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An efficient high-order compact finite difference scheme for Lane-Emden type equations

Reza Doostaki[*,1,2], **Mohammad Mehdi Hosseini**[1,2], **and Abbas Salemi**[1,2]
[1]Department of Applied Mathematics, Faculty of Mathematics and Computer, Shahid Bahonar University of Kerman, Kerman, Iran.
[2]Mahani Mathematical Research Center, Shahid Bahonar University of Kerman, Kerman, Iran.

Abstract

In this paper, an efficient high-order compact finite difference (HOCFD) scheme is introduced for solving generalized Lane-Emden equations. For nonlinear types, it is shown that a combined quasilinearization and HOCFD scheme gives excellent results while a few quasilinear iterations is needed. Then the proposed method is developed for solving the system of linear and nonlinear Lane-Emden equations. Some numerical examples are provided, and obtained results of the proposed method are then compared with previous well-established methods. The numerical experiments show the accuracy and efficiency of the proposed method.

Keywords. Lane-Emden equations, Compact finite difference scheme, Quasilinearization method, High-order accuracy.
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1. Introduction

The main objective in this paper is to find the numerical solution of generalized Lane-Emden equation

$$u''(x) + p(x)u'(x) + F(u,x) = h(x), \quad a \le x \le b, \tag{1}$$

with boundary conditions

$$u(a) = u_a, \quad u(b) = u_b, \tag{2}$$

and the system of Lane-Emden equations

$$u'' + p_1(x)u' + F_1(u,v) = h_1(x), \tag{3}$$

$$v'' + p_2(x)v' + F_2(u,v) = h_2(x), \tag{4}$$

with known boundary conditions $u(a_1, y)$, $u(b_1, y)$, $u(x, a_2)$ and $u(x, b_2)$ for $a_1 \le x \le b_1$ and $a_2 \le y \le b_2$.
The Lane-Emden equation was first studied by astrophysicists J. H. Lane and R. Emden [9] and is categorized as a singular initial value problem. A lot of researches containing both analytical as well as numerical techniques have been presented for the solution of Lane-Emden equations and system of Lane-Emden equations, such as Adomian decomposition method [30, 31], series solutions [25], wavelets methods [28, 33], differential transform method [12], Bernstein and Legendre operational matrix of differentiation [20, 21], rational Legendre pseudospectral approach [24], Homotopy analysis method [29], modified Adomian decomposition method [8], Hermit functions collocation method [23], B-spline expansion and collocation approach [13, 27], a Jacobi-Gauss collocation method [3],collocation method based on cubic Hermit spline functions [18], Chebyshev neural network based model approach [15], Picard-reproducing kernel Hilbert space method [1], generalized Chebyshev function methods [22], compact finite difference method [4], and Laguerre collocation method [35].
High-Order Compact Finite Difference (HOCFD) schemes [7, 14, 17] have been studied to approximate the function derivatives in grid points. The HOCFD schemes give high and better resolution characteristics as compared to classical

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∗ Corresponding author. Email: rdoostaki@math.uk.ac.ir; rdoostaki@yahoo.com.

finite difference schemes for the same number of grid points. This feature brings them closer to the spectral methods while the freedom in choosing the mesh geometry and the boundary conditions is maintained. Moreover, as compared to classical finite difference schemes, the HOCFD schemes have simpler stencil, less computation cost and higher efficiency. This paper is considered the HOCFD scheme for solving the generalized Lane-Emden equation and the system of Lane-Emden equations with known boundary conditions. In order to solve boundary value problems, we need to adjust HOCFD formulas with known boundary conditions. Hence, an HOCFD scheme is presented such that the function derivatives are considered only in grid points while the function values are known at boundary points. We show that applying the HOCFD scheme on a linear Lane-Emden equation leads to solving a linear system. The quasilinearization method (QLM) is an iterative method and was originally introduced by Bellman and Kalaba [2, 11] as a generalization of the Newton–Raphson method [5] to solve individual or systems of nonlinear ordinary and partial differential equations. Hence, in this paper, the nonlinear Lane-Emden equation is linearized by the qausilinearization method. Then, the proposed HOCFD method is used to solve the nonlinear Lane-Emden equations. However, the proposed method applied on nonlinear Lane-Emden equations yields to a linear system in every QLM iteration, but it is shown that a few iterations already provide suitable solutions. The second main goal in this paper is to extend the proposed method for solving system of Lane-Emden equations. The merit of the proposed method is simplicity in implementation, high accuracy and high convergence speed. The numerical experiments show the efficiency of the proposed method.

2. Compact finite difference scheme

Consider the function $u(x)$ on the interval $[a, b]$ with grid points

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b, \tag{5}$$

with equal distance $h = (b-a)/n$. The classical HOCFD schemes are implicit and have a form of [14, 17]

$$A_i^{(k)} \begin{pmatrix} u^{(k)}(x_0) \\ \vdots \\ u^{(k)}(x_n) \end{pmatrix} = B_i^{(k)} \begin{pmatrix} u(x_0) \\ \vdots \\ u(x_n) \end{pmatrix}, \tag{6}$$

where $A_i^{(k)}$ and $B_i^{(k)}$ are corresponding $(n+1)\times(n+1)$ matrices to the k-th derivative of $u(x)$ of order i. But for solving the boundary value problems, we have to obtain the HOCFD formulas as

$$A_i^{(k)} \begin{pmatrix} u^{(k)}(x_1) \\ \vdots \\ u^{(k)}(x_{n-1}) \end{pmatrix} = B_i^{(k)} \begin{pmatrix} u(x_1) \\ \vdots \\ u(x_{n-1}) \end{pmatrix} + b_i^{(k)}, \tag{7}$$

where $A_i^{(k)}$ and $B_i^{(k)}$ are the corresponding $(n-1)\times(n-1)$ matrices to the k-th derivative of $u(x)$ of order i, and the $(n-1)$ vector $b_i^{(k)}$ is a known vector contain the boundary values $u(x_0)$ and $u(x_n)$. Moreover, for simplicity in numerical computations and also stability of the method, we can choose the coefficient matrix $A_i^{(k)}$ as a symmetric diagonally dominant Toeplitz matrix. For this purpose, we use the method of undetermined coefficients which was introduced by Lele [14]. For first derivative of order 4 in the interior points (5), we have [14]

$$\frac{1}{4}u'_{i-1} + u'_i + \frac{1}{4}u'_{i+1} = \frac{3}{4h}(u_{i+1} - u_{i-1}), \quad i = 2, 3, \ldots, n-2.$$

For imposing the boundary conditions, we can write [7]

$$u'_1 + \frac{1}{4}u'_2 = \frac{1}{h}\left[a_0u_0 + a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4\right] + o(h^4),$$

$$\frac{1}{4}u'_{n-2} + u'_{n-1} = \frac{1}{h}\left[b_nu_n + b_{n-1}u_{n-1} + b_{n-2}u_{n-2} + b_{n-3}u_{n-3} + b_{n-4}u_{n-4}\right] + o(h^4).$$

By matching the Taylor expansion coefficients of both sides of the above equations, we have $A_4^{(1)}\mathbf{u}' = B_4^{(1)}\mathbf{u} + b_4^{(1)}$, where

$$A_4^{(1)} = \begin{pmatrix} 1 & \frac{1}{4} & & & \\ \frac{1}{4} & 1 & \frac{1}{4} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{4} & 1 & \frac{1}{4} \\ & & & \frac{1}{4} & 1 \end{pmatrix}, \quad B_4^{(1)} = \frac{1}{h}\begin{pmatrix} -1 & \frac{3}{2} & -\frac{1}{3} & \frac{1}{16} & \\ -\frac{3}{4} & 0 & \frac{3}{4} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{3}{4} & 0 & \frac{3}{4} \\ & -\frac{1}{16} & \frac{1}{3} & -\frac{3}{2} & 1 \end{pmatrix},$$

and

$$b_4^{(1)} = \frac{1}{h}\begin{pmatrix} -\frac{11}{48}u_0 & 0 & \dots & 0 & \frac{11}{48}u_n \end{pmatrix}^T.$$

Similar to above scheme, for the second derivative, we can write $A_4^{(2)}\mathbf{u}'' = B_4^{(2)}\mathbf{u} + b_4^{(2)}$ where

$$A_4^{(2)} = \begin{pmatrix} 1 & \frac{1}{10} & & & \\ \frac{1}{10} & 1 & \frac{1}{10} & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 1 & \frac{1}{10} \\ & & & \frac{1}{10} & 1 \end{pmatrix}, \quad B_4^{(2)} = \frac{1}{h^2}\begin{pmatrix} -\frac{23}{15} & \frac{1}{4} & \frac{7}{15} & -\frac{11}{120} & & & \\ \frac{6}{5} & -\frac{12}{5} & \frac{6}{5} & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & \frac{6}{5} & -\frac{12}{5} & \frac{6}{5} \\ & & & -\frac{11}{120} & \frac{7}{15} & \frac{1}{4} & -\frac{23}{15} \end{pmatrix},$$

and $b_4^{(2)} = \frac{1}{h^2}\begin{pmatrix} \frac{109}{120}u_0 & 0 & \dots & 0 & \frac{109}{120}u_n \end{pmatrix}^T$. Also $A_6^{(1)}\mathbf{u}' = B_6^{(1)}\mathbf{u} + b_6^{(1)}$, where

$$A_6^{(1)} = \begin{pmatrix} 1 & \frac{1}{3} & & & \\ \frac{1}{3} & 1 & \frac{1}{3} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{3} & 1 & \frac{1}{3} \\ & & & \frac{1}{3} & 1 \end{pmatrix}, \quad B_6^{(1)} = \frac{1}{h}\begin{pmatrix} -\frac{17}{12} & \frac{83}{36} & -\frac{11}{9} & \frac{2}{3} & -\frac{37}{180} & \frac{1}{36} & & \\ -\frac{7}{9} & 0 & \frac{7}{9} & \frac{1}{36} & & & & \\ -\frac{1}{36} & -\frac{7}{9} & 0 & \frac{7}{9} & \frac{1}{36} & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & -\frac{1}{36} & -\frac{7}{9} & 0 & \frac{7}{9} & \frac{1}{36} \\ & & & & -\frac{1}{36} & -\frac{7}{9} & 0 & \frac{7}{9} \\ & & -\frac{1}{36} & \frac{37}{180} & -\frac{2}{3} & \frac{11}{9} & -\frac{83}{36} & \frac{17}{12} \end{pmatrix},$$

and $b_6^{(1)} = \frac{1}{h}\begin{pmatrix} -\frac{7}{45}u_0 & -\frac{1}{36}u_0 & 0 & \dots & 0 & \frac{1}{36}u_n & \frac{7}{45}u_n \end{pmatrix}^T$. Also

$$A_6^{(2)}\mathbf{u}'' = B_6^{(2)}\mathbf{u} + b_6^{(2)},$$

where

$$A_6^{(2)} = \begin{pmatrix} 1 & \frac{2}{11} & & & \\ \frac{2}{11} & 1 & \frac{2}{11} & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 1 & \frac{2}{11} \\ & & & \frac{2}{11} & 1 \end{pmatrix},$$

$$B_6^{(2)} = \frac{1}{h^2}\begin{pmatrix}
-\frac{129}{220} & -\frac{81}{44} & \frac{557}{198} & -\frac{69}{44} & \frac{111}{220} & -\frac{39}{1980} & & & & \\
\frac{12}{11} & -\frac{51}{22} & \frac{12}{11} & \frac{3}{44} & & & & & & \\
\frac{3}{44} & \frac{12}{11} & -\frac{51}{22} & \frac{12}{11} & \frac{3}{44} & & & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\
 & & & & & & & & & \\
 & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & & \frac{3}{44} & \frac{12}{11} & -\frac{51}{22} & \frac{12}{11} & \frac{3}{44} \\
 & & & & & & \frac{3}{44} & \frac{12}{11} & -\frac{51}{22} & \frac{12}{11} \\
 & & & & -\frac{39}{1980} & \frac{111}{220} & -\frac{69}{44} & \frac{557}{198} & -\frac{81}{44} & -\frac{129}{220}
\end{pmatrix},$$

and

$$b_6^{(2)} = \frac{1}{h^2}\begin{pmatrix} \frac{1481}{1980}u_0 & \frac{3}{44}u_0 & 0 & \dots & 0 & \frac{3}{44}u_n & \frac{1481}{1980}u_n \end{pmatrix}^T.$$

Moreover, we have $A_8^{(1)}\mathbf{u}' = B_8^{(1)}\mathbf{u} + b_8^{(1)}$ where

$$A_8^{(1)} = \begin{pmatrix}
1 & \frac{4}{9} & \frac{1}{36} & & & & & \\
\frac{4}{9} & 1 & \frac{4}{9} & \frac{1}{36} & & & & \\
\frac{1}{36} & \frac{4}{9} & 1 & \frac{4}{9} & \frac{1}{36} & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & \frac{1}{36} & \frac{4}{9} & 1 & \frac{4}{9} & \frac{1}{36} \\
 & & & & \frac{1}{36} & \frac{4}{9} & 1 & \frac{4}{9} \\
 & & & & & \frac{1}{36} & \frac{4}{9} & 1
\end{pmatrix},$$

$$B_8^{(1)} = \frac{1}{h}\begin{pmatrix}
-\frac{481}{280} & \frac{1103}{360} & -\frac{1889}{720} & \frac{115}{48} & -\frac{317}{216} & \frac{641}{1080} & -\frac{239}{1680} & \frac{99}{6466} & & & \\
-\frac{26}{27} & \frac{7}{18} & \frac{2}{9} & \frac{65}{108} & -\frac{14}{45} & \frac{7}{54} & -\frac{2}{63} & \frac{1}{288} & & & \\
-\frac{25}{216} & -\frac{20}{27} & 0 & \frac{20}{27} & \frac{25}{216} & & & & & & \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & & & & \\
 & & & & & & & & & & \\
 & & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
 & & & & & & -\frac{25}{216} & -\frac{20}{27} & 0 & \frac{20}{27} & \frac{25}{216} \\
 & & & -\frac{1}{288} & \frac{2}{63} & -\frac{7}{54} & \frac{14}{45} & -\frac{65}{108} & -\frac{2}{9} & -\frac{7}{18} & \frac{26}{27} \\
 & & & -\frac{99}{6466} & \frac{239}{1680} & -\frac{641}{1080} & \frac{317}{216} & -\frac{115}{48} & \frac{1889}{720} & -\frac{1103}{360} & \frac{481}{280}
\end{pmatrix},$$

and

$$b_8^{(1)} = \frac{1}{h}\begin{pmatrix} -\frac{709}{6048}u_0 & -\frac{1217}{30240}u_0 & 0 & \dots & 0 & \frac{1217}{30240}u_n & \frac{709}{6048}u_n \end{pmatrix}^T.$$

Also, we have $A_8^{(2)}\mathbf{u}'' = B_8^{(2)}\mathbf{u} + b_8^{(2)}$ where

$$A_8^{(2)} = \begin{pmatrix} 1 & \frac{344}{11799} & \frac{23}{2358} & & & & & \\ \frac{344}{1179} & 1 & \frac{344}{1179} & \frac{23}{2358} & & & & \\ \frac{23}{2358} & \frac{344}{1179} & 1 & \frac{344}{1179} & \frac{23}{2358} & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & \frac{23}{2358} & \frac{344}{1179} & 1 & \frac{344}{1179} & \frac{23}{2358} \\ & & & & \frac{23}{2358} & \frac{344}{1179} & 1 & \frac{344}{1179} \\ & & & & & \frac{23}{2358} & \frac{344}{1179} & 1 \end{pmatrix},$$

$$B_8^{(2)} = \frac{1}{h^2}\begin{pmatrix} \frac{7685}{22008} & -\frac{1983151}{424440} & \frac{1630439}{212220} & \frac{44051}{6288} & \frac{1892837}{424440} & -\frac{156895}{84888} & \frac{6197}{13755} & -\frac{583309}{11884320} & & \\ \frac{44663}{41265} & -\frac{6887}{2620} & \frac{89269}{53055} & -\frac{12173}{18864} & \frac{1081}{1965} & -\frac{49289}{212220} & \frac{2369}{41265} & -\frac{2783}{440160} & & \\ \frac{155}{786} & \frac{320}{393} & -\frac{265}{131} & \frac{320}{393} & \frac{155}{786} & & & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\ & & & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & & \frac{155}{786} & \frac{320}{393} & -\frac{265}{131} & \frac{320}{393} & \frac{155}{786} \\ & & -\frac{2783}{440160} & \frac{2369}{41265} & -\frac{49289}{212220} & \frac{1081}{1965} & -\frac{12173}{18864} & \frac{89269}{53055} & -\frac{6887}{2620} & \frac{44663}{41265} \\ & & -\frac{583309}{11884320} & \frac{6197}{13755} & -\frac{156895}{84888} & \frac{1892837}{424440} & \frac{44051}{6288} & \frac{1630439}{212220} & -\frac{1983151}{424440} & \frac{7685}{22008} \end{pmatrix},$$

and

$$b_8^{(2)} = \frac{1}{h^2}\left(\frac{7525099}{11884320}u_0 \quad \frac{1664387}{11884320}u_0 \quad 0 \quad \dots \quad 0 \quad \frac{1664387}{11884320}u_n \quad \frac{7525099}{11884320}u_n \right)^T.$$

3. Fourier analysis of error

Fourier analysis provides an effective way to analysis the error of difference schemes and is widely used to obtain the dispersion and dissipation errors which quantifies the resolution characteristics of difference approximations [14]. We assume that $u(x)$ is periodic over the domain $[0, L]$, then it can be written as

$$u(x) = \sum_{k=-N/2}^{N/2} \hat{u}_k e^{\frac{2\pi lkx}{L}}, \tag{8}$$

where l is imaginary unit and N is the number of mesh points with domain step size $h = L/N$. By defining a scaled wave number $\omega = \frac{2\pi kh}{L} = \frac{2\pi k}{N}$ and a scaled coordinate $s = \frac{x}{h}$, the Fourier modes simplify to $e^{i\omega s}$. The domain of ω is then $[0, \pi]$. The exact first and second derivatives of (8) (with respect to s) provide a function with Fourier coefficients

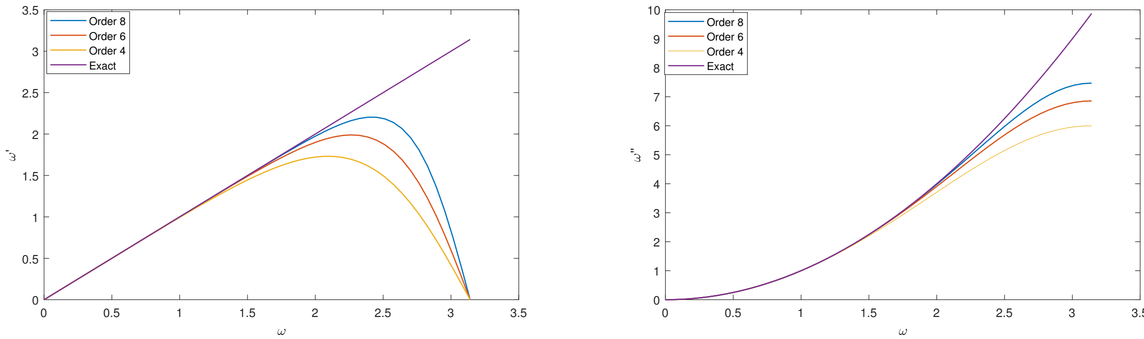
$$\hat{u}'_k = l\omega\hat{u}_k, \quad \hat{u}''_k = -\omega^2\hat{u}_k, \tag{9}$$

respectively. The difference errors of the first and second derivative schemes are obtained by comparing the Fourier coefficients of the derivatives obtained from the differencing schemes

$$(\hat{u}'_k)_{fd} = l\omega'\hat{u}_k, \quad (\hat{u}''_k)_{fd} = -\omega''\hat{u}_k,$$

with the exact Fourier coefficients (9), where ω' and ω'' represent the modified wavenumbers for tha first and second derivatives, respectively. The finite difference scheme for the first and second derivatives, correspond to functions $\omega'(\omega)$ and $\omega''(\omega)$, respectively. Spectral methods provides $\omega' = \omega$ for $\omega \neq \pi$ ($\omega' = 0$ for $\omega = \pi$) and also $\omega'' = \omega^2$. Hence difference between $\omega'(\omega)$ and ω, and also difference between $\omega''(\omega)$ and ω^2 are two measures of errors.
The compact finite difference schemes for the first derivative is corresponded to

$$\omega'(\omega) = \frac{a\sin(\omega) + (b/2)\sin(2\omega) + (c/3)\sin(3\omega)}{1 + 2\alpha\cos(\omega) + 2\beta\cos(2\omega)},$$



(A) (B)

Figure 1. The modified wavenumbers for fourth, sixth and eighth order compact finite difference schemes for first derivative approximation (A) and secod derivative approximation (B).

where

$$\begin{cases} \alpha = \frac{1}{4}, \beta = 0, a = \frac{3}{2}, b = 0, c = 0, & \text{for fourth order compact scheme,} \\ \alpha = \frac{1}{3}, \beta = 0, a = \frac{14}{9}, b = \frac{1}{9}, c = 0, & \text{for sixth order compact scheme,} \\ \alpha = \frac{4}{9}, \beta = \frac{1}{36}, a = \frac{40}{27}, b = \frac{25}{54}, c = 0, & \text{for eighth order compact scheme,} \end{cases}$$

Also for the second derivative, we can write

$$\omega''(\omega) = \frac{2a(1-\cos(\omega)) + (b/2)(1-\cos(2\omega)) + (2c/9)(1-\cos(3\omega))}{1 + 2\alpha\cos(\omega) + 2\beta\cos(2\omega)},$$

where

$$\begin{cases} \alpha = \frac{1}{10}, \beta = 0, a = \frac{6}{5}, b = 0, c = 0, & \text{for fourth order compact scheme,} \\ \alpha = \frac{2}{11}, \beta = 0, a = \frac{12}{11}, b = \frac{3}{11}, c = 0, & \text{for sixth order compact scheme,} \\ \alpha = \frac{344}{1179}, \beta = \frac{23}{2358}, a = \frac{320}{393}, b = \frac{310}{393}, c = 0, & \text{for eighth order compact scheme.} \end{cases}$$

Figure 1 shows the modified wavenumbers ω' and ω'' for the fourth, sixth and eighth compact finite difference schemes.

In the following, we use the HOCFD formulas (7) to solve the Lane-Emden Equation (1) with boundary conditions (2).

4. Lane-Emden equations

Consider the generalized Lane-Emden equation

$$u''(x) + p(x)u'(x) + F(u,x) = h(x), \quad a \le x \le b, \tag{10}$$

with boundary conditions

$$u(a) = u_a, \quad u(b) = u_b. \tag{11}$$

We discreitize the interval $[a,b]$ to $(n+1)$ grid points

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b, \tag{12}$$

with equal distance $h = (b-a)/n$. We consider the Lane-Emden Equation (10) in cases of linear and nonlinear function $F(u,x)$.

Table 1. The maximum absolute error and CPU time for Example 4.1.

h	Order 4		Order 6		Order 8	
	Error	CPU time	Error	CPU time	Error	CPU time
$h=\frac{1}{10}$	6.3297e-16	0.002068 s	1.5266e-16	0.002143 s	6.4705e-16	0.002377 s

4.1. **Linear Lane-Emden equations.** Without lost of generality, we consider the linear Lane-Emden Equation (10) as

$$u''(x)+p(x)u'(x)+g(x)u(x)=h(x), \quad a\le x\le b, \tag{13}$$

with known boundary conditions $u(a)=u_a$ and $u(b)=u_b$. Let us U be the vector of function values $u(x)$ in the interior grid points (12) as

$$U=\begin{pmatrix} u(x_1), & \dots, & u(x_{n-1}) \end{pmatrix}^T. \tag{14}$$

Then, the Equation (13) can be written in interior grid points (12) as

$$U''+diag_{i=1}^{n-1}(p(x_i))U'+diag_{i=1}^{n-1}(g(x_i))U=diag_{i=1}^{n-1}(h(x_i)), \tag{15}$$

where

$$diag_{i=1}^{n-1}(f(x_i))=\begin{pmatrix} f(x_1) & & \\ & \ddots & \\ & & f(x_{n-1}) \end{pmatrix}. \tag{16}$$

By using HOCFD formulas (7) for the first and second derivatives of U and substituting in (15), we have the following linear system

$$M^{(k)}U=R^{(k)}, \tag{17}$$

where $M^{(k)}$ and $R^{(k)}$ are the matrices corresponding to approximate solution U of order $k=4,6,8$ as

$$M^{(k)}=\left(A_2^{(k)}\right)^{-1}B_2^{(k)}+diag_{i=1}^{n-1}(p(x_i))\left(A_1^{(k)}\right)^{-1}B_1^{(k)}+diag_{i=1}^{n-1}(g(x_i)), \tag{18}$$

$$R^{(k)}=diag_{i=1}^{n-1}(h(x_i))-\left(A_2^{(k)}\right)^{-1}b_k^{(2)}-diag_{i=1}^{n-1}(p(x_i))\left(A_1^{(k)}\right)^{-1}b_k^{(1)}. \tag{19}$$

The linear system (17) gives approximate solution of the Lane-Emden Equation (13) of orders $k=4,6,8$. Here, we consider some linear Lane-Emden equations from the literature. Through this paper, all numerical experiments where done using MATLAB 2018a on a computer with configuration: Intel(R) Core(TM) i5-5300U @2.30 GHz processor. For the same spatial and time step sizes, the rate of convergence of the proposed method is defined as

$$R_{oc}=\frac{\log(\frac{Error_{new}}{Error_{old}})}{\log(\frac{h_{new}}{h_{old}})}.$$

Example 4.1. *[10, 31, 33] Consider the Lane-Emden Equation (13) on $x\in[0,1]$ with parameters $p(x)=\frac{8}{x}$, $g(x)=x$, $h(x)=-30x+44x^2-x^4+x^5$ and boundary conditions $u_0=u_1=0$. The exact solution of this boundary-value problem is $u(x)=x^4-x^3$. Table 1 shows the maximum absolute error and CPU times for $h=1/10$. It can be seen that the obtained solutions by the proposed method are in excellent agreement with the exact solution (up to machine epsilon).*

Example 4.2. *[1, 9, 10, 20–22, 30] We consider the linear Lane-Emden equation*

$$u''(x)+\frac{2}{x}u'(x)+u^m(x)=0, \quad 0\le x\le 1, \tag{20}$$

with known boundary conditions u_0 and u_1. This problem is modeled in thermal behavior of a spherical cloud of gas action under the mutual attraction of its molecules and subject to classical laws of thermodynamics.
For $m=0$, the Equation (20) is a version of (13) with parameters $p(x)=\frac{2}{x}$, $g(x)=0$, $h(x)=-1$. The exact solution

TABLE 2. The maximum absolute error and CPU time for $m = 0$ for Example 4.2 by the proposed method for given step size h.

h	Order 4		Order 6		Order 8	
	Error	CPU time	Error	CPU time	Error	CPU time
$h = \frac{1}{10}$	4.8850e-15	0.000331 s	3.4417e-15	0.000348 s	8.6597e-15	0.000542 s

TABLE 3. The maximum absolute error and CPU time for $m = 1$ for Example 4.2 by the proposed method with given step sizes h.

h	Order 4			Order 6			Order 8		
	Error	CPU time	R_{oc}	Error	CPU time	R_{oc}	Error	CPU time	R_{oc}
$h = \frac{1}{10}$	1.5540e-07	0.000416 s	–	7.0026e-10	0.000450 s	–	4.6463e-12	0.000496 s	–
$h = \frac{1}{20}$	4.5213e-09	0.000465 s	5.1	6.3736e-12	0.000468 s	6.7	4.6074e-14	0.000469 s	6.6
$h = \frac{1}{40}$	4.3651e-10	0.000489 s	3.3	4.3077e-14	0.000611 s	7.2	4.6629e-15	0.000624 s	3.3

TABLE 4. Comparison of the maximum absolute error for $m = 1$ in Example 4.2 acquired by the proposed method of order 8 with step size $h = \frac{1}{20}$ with previous works [10, 20, 21].

Proposed method	BOMD [21]	LOMD [20]	LDG [10]
4.6074e-14	5.0e-10	3.0e-07	1e-08

for this problem is $u(x) = 1 - \frac{x^2}{6}$. Table 2 gives the maximum absolute errors, CPU times and the rate of convergence by the proposed method and shows that the obtained solutions are in excellent agreement with the exact solution (up to machine epsilon). As shown in Equation (17), the proposed method for solving linear Lane-Emden equations leads to a linear system. But as it can be seen in (18), the coefficient matrix $M^{(k)}$ is depended to functions $p(x)$ and $g(x)$. Hence, for any linear Lane-Emden problem, we have to consider eigenvalues of the corresponding coefficient matrix $M^{(k)}$. For existing the numerical solution in this example, we plot the eigenvalues of coefficient matrix $M^{(k)}$ for orders $k = 4, 6, 8$ in Figure 2 that show the real parts of eigenvalues are negative.
Also, for case $m = 1$, the Equation (20) is a homogeneous version of (13) on $x \in [0, 1]$ with parameters $p(x) = \frac{2}{x}$, $g(x) = 1$, $h(x) = 0$. The exact solution for this problem is $\frac{sinx}{x}$. We set the initial condition as $u_0 = \lim_{x \longrightarrow 0} \frac{sinx}{x}$. Tables 3 shows the maximum absolute errors, CPU times and the rate of convergence by the proposed method. Moreover, for $m = 1$, comparing between the exact and approximate solutions by the proposed method for $h = \frac{1}{10}$ is shown in Figure 3. In Table 4, the maximum absolute error by the proposed method is compared to existing numerical methods in literature, such as Bernestien operational matrix of differentiation (BOMD) method from [21], Legendre operational matrix of differentiation (LOMD) method from [20] and local discontinuous Galerkin (LDG) method from [10].

4.2. **Non-Linear Lane-Emden equations.** Consider the following nonlinear Lane-Emden equation

$$u''(x) + p(x)u'(x) + F(u, x) = h(x), \quad a \leq x \leq b, \tag{21}$$

with boundary conditions $u(a) = u_a$ and $u(b) = u_b$. Without lost of generality, we consider $F(u, x) = F(u)$ as

$$F(u) = L(u) + N(u),$$

where L and N are linear and nonlinear operators, respectively. The quasilinearization method (QLM) was originally introduced by Bellman and Kalaba [2, 11] as a generalization of the Newton–Raphson method [5] to solve individual or systems of nonlinear ordinary and partial differential equations. It was shown that the difference between the exact solution $u(x)$ and rth iteration $u_r(x)$ of the QLM is decreasing quadratically and the QLM iterations converge uniformly to the exact solution [16]. It is important to stress that in view of the quadratic convergence of the QLM, convergence of two subsequent QLM iterations leads to convergence of the QLM iteration sequence to the exact solution. Also, Once the quasilinear iteration sequence at some interval starts to converge, it will always continue to do so. Unlike an asymptotic perturbation series, the QLM yield the required precision once a successful initial guess generates convergence after a few steps. In order to solve the nonlinear Lane-Emden equation (21), we linearize

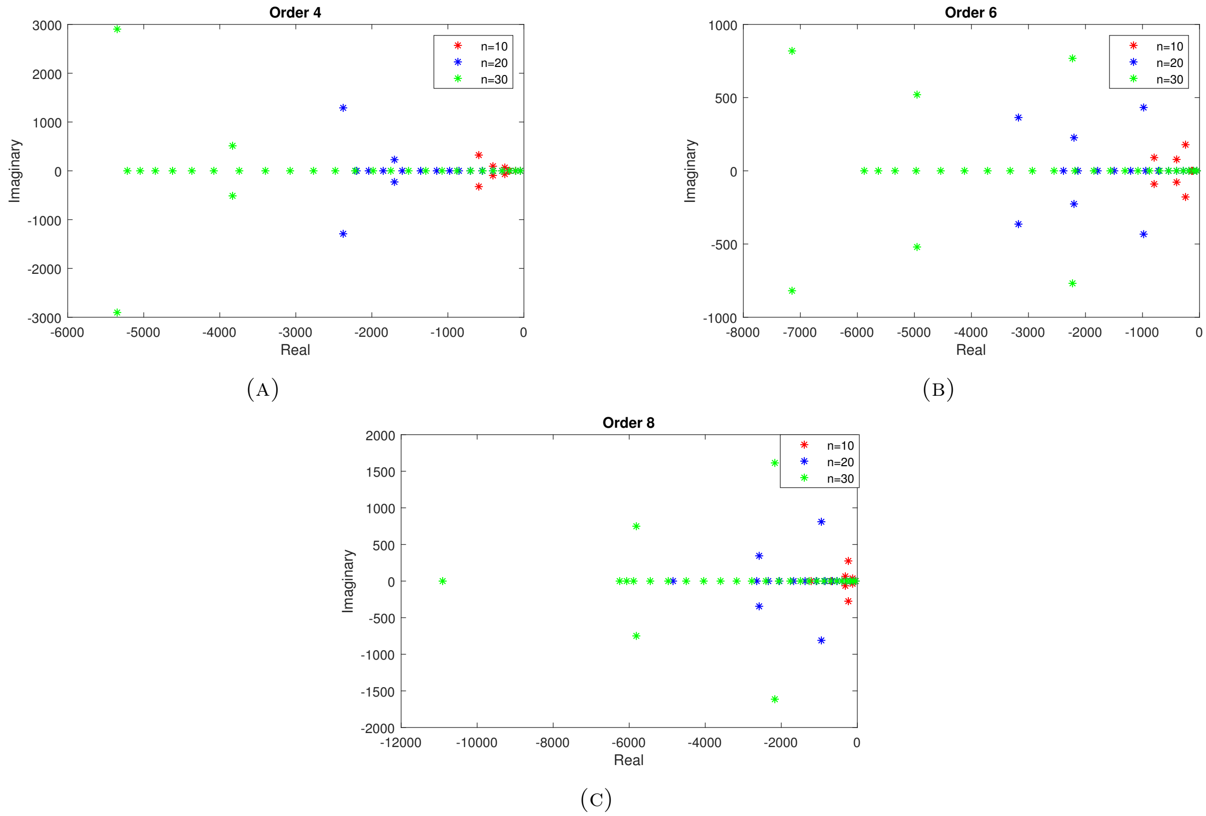


FIGURE 2. The eigenvalues of coefficient matrix by the proposed method for the case $m = 0$ in Example 4.2.

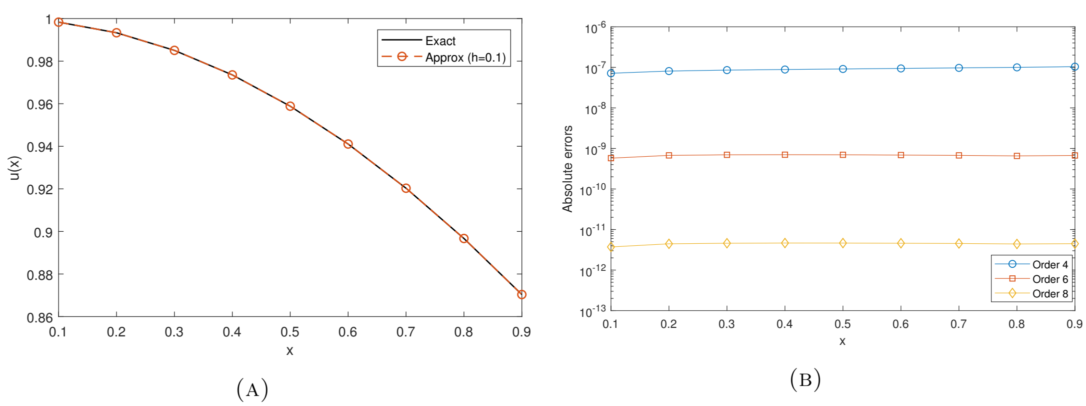


FIGURE 3. Comparing the obtained solution by the HOCFD and exact solutions (A) and the corresponding absolute errors (B) for $m = 1$ in Example 2 using $h = \frac{1}{10}$.

the nonlinear operator N by QLM. Then, the HOCFD scheme is used to solve the problem. Let us assume that the difference between $u_{r+1} - u_r$ is small, then we can approximate the nonlinear operator N using the linear terms of

Table 5. The maximum absolute error and CPU time for Example 4.3.

h	Order 4			Order 6			Order 8		
	Error	CPU time	R_{oc}	Error	CPU time	R_{oc}	Error	CPU time	R_{oc}
$h=\frac{1}{10}$	9.9914e-07	–	0.000487 s	4.5128e-08	0.000503 s	–	9.8868e-09	0.000515 s	–
$h=\frac{1}{20}$	3.3462e-08	0.000548 s	4.9	4.7650e-10	0.000612 s	6.5	6.7806e-11	0.000621 s	7.1
$h=\frac{1}{40}$	3.7025e-09	0.001609 s	3.1	5.1402e-12	0.002987 s	6.5	1.1535e-13	0.002998 s	9.1

Table 6. Comparison of the maximum absolute error by the proposed method of order 8 with step size $h=\frac{1}{20}$ for Example 4.3.

Proposed method	ADM [30]	BOMD [21]	LOMD [20]	LDG [10]
4.6074e-14	3.0000e-8	5.0e-10	3.0e-07	1e-08

Taylor series as

$$N(u) \approx N(u_r) + (\frac{\partial N}{\partial u})_r (u_{r+1} - u_r), \tag{22}$$

where r and $r+1$ denote previous and current iterations, respectively. Hence, we can write the nonlinear Lane-Emden equation (21) in current iteration $r+1$ as

$$u''_{r+1} + p(x)u'_{r+1} + g(x)L(u_{r+1}) + g(x)N(u_{r+1}) = h(x).$$

By (22) we have

$$u''_{r+1} + p(x)u'_{r+1} + g(x)L(u_{r+1}) + g(x)N(u_r) + g(x)(\frac{\partial N}{\partial u})_r (u_{r+1} - u_r) = h(x),$$

thus

$$u''_{r+1} + p(x)u'_{r+1} + g(x)\left(L(u_{r+1}) + (\frac{\partial N}{\partial u})_r u_{r+1}\right) = h(x) - g(x)N(u_r) + g(x)(\frac{\partial N}{\partial u})_r u_r. \tag{23}$$

Hence, by substituting the grid points and by using the HOCFD formulas (7) in above equation, the following linear system is derived in each iteration of QLM:

$$M^{(k)}U_{r+1} = R_r^{(k)}(U_r), \quad r = 0, 1, \dots \tag{24}$$

where $M^{(k)}$ and $R^{(k)}$ are the matrices corresponding to approximate solution U of order $k = 4, 6, 8$. Hence, by using an initial vector U_0, we can obtain the solutions of (23) for $r = 0, 1, \dots$. However, the HOCFD scheme with QLM for nonlinear Lane-Emden equation leads to an iterative method, but in numerical experiments, we show that a few iterations is needed to obtain a suitable solution.

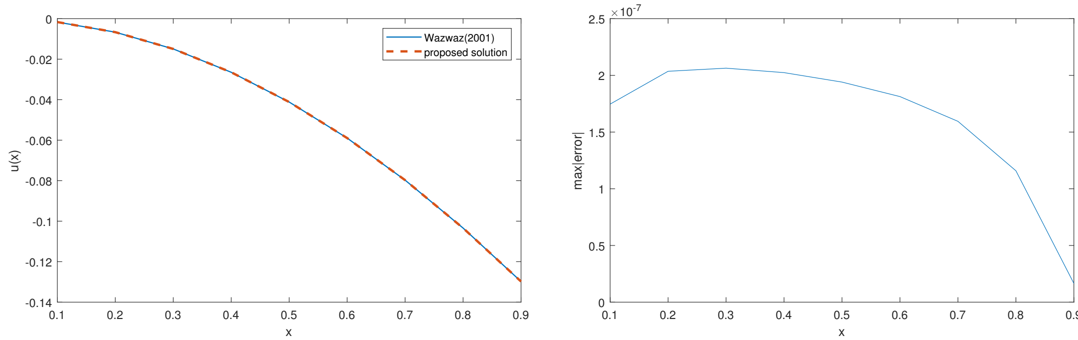
Example 4.3. *[1, 9, 10, 20–22] We consider the nonlinear Lane-Emden equation*

$$u''(x) + \frac{2}{x}u'(x) + u^5(x) = 0, \quad 0 \le x \le 1, \tag{25}$$

with known boundary conditions u_0 and u_1. The exact solution is $\frac{1}{\sqrt{1+\frac{x^2}{3}}}$. Table 5 shows the maximum absolute error and CPU time by the proposed method of orders 4, 6 and 8 for given step size h and iteration numbers $r = 5$. In Table 6, the maximum absolute error by the proposed method is compared to Adomian decomposition method (ADM) from [30], BOMD [21], LOMD [20] and LDG [10].

Example 4.4. *We consider the Isothermal gas spheres equation [6, 10, 20, 21, 30]*

$$u''(x) + \frac{2}{x}u'(x) + e^{u(x)} = 0, \quad 0 \le x \le 1, \tag{26}$$



(A) (B)

FIGURE 4. The graph of isothermal gas sphere equation in comparison with [30] (A), and corresponding maximum absolute error (B).

TABLE 7. The maximum absolute error and CPU time for Example 4.5.

h	Order 4			Order 6			Order 8		
	Error	CPU time	R_{oc}	Error	CPU time	R_{oc}	Error	CPU time	R_{oc}
$h=\frac{1}{10}$	3.1187e-04	0.000972 s	–	3.4750e-05	0.002730 s	–	4.5614e-06	0.003461 s	–
$h=\frac{1}{20}$	1.4932e-05	0.002224 s	4.3	5.3850e-07	0.002812 s	6.0	2.3554e-08	0.004372 s	7.5
$h=\frac{1}{40}$	5.8248e-07	0.005492 s	4.6	6.0244e-09	0.006216 s	6.4	7.6953e-11	0.006503 s	8.2

with boundary conditions u_0 and u_1. For this problem, the series solution using the ADM [30] is given as

$$u(x) \approx -\frac{x^2}{6} + \frac{x^4}{5 \times 4!} - \frac{8x^6}{21 \times 6!} + \frac{122x^8}{81 \times 8!} - \frac{61 \times 67x^{10}}{495 \times 10!} + \dots \tag{27}$$

The graph of isothermal gas sphere Equation (26) by the proposed method of order 4 for $h = \frac{1}{10}$ and $r = 5$ in comparison with approximate series solution (27) is shown in Figure 4(a). Also, Figure 4(b) shows the corresponding absolute error.

Example 4.5. *[23, 26, 34] Consider the nonlinear Lane-Emden equation*

$$u''(x) + \frac{2}{x}u'(x) - 6u(x) = 4u(x)\ln(u(x)), \quad 0 \le x \le 1, \tag{28}$$

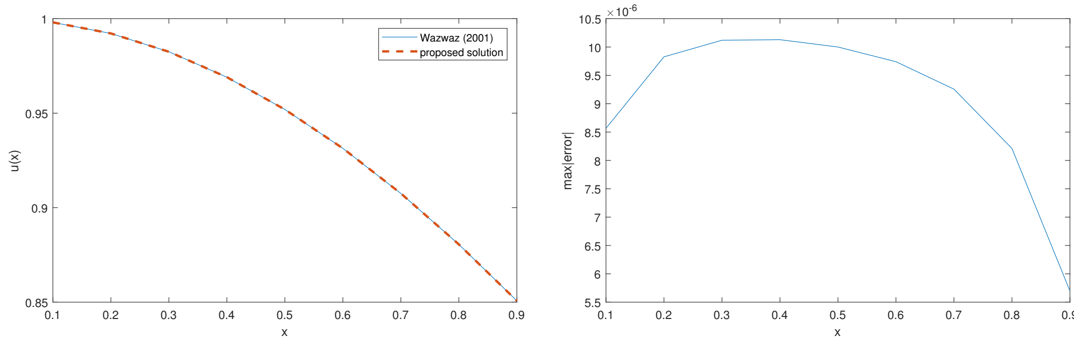
with known boundary conditions $u_0 = 1$ and $u_1 = e$, which has the exact solution $u(x) = e^{x^2}$. Table 7 shows the maximum absolute error, CPU time and the rate of convergence by the proposed method of orders 4, 6 and 8 for given step size h and iteration numbers $r = 5$.

Example 4.6. *This example corresponds to the following Lane-Emden equation [13, 23, 30]*

$$u''(x) + \frac{2}{x}u'(x) + \sinh(u(x)) = 0, \quad 0 \le x \le 1, \tag{29}$$

with boundary conditions u_0 and u_1. For this problem, the series solution using the ADM [30] is given as

$$\begin{aligned} u(x) \quad \approx \quad & 1 - \frac{(e^2-1)x^2}{12e} + \frac{(e^4-1)x^4}{480e^2} - \frac{(2e^6+3e^2-3e^4-2)x^6}{30240e^3} \\ & + \frac{(61e^8-104e^6+104e^2-61)x^8}{26127360e^4}. \end{aligned} \tag{30}$$



(A) (B)

FIGURE 5. Comparison between the solutions obtained by the proposed method and ADM [30] (A), and corresponding maximum absolute error (B).

TABLE 8. The maximum absolute error and CPU time for Example 5.1.

h	For	Order 4		Order 6		Order 8	
		Error	CPU time	Error	CPU time	Error	CPU time
$h=\frac{1}{10}$	$u(x)$	3.3079e-04	0.000822 s	3.6939e-05	0.001275 s	4.8520e-06	0.001279 s
	$v(x)$	1.1231e-05		1.4332e-06		1.8836e-07	
$h=\frac{1}{20}$	$u(x)$	1.5354e-05	0.000773 s	5.5503e-07	0.001465 s	2.4281e-08	0.001449 s
	$v(x)$	4.6211e-07		1.9751e-08		8.5071e-10	
$h=\frac{1}{40}$	$u(x)$	5.8976e-07	0.001556 s	6.1161e-09	0.002700 s	7.8109e-11	0.002973 s
	$v(x)$	1.3063e-08		2.1111e-10		2.6114e-12	

The graph of approximate solution of Equation (29) by the proposed method of order 4 for $h=\frac{1}{10}$ and $r=5$ in comparison with approximate series solution (30) is shown in Figure 5(a) . Also, Figure 5(b) shows the corresponding absolute error.

In next section, we will study the linear and nonlinear systems of Lane-Emden equations.

5. System of Lane-Emden equations

In this section, we give some examples for systems of Lane–Emden equations (linear and nonlinear) that show the proposed method can be easily extended to solve the system of Lane-Emden equations (linear and nonlinear).

Example 5.1. *We consider the non-homogeneous linear system of Lane–Emden equations, which describes polytropes in hydrostatic equilibrium as simple models of a star [19]*

$$u''+\frac{2}{x}u'-(4x^2+6)u+v = x^4-x^3, \tag{31}$$

$$v''+\frac{8}{x}v'+xv+u = e^{x^2}+x^5-x^4+44x^2-30x, \tag{32}$$

where $x,y\in[0,1]$ and u_0, u_1, v_0 and v_1 are known.

The exact solutions are $u(x)=e^{x^2}$ and $v(x)=x^4-x^3$. Table 8 shows the maximum absolute error with corresponding CPU time by the proposed method for given step sizes of orders 4, 6 and 8. Also, comparison between exact and approximate solutions for $u(x)$ and $v(x)$ are shown in Tables 9 and 10, respectively.

TABLE 9. Comparison between exact and approximate solution $u(x)$ with $h = \frac{1}{40}$ for Example 5.1.

x	Order 4	Order 6	Order 8	Exact
0.1	1.010050236235647	1.010050169393248	1.010050167112353	1.010050167084168
0.2	1.040810851077961	1.040810776633562	1.040810774222218	1.040810774192388
0.3	1.094174371892875	1.094174286285983	1.094174283736813	1.094174283705210
0.4	1.173510976475032	1.173510873759354	1.173510871025806	1.173510870991810
0.5	1.284025548102475	1.284025419706214	1.284025416724985	1.284025416687741
0.6	1.433329584541152	1.433329417910964	1.433329414601939	1.433329414560340
0.7	1.632316447425718	1.632316223740882	1.632316220002771	1.632316219955379
0.8	1.896481193204513	1.896480883656438	1.896480879360031	1.896480879304952
0.9	2.247908432063972	2.247907991762387	2.247907986741762	2.247907986676472

TABLE 10. Comparison between exact and aproximate solution $v(x)$ with $h = \frac{1}{40}$ for Example 5.1.

x	Order 4	Order 6	Order 8	Exact
0.1	-0.000899987752263	-0.000899999797720	-0.000899999997460	-0.0009
0.2	-0.006399987798657	-0.006399999801294	-0.006399999997508	-0.0064
0.3	-0.018899988027302	-0.018899999808291	-0.018899999997593	-0.0189
0.4	-0.038399988396586	-0.038399999818686	-0.038399999997721	-0.0384
0.5	-0.062499988959514	-0.062499999832999	-0.062499999997897	-0.0625
0.6	-0.086399989800275	-0.086399999851960	-0.086399999998131	-0.0864
0.7	-0.102899991050239	-0.102899999876557	-0.102899999998437	-0.1029
0.8	-0.102399992915670	-0.102399999908112	-0.102399999998832	-0.1024
0.9	-0.072899995723731	-0.072899999948388	-0.072899999999340	-0.0729

TABLE 11. The maximum absolute error and CPU time for Example 5.2.

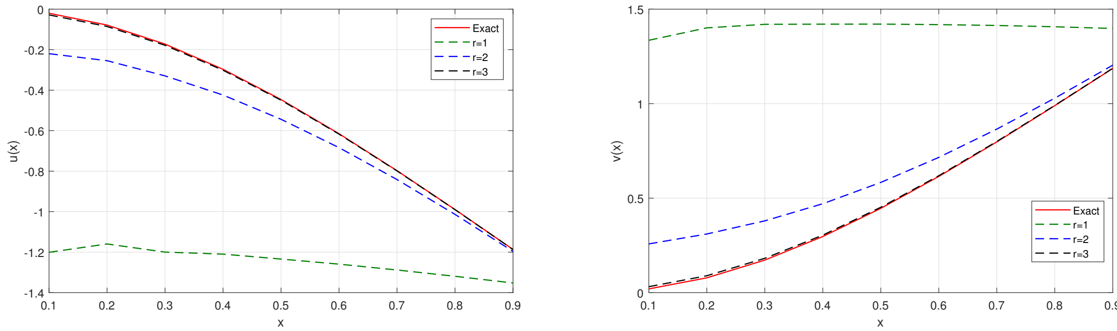
h	For	Order 4		Order 6		Order 8	
		Error	CPU time	Error	CPU time	Error	CPU time
$h = \frac{1}{10}$	$u(x)$	4.4490e-05	0.001405 s	1.414e-05	0.001464 s	7.5261e-06	0.001657 s
	$v(x)$	4.4848e-05		1.4955e-05		7.84283-06	
$h = \frac{1}{20}$	$u(x)$	2.4205e-06	0.002059 s	7.1466e-08	0.001638 s	1.0758e-09	0.001777 s
	$v(x)$	2.6123e-06		7.4963e-08		1.1073e-09	
$h = \frac{1}{40}$	$u(x)$	1.2389e-07	0.005400 s	3.5722e-10	0.005563 s	4.2613e-12	0.005691 s
	$v(x)$	1.3539e-07		3.9147e-10		3.7076e-12	

Example 5.2. *Consider the nonlinear system of Lane–Emden equations [32]*

$$u'' + \frac{5}{x}u' + 8(e^{u} + 2e^{-\frac{v}{2}}) = 0, \tag{33}$$

$$v'' + \frac{3}{x}v' - 8(e^{-v} + e^{\frac{u}{2}}) = 0, \tag{34}$$

where $x, y \in [0,1]$ and with known boundary conditions. The exact solutions are $u(x) = -2ln(1+x^2)$ and $v(x) = 2ln(1+x^2)$. Table 11 shows the maximum absolute error with corresponding CPU time by the proposed method for $r = 6$ and given step sizes. Also Figure 6 shows the convergence of the proposed method with increase in the iterations r of QLM for $h = 1/10$ for order 4. Moreover, comparison between the exact and approximate solutions for $r = 6$ and $h = \frac{1}{40}$ for $u(x)$ and $v(x)$ are shown in tables 12 and 13, respectively.



(A) (B)

FIGURE 6. Convergence of the proposed method with increase in the QLM iterations r for Example 5.2.

TABLE 12. Comparison between exact and approximate solution $u(x)$ with $h = \frac{1}{40}$ for Example 5.2.

x	Order 4	Order 6	Order 8	Exact
0.1	-0.019900543321239	-0.019900662050199	-0.019900661709269	-0.019900661706336
0.2	-0.078441325283406	-0.078441426659778	-0.078441426309448	-0.078441426306563
0.3	-0.172355314327941	-0.172355392838920	-0.172355392484839	-0.172355392482105
0.4	-0.296839954615990	-0.296840010580882	-0.296840010239055	-0.296840010236547
0.5	-0.446287065352555	-0.446287102944218	-0.446287102630656	-0.446287102628420
0.6	-0.614969375024299	-0.614969399773190	-0.614969399497878	-0.614969399495921
0.7	-0.797552223180955	-0.797552240150563	-0.797552239916426	-0.797552239914736
0.8	-0.989392470901493	-0.989392483868766	-0.989392483673663	-0.989392483672214
0.9	-1.186653679338796	-1.186653690717454	-1.186653690556702	-1.186653690555469

TABLE 13. Comparison between exact and approximate solution $v(x)$ with $h = \frac{1}{40}$ for Example 5.2.

x	Order 4	Order 6	Order 8	Exact
0.1	0.019900529460760	0.019900662085841	0.019900661709699	0.019900661706336
0.2	0.078441311559095	0.078441426696000	0.078441426309791	0.078441426306563
0.3	0.172355301438630	0.172355392872056	0.172355392485144	0.172355392482105
0.4	0.296839943070497	0.296840010610382	0.296840010239320	0.296840010236547
0.5	0.446287055568406	0.446287102969845	0.446287102630880	0.446287102628420
0.6	0.614969367273717	0.614969399794541	0.614969399498056	0.614969399495921
0.7	0.797552217566634	0.797552240167020	0.797552239916556	0.797552239914736
0.8	0.989392467377548	0.989392483879650	0.989392483673740	0.989392483672214
0.9	1.186653677756701	1.186653690722175	1.186653690556727	1.186653690555469

Example 5.3. *Consider the nonlinear systems of Lane–Emden equations [32]*

$$u'' + \frac{8}{x}u' + (18u - 4ln(v)) = 0, \tag{35}$$

$$v'' + \frac{4}{x}v' + (4vln(u) - 10v) = 0, \tag{36}$$

where $x, y \in [0, 1]$ and with known boundary conditions. The exact solutions are $u(x) = e^{-x^2}$ and $v(x) = e^{x^2}$. Table 14 shows the maximum absolute error with corresponding CPU time by the proposed method for $r = 6$ and given step sizes.

Table 14. The maximum absolute error and CPU time for Example 5.3.

h	For	Order 4		Order 6		Order 8	
		Error	CPU time	Error	CPU time	Error	CPU time
$h=\frac{1}{10}$	$u(x)$ $v(x)$	5.5210e-05 3.4417e-04	0.001422 s	1.4777e-06 3.9047e-05	0.001508 s	1.2227e-06 5.0273e-06	0.001592 s
$h=\frac{1}{20}$	$u(x)$ $v(x)$	6.4219e-07 1.5732e-05	0.002168 s	6.8614e-08 5.6870e-07	0.003951 s	1.8672e-09 2.4940e-08	0.004517 s
$h=\frac{1}{40}$	$u(x)$ $v(x)$	3.0090e-08 5.9655e-07	0.007460 s	7.7963e-010 6.1932e-09	0.007847 s	6.7902e-012 7.9213e-11	0.007945 s

6. Conclusions

In this paper, we have considered an efficient high-order compact finite difference (HOCFD) scheme for solving generalized Lane-Emden and system of Lane-Emden equations. For nonlinear types, it is shown that a combined quasilinearization and HOCFD scheme gives excellent results while a few quasilinear iterations is needed. Some numerical examples have been provided, and obtained results of the proposed method have been compared with previous well-established methods. The numerical experiments with low CPU time show the accuracy and efficiency of the proposed method.

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