{A}(h, h) \in \rho_{h, e_1}$ ,  $\mathcal{B}(h) \in \rho_{h, e_1}$  and  $\mathcal{C}(h) \in \rho_{h, e_1}$ . So we need to prove  $\mathcal{A}(h, h) + e_1 \in \rho_h$ ,  $\mathcal{B}(h) + e_1 \in \rho_h$  and  $\mathcal{C}(h) + e_1 \in \rho_h$ . By using of  $(J'_1)$  and  $(J'_2)$ , we have:

$$\begin{aligned} \mathcal{A}(h, h)(\tau) + e_1 &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \chi(\varsigma, h(\varsigma), h(\varsigma)) d\varsigma \\ &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \chi(\varsigma, \varsigma^{\gamma-1}, -I_{q,0} \varsigma^{\gamma-1}) d\varsigma \end{aligned}$$

According to 2.6, we know that if  $(\lambda = \gamma - 1, a = 0, \alpha = 1)$ , thus,

$-I_{q,0} \varsigma^{\gamma-1} = \frac{-\Gamma_q(\gamma)}{\Gamma_q(\gamma+1)} \varsigma^\gamma$ . Therefore, by using of 2.12 we get:

$$\begin{aligned} \mathcal{A}(h, h) + e_1 &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \chi(\varsigma, \varsigma^{\gamma-1}, \frac{-\Gamma_q(\gamma)}{\Gamma_q(\gamma+1)} \varsigma^\gamma) d\varsigma \\ &\leq E_q(\lambda\tau) \tau^{\gamma-1} \int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}}{\Gamma(\gamma)} I_{q,0} \chi(\varsigma, \varsigma^{\gamma-1}, \frac{-\Gamma_q(\gamma)}{\Gamma_q(\gamma+1)} \varsigma^\gamma) d\varsigma \\ &\leq E_q(\lambda\tau) \tau^{\gamma-1} \int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}}{\Gamma(\gamma)} I_{q,0} \chi(\varsigma, 1, 0) d\varsigma, \end{aligned}$$

$$\begin{aligned} \mathcal{A}(h, h)(\tau) + e_1 &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \chi(\varsigma, h(\varsigma), h(\varsigma)) d\varsigma \\ &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \chi(\varsigma, \varsigma^{\gamma-1}, -I_{q,0} \varsigma^{\gamma-1}) d\varsigma \\ &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \chi(\varsigma, \varsigma^{\gamma-1}, \frac{-\Gamma_q(\gamma)}{\Gamma_q(\gamma+1)} \varsigma^\gamma) d\varsigma \\ &\geq E_q(\lambda\tau) \tau^{\gamma-1} \int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1} - (1-\varsigma)^{\gamma-1}}{\Gamma(\gamma)} I_{q,0} \chi(\varsigma, \varsigma^{\gamma-1}, \frac{-\Gamma_q(\gamma)}{\Gamma_q(\gamma+1)} \varsigma^\gamma) d\varsigma \\ &\geq E_q(\lambda\tau) \tau^{\gamma-1} \int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1} - (1-\varsigma)^{\gamma-1}}{\Gamma(\gamma)} I_{q,0} \chi(\varsigma, 0, 1) d\varsigma. \end{aligned}$$



From  $(J'_1)$  and  $(J'_2)$ , we obtain:

$$\chi(\varsigma, 0, 1) \neq 0, \chi(\varsigma, 1, 0) \geq \chi(\varsigma, 0, 1).$$

So,  $\int_0^1 \chi(\varsigma, 1, 0) d\varsigma \geq \int_0^1 \chi(\varsigma, 0, 1) d\varsigma$ . Consequently,

$$\begin{aligned} E_q(\lambda\tau)\tau^{\gamma-1} \int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1} - (1-\varsigma)^{\gamma-1}}{\Gamma(\gamma)} I_{q,0}\chi(\varsigma, 0, 1) d\varsigma \\ \leq \mathcal{A}(h, h)(\tau) + e_1 \leq E_q(\lambda\tau)\tau^{\gamma-1} \int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}}{\Gamma(\gamma)} I_{q,0}\chi(\varsigma, 1, 0) d\varsigma. \end{aligned}$$

So, we define that:

$$\begin{aligned} L_1 &:= E_q(\lambda\tau)\tau^{\gamma-1} \int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1} - (1-\varsigma)^{\gamma-1}}{\Gamma(\gamma)} I_{q,0}\chi(\varsigma, 0, 1) d\varsigma \\ L_2 &:= E_q(\lambda\tau)\tau^{\gamma-1} \int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}}{\Gamma(\gamma)} I_{q,0}\chi(\varsigma, 1, 0) d\varsigma. \end{aligned}$$

Therefore, for  $L_1\tau^{\gamma-1} \leq \mathcal{A}(h, h)(\tau) + e_1 \leq L_2\tau^{\gamma-1}$ ,  $\tau \in O$  where  $h(\tau) = \tau^{\gamma-1}$ . Then, we obtain  $L_1h(\tau) \leq \mathcal{A}(h, h)(\tau) + e_1 \leq L_2h(\tau)$  that's mean,  $\mathcal{A}(h, h) + e_1 \in \rho_h$ . Thus, according to define  $\rho_{h, e_1}$ , we conclude that  $\mathcal{A}(h, h) \in \rho_{h, e_1}$ . By using of  $(J'_1)$ ,  $(J'_2)$  and Lemma 2.6, we get:

$$\begin{aligned} \mathcal{B}(h) + e_1 &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0}\varrho(\varsigma, h(\varsigma)) d\varsigma \\ &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0}\varrho(\varsigma, \varsigma^{\gamma-1}) d\varsigma \\ &\geq E_q(\lambda\tau)\tau^{\gamma-1} \int_0^1 \mathfrak{G}(\tau, \varsigma) \frac{(1-\varsigma)^{\gamma+\alpha-1} - (1-\varsigma)^{\gamma-1}}{\Gamma(\gamma)} I_{q,0}\varrho(\varsigma, 0) d\varsigma, \end{aligned}$$

$$\begin{aligned} \mathcal{B}(h) + e_1 &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0}\varrho(\varsigma, h(\varsigma)) d\varsigma \\ &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0}\varrho(\varsigma, \varsigma^{\gamma-1}) d\varsigma \\ &\leq E_q(\lambda\tau)\tau^{\gamma-1} \int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}}{\Gamma(\gamma)} I_{q,0}\varrho(\varsigma, 1) d\varsigma. \end{aligned}$$

Because  $\varrho(\varsigma, 0) \neq 0$  and  $\varrho(\varsigma, 1) \geq \varrho(\varsigma, 0)$ , consequently

$$\begin{aligned} E_q(\lambda\tau)\tau^{\gamma-1} \int_0^1 \mathfrak{G}(\tau, \varsigma) \frac{(1-\varsigma)^{\gamma+\alpha-1} - (1-\varsigma)^{\gamma-1}}{\Gamma(\gamma)} I_{q,0}\varrho(\varsigma, 0) d\varsigma \\ \leq \mathcal{B}(h)(\tau) + e_1 \leq E_q(\lambda\tau)\tau^{\gamma-1} \int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}}{\Gamma(\gamma)} I_{q,0}\varrho(\varsigma, 1) d\varsigma, \end{aligned}$$

that's mean,  $L_1h(\tau) \leq \mathcal{B}(h)(\tau) + e_1 \leq L_2h(\tau)$ . So, according to define  $\rho_{h, e_1}$ , we get  $\mathcal{B}(h) \in \rho_{h, e_1}$ . Similarly, since  $\phi(\varsigma, 1) \neq 0$ , we have:

$$\begin{aligned} E_q(\lambda\tau)\tau^{\gamma-1} \int_0^1 \mathfrak{G}(\tau, \varsigma) \frac{(1-\varsigma)^{\gamma+\alpha-1} - (1-\varsigma)^{\gamma-1}}{\Gamma(\gamma)} I_{q,0}\phi(\varsigma, 1) d\varsigma \\ \leq \mathcal{C}(h)(\tau) + e_1 \leq E_q(\lambda\tau)\tau^{\gamma-1} \int_0^1 \frac{(1-\varsigma)^{\gamma+\alpha-1}}{\Gamma(\gamma)} I_{q,0}\phi(\varsigma, 0) d\varsigma, \end{aligned}$$



that's mean,  $L_1 h(\tau) \leq \mathcal{C}(h)(\tau) + e_1 \leq L_2 h(\tau)$  and  $\mathcal{C}(h) + e_1 \in \rho_h$  consequently,  $\mathcal{C}(h) \in \rho_{h, e_1}$ . For any  $w, \vartheta \in \rho_{h, e_1}$  and every  $\tau \in O$  by using of  $(J'_5)$  there is a constant  $\delta > 0$  S.T

$$\begin{aligned} \mathcal{A}(w, \vartheta)(\tau) + e_1 &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \chi(\varsigma, w(\varsigma), \vartheta(\varsigma)) d\varsigma \\ &\geq \delta [E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} \{\varrho(\varsigma, w(\varsigma)) + \phi(\varsigma, \vartheta(\varsigma))\} d\varsigma] + (\delta - 1)e_1 \\ &= \delta [\mathcal{B}(w)(\tau) + \mathcal{C}(\vartheta)(\tau) + 2e_1] + (\delta - 1)e_1, \end{aligned}$$

so, we obtain

$$\mathcal{A}(w, \vartheta)(\tau) + e_1 \geq \delta [\mathcal{B}(w)(\tau) + \mathcal{C}(\vartheta)(\tau) + 2e_1] + (\delta - 1)e_1.$$

Finally, using 3.7, we conclude that there exists  $w_0, \vartheta_0 \in \rho_{h, e_1}$  and  $\epsilon \in (0, 1)$  S.T  $\epsilon \vartheta_0 \leq w_0 < E_q(\lambda\tau) \vartheta_0$  and

$$\begin{aligned} w_0(\tau) &\leq E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} [\chi(\varsigma, w_0(\varsigma), \vartheta_0(\varsigma)) + \varrho(\varsigma, w_0(\varsigma)) + \phi(\varsigma, \vartheta_0(\varsigma))] d\varsigma - 3e_1 \\ &\leq E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} [\chi(\varsigma, \vartheta_0(\varsigma), w_0(\varsigma)) + \varrho(\varsigma, \vartheta_0(\varsigma)) + \phi(\varsigma, w_0(\varsigma))] d\varsigma - 3e_1 \\ &\leq E_q(\lambda\tau) \vartheta_0. \end{aligned}$$

Operator equation  $w = \mathcal{A}(w, \vartheta) + \mathcal{B}(w) + \mathcal{C}(\vartheta) + 3e_1 = E_q(\lambda\tau) \mathbf{T}(w, \vartheta)$  has a unique solution q-EFP  $w^*$  in  $\rho_{h, e_1}$ . For any primitive value  $u_0, v_0 \in \rho_{h, e_1}$  by making recursive sequences

$$\begin{aligned} u_{\mathfrak{N}} &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} [\chi(\varsigma, u_{\mathfrak{N}-1}(\varsigma), v_{\mathfrak{N}-1}(\varsigma)) + \varrho(\varsigma, u_{\mathfrak{N}-1}(\varsigma)) + \phi(\varsigma, v_{\mathfrak{N}-1}(\varsigma))] d\varsigma - 3e_1, \\ v_{\mathfrak{N}} &= E_q(\lambda\tau) \int_0^1 \mathfrak{G}(\tau, \varsigma) I_{q,0} [\chi(\varsigma, v_{\mathfrak{N}-1}(\varsigma), u_{\mathfrak{N}-1}(\varsigma)) + \varrho(\varsigma, v_{\mathfrak{N}-1}(\varsigma)) + \phi(\varsigma, u_{\mathfrak{N}-1}(\varsigma))] d\varsigma - 3e_1, \\ \mathfrak{N} &= 1, 2, \dots \end{aligned}$$

We obtain  $u_{\mathfrak{N}} \rightarrow w^*$  and  $v_{\mathfrak{N}} \rightarrow w^*$  as  $\mathfrak{N} \rightarrow \infty$ . Therefore, conditions 3.8 is satisfied.  $\square$

#### 4. EXAMPLES

**Example 4.1.** We consider the following problem in the case where  $q > \frac{1}{2}$  and  $\lambda = 2$ .

$$\begin{aligned} D_q D_{0^+}^{\frac{5}{2}} \left( \frac{w(\tau) + 3e_1(\tau)}{E_q(2\tau)} \right) &= [w(\tau) + e_1(\tau)]^{\frac{1}{3} \cos \frac{\pi}{4} \tau} + \left[ \left( 1 - \frac{I_{q,0} w(\tau)}{\tau} \right) + e_1(\tau) \right]^{\frac{-1}{3} \sin \frac{\pi}{4} \tau} \\ &\quad + [w(\tau) + e_1(\tau)]^{\frac{1}{3} \sin \frac{\pi}{4} \tau} + [\vartheta(\tau) + e_1(\tau)]^{\frac{-1}{3} \sin \frac{\pi}{4} \tau}, \\ w(0) = w'(0) = e_1(0) = e_1'(0) &= 0, \\ I_{0^+}^{\frac{1}{2}} \left( \frac{w(1) + 3e_1(1)}{E_q(2)} \right) &= 0. \end{aligned} \tag{4.1}$$

In this problem  $\gamma = \frac{5}{2}$ ,  $\alpha = \frac{1}{2}$ ,  $\alpha(\tau) = \frac{1}{3} \cos \frac{\pi}{4} \tau$ ,  $\beta(\tau) = \sigma(\tau) = \frac{1}{3} \sin \frac{\pi}{4} \tau$  and  $\vartheta(\tau) = 1 - \frac{I_{q,0} w(\tau)}{\tau}$ . Thus, we have:

$$\begin{aligned} \chi(\tau, w, \vartheta) &= [w(\tau) + e_1(\tau)]^{\frac{1}{3} \cos \frac{\pi}{4} \tau} + [\vartheta(\tau) + e_1(\tau)]^{\frac{-1}{3} \sin \frac{\pi}{4} \tau}, \\ \varrho(\tau, w) &= [e_1(\tau)w(\tau) + e_1(\tau)]^{\frac{1}{3} \sin \frac{\pi}{4} \tau}, \\ \phi(\tau, \vartheta) &= [e_1(\tau)\vartheta(\tau) + e_1(\tau)]^{\frac{-1}{3} \sin \frac{\pi}{4} \tau}. \end{aligned}$$

Obviously, the functions  $\chi : O \times \Upsilon \times \Upsilon \rightarrow \Upsilon$  and  $\varrho, \phi : O \times \Upsilon \rightarrow \Upsilon$  are continuous with  $\chi(\tau, 0, 1) \neq 0$ ,  $\varrho(\tau, 0) \neq 0$  and  $\phi(\tau, 1) \neq 0$ . The function  $\chi(\tau, w, \vartheta)$  is increasing with respect to  $w \in \Upsilon$  for fixed  $\tau \in O$  and  $\vartheta \in \Upsilon$ , and function  $\chi$  is



decreasing with respect to  $\vartheta \in \Upsilon$  for fixed  $\tau \in O$  and  $w \in \Upsilon$ . The function  $\varrho(\tau, w)$  is increasing with respect to  $w \in \Upsilon$  for fixed  $\tau \in O$  and  $\phi(\tau, \vartheta)$  is decreasing with respect to  $\vartheta \in \Upsilon$  for fixed  $\tau \in O$ . For each  $\eta \in (0, 1)$ ,  $\tau \in O$ ,  $w, \vartheta \geq 0$ , we have

$$\begin{aligned} \chi(\tau, \eta w + (\eta - 1)e_1, \eta^{-1}\vartheta + (\eta^{-1} - 1)e_1) &= [e_1(\tau) (\eta w(\tau) + (\eta - 1)e_1) + e_1(\tau)]^{\frac{1}{3} \cos \frac{\pi}{4} \tau} \\ &\quad + [e_1(\tau) (\eta^{-1}\vartheta(\tau) + (\eta^{-1} - 1)e_1) + e_1(\tau)]^{\frac{-1}{3} \sin \frac{\pi}{4} \tau} \\ &\geq \eta^{\frac{1}{3}(\sin \frac{\pi}{4} \tau + \cos \frac{\pi}{4} \tau)} \{ [e_1(\tau)w(\tau) + e_1(\tau)]^{\frac{1}{3} \cos \frac{\pi}{4} \tau} \\ &\quad + [e_1(\tau)(\vartheta(\tau)) + e_1(\tau)]^{\frac{-1}{3} \sin \frac{\pi}{4} \tau} \} + (\eta^{\frac{1}{3}(\sin \frac{\pi}{4} \tau + \cos \frac{\pi}{4} \tau)} - 1)e_1, \end{aligned}$$

$$\begin{aligned} \varrho(\tau, \eta w + (\eta - 1)e_1) &= \{e_1(\tau) [\eta w(\tau) + (\eta - 1)e_1] + e_1(\tau)\}^{\frac{1}{3} \sin \frac{\pi}{4} \tau} \\ &\geq \eta^{\frac{1}{3} \sin \frac{\pi}{4} \tau} \left[ (e_1(\tau)w(\tau) + e_1(\tau))^{\frac{1}{3} \sin \frac{\pi}{4} \tau} \right] + (\eta^{\frac{1}{3} \sin \frac{\pi}{4} \tau} - 1)e_1, \end{aligned}$$

$$\begin{aligned} \phi(\tau, \eta^{-1}\vartheta + (\eta^{-1} - 1)e_1) &= [e_1(\tau) (\eta^{-1}\vartheta(\tau) + (\eta^{-1} - 1)e_1) + e_1(\tau)]^{\frac{-1}{3} \sin \frac{\pi}{4} \tau}, \\ &\geq \eta^{\frac{1}{3} \sin \frac{\pi}{4} \tau} \left[ (e_1(\tau)\vartheta(\tau) + e_1(\tau))^{\frac{-1}{3} \sin \frac{\pi}{4} \tau} \right] + (\eta^{\frac{1}{3} \sin \frac{\pi}{4} \tau} - 1)e_1. \end{aligned}$$

For  $w, \vartheta \geq 0$  there is a constant  $\delta = \frac{1}{6}$  S.T

$$\begin{aligned} \chi(\tau, w, \vartheta) &= [e_1(\tau)w(\tau) + e_1(\tau)]^{\frac{1}{3} \cos \frac{\pi}{4} \tau} + [e_1(\tau)\vartheta(\tau) + e_1(\tau)]^{\frac{-1}{3} \sin \frac{\pi}{4} \tau} \\ &\geq \frac{1}{6} \left[ (e_1(\tau)w(\tau) + e_1(\tau))^{\frac{1}{3} \sin \frac{\pi}{4} \tau} + (e_1(\tau)\vartheta(\tau) + e_1(\tau))^{\frac{-1}{3} \sin \frac{\pi}{4} \tau} \right] - \frac{5}{6}e_1 \\ &= \delta [\varrho(\tau, w) + \phi(\tau, \vartheta)] - \frac{5}{6}e_1. \end{aligned}$$

Therefore, we conclude that the condition of 3.7 is satisfied and 3.8 guarantees that the problem (4.1) has a unique q-EPS in  $\rho_{h, e_1}$ , where  $h(\tau) = \tau^{\frac{3}{2}}$ ,  $\tau \in O$ .

**Example 4.2.** Now, for a better understanding of the issue, we consider another example with a different lambda and q and check the existence and uniqueness of its positive answer.

We consider the following problem assuming that  $\lambda = \frac{3}{2}$  and  $q > \frac{1}{3}$ .

$$\begin{aligned} D_q D_{0+}^{\frac{7}{3}} \left( \frac{w(\tau) + 3e_1(\tau)}{E_q(\frac{3}{2}\tau)} \right) &= [w(\tau) + e_1(\tau)]^{\frac{1}{3}\tau} + \left[ \left( 1 - \frac{I_{q,0} w(\tau)}{\tau} \right) + e_1(\tau) \right]^{\frac{-1}{6}\tau} \\ &\quad + [w(\tau) + e_1(\tau)]^{\frac{1}{6}\tau} + [\vartheta(\tau) + e_1(\tau)]^{\frac{-1}{6}\tau}, \\ w(0) = w'(0) = e_1(0) = e_1'(0) &= 0, \\ I_{0+}^{\frac{4}{3}} \left( \frac{w(1) + 3e_1(1)}{E_q(\frac{3}{2})} \right) &= 0. \end{aligned} \tag{4.2}$$

In this problem  $\gamma = \frac{7}{3}$ ,  $\alpha = \frac{4}{3}$ ,  $\alpha(\tau) = \frac{1}{3}\tau$ ,  $\beta(\tau) = \sigma(\tau) = \frac{1}{6}\tau$  and  $\vartheta(\tau) = 1 - \frac{I_{q,0} w(\tau)}{\tau}$ . Thus, we have:

$$\begin{aligned} \chi(\tau, w, \vartheta) &= [w(\tau) + e_1(\tau)]^{\frac{1}{3}\tau} + [\vartheta(\tau) + e_1(\tau)]^{\frac{-1}{6}\tau} \\ \varrho(\tau, w) &= [e_1(\tau)w(\tau) + e_1(\tau)]^{\frac{1}{6}\tau} \\ \phi(\tau, \vartheta) &= [e_1(\tau)\vartheta(\tau) + e_1(\tau)]^{\frac{-1}{6}\tau}. \end{aligned}$$

Obviously, the functions  $\chi : O \times \Upsilon \times \Upsilon \rightarrow \Upsilon$  and  $\varrho, \phi : O \times \Upsilon \rightarrow \Upsilon$  are continuous with  $\chi(\tau, 0, 1) \neq 0$ ,  $\varrho(\tau, 0) \neq 0$  and  $\phi(\tau, 1) \neq 0$ . The function  $\chi(\tau, w, \vartheta)$  is increasing with respect to  $w \in \Upsilon$  for fixed  $\tau \in O$  and  $\vartheta \in \Upsilon$ , and function  $\chi$  is decreasing with respect to  $\vartheta \in \Upsilon$  for fixed  $\tau \in O$  and  $w \in \Upsilon$ . The function  $\varrho(\tau, w)$  is increasing with respect to  $w \in \Upsilon$



for fixed  $\tau \in O$  and  $\phi(\tau, \vartheta)$  is decreasing with respect to  $\vartheta \in \Upsilon$  for fixed  $\tau \in O$ . For each  $\eta \in (0, 1)$ ,  $\tau \in O$ ,  $w, \vartheta \geq 0$ , we have

$$\begin{aligned} \chi(\tau, \eta w + (\eta - 1)e_1, \eta^{-1}\vartheta + (\eta^{-1} - 1)e_1) &= [e_1(\tau)(\eta w(\tau) + (\eta - 1)e_1) + e_1(\tau)]^{\frac{1}{3}\tau} \\ &\quad + [e_1(\tau)(\eta^{-1}\vartheta(\tau) + (\eta^{-1} - 1)e_1) + e_1(\tau)]^{\frac{-1}{6}\tau} \\ &\geq \eta^{\frac{1}{2}\tau} \{ [e_1(\tau)w(\tau) + e_1(\tau)]^{\frac{1}{3}\tau} \\ &\quad + [e_1(\tau)(\vartheta(\tau) + e_1(\tau))]^{\frac{-1}{6}\tau} \} + (\eta^{\frac{1}{2}\tau} - 1)e_1, \end{aligned}$$

$$\begin{aligned} \varrho(\tau, \eta w + (\eta - 1)e_1) &= \{e_1(\tau)[\eta w(\tau) + (\eta - 1)e_1] + e_1(\tau)\}^{\frac{1}{6}\tau} \\ &\geq \eta^{\frac{1}{6}\tau} \left[ (e_1(\tau)w(\tau) + e_1(\tau))^{\frac{1}{6}} \right] + (\eta^{\frac{1}{6}\tau} - 1)e_1, \end{aligned}$$

$$\begin{aligned} \phi(\tau, \eta^{-1}\vartheta + (\eta^{-1} - 1)e_1) &= [e_1(\tau)(\eta^{-1}\vartheta(\tau) + (\eta^{-1} - 1)e_1) + e_1(\tau)]^{\frac{-1}{6}\tau}, \\ &\geq \eta^{\frac{1}{6}\tau} \left[ (e_1(\tau)\vartheta(\tau) + e_1(\tau))^{\frac{-1}{6}\tau} \right] + (\eta^{\frac{1}{6}\tau} - 1)e_1. \end{aligned}$$

For  $w, \vartheta \geq 0$  there is a constant  $\delta = \frac{1}{5}$  S.T

$$\begin{aligned} \chi(\tau, w, \vartheta) &= [e_1(\tau)w(\tau) + e_1(\tau)]^{\frac{1}{3}\tau} + [e_1(\tau)\vartheta(\tau) + e_1(\tau)]^{\frac{-1}{6}\tau} \\ &\geq \frac{1}{6} \left[ (e_1(\tau)w(\tau) + e_1(\tau))^{\frac{1}{6}\tau} + (e_1(\tau)\vartheta(\tau) + e_1(\tau))^{\frac{-1}{6}\tau} \right] - \frac{4}{5}e_1 \\ &= \delta [\varrho(\tau, w) + \phi(\tau, \vartheta)] - \frac{4}{5}e_1. \end{aligned}$$

Therefore, we conclude that the condition of 3.7 is satisfied and 3.8 guarantees that the problem (4.2) has a unique q-EPS in  $\rho_{h, e_1}$ , where  $h(\tau) = \tau^{\frac{4}{3}}$ ,  $\tau \in O$ .

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