Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 13, No. 1, 2025, pp. 25-40
DOI:10.22034/cmde.2024.58138.2452

A Green's function-based computationally efficient approach for solving a kind of nonlocal BVPs

Jyoti and Mandeep Singh*
Department of Mathematics, JUIT Solan, Waknaghat–173234, Himachal Pradesh, India.

Abstract

This study attempts to find approximate numerical solutions for a kind of second-order nonlinear differential problem subject to some Dirichlet and mixed-type nonlocal (specifically three-point) boundary conditions, appearing in various realistic physical phenomena, such as bridge design, control theory, thermal explosion, thermostat model, and the theory of elastic stability. The proposed approach offers an efficient and rapid solution for addressing the inherent complexity of nonlinear differential problems with nonlocal boundary conditions. Picard's iterative technique and quasilinearization method are the basis for the proposed coupled iterative methodology. In order to convert nonlinear boundary value problems to linearized form, the quasilinearization approach (with convergence controller parameters) is implemented. Making use of Picard's iteration method with the assistance of Green's function, an equivalent integral representation for the linearized problems is derived. Discussion is also had over the proposed method's convergence analysis. In order to determine its efficiency and effectiveness, the coupled iterative technique is tested on some numerical examples. Results are also compared with the existing techniques and documented (in terms of absolute errors) to validate the accuracy and precision of the proposed iterative technique.

Keywords. Nonlocal conditions, Three-point boundary value problem, Quasilinearization method, Green's function, Convergence analysis, Picard's iteration method.

2010 Mathematics Subject Classification. 65L10, 34B10, 34K28.

1. Introduction

Non-classical boundary conditions, which occur when data cannot be measured directly at the boundary, are being used to study a range of real-world events. In several situations, it is often preferable to have more information regarding the evolution of the investigated process. By replacing the local conditions with nonlocal ones, a differential model produces better results and measurements than the local ones. The formulation of differential problems with nonlocal boundary conditions (NBCs) can be classified as "nonlocal" boundary value problems (BVPs). In nonlocal BVPs, the solution at the boundary points is connected to the solution within the given domain.

The study of nonlocal (especially multi-point/three-point) BVPs has grown rapidly over the past few decades. It is not just a theoretical interest that propels the study of this type of problems, but also the fact that many real-world physical phenomena that appear in various fields can be modeled by them. Such as optimal bridge designing, control theory, the process of heat conduction, the chemical diffusion process, underground water flow, the fluid flow in the porous medium, thermal explosion, etc. [1, 11, 15, 18, 22, 28, 40, 46]. Studies reveal that, in many cases, nonlocal BVPs are often the most scientifically rational choice for mathematical modeling of diverse biological and physical processes. For instance, Infante and Webb [19] investigated the existence of solutions to the following nonlinear nonlocal boundary value problems, which occur in a thermostat model

$$-u''(x) = f(x, u(x)), \quad x \in (0,1), \tag{1.1}$$

Received: 25 August 2023 ; Accepted: 09 February 2024.
∗ Corresponding author. Email: mandeep04may@yahoo.in .

having nonlocal BCs

$$u'(0) = 0, \quad \beta u'(1) + u(\eta) = 0, \quad \eta \in [0, 1], \tag{1.2}$$

where $\beta > 0$ and f is a non-negative function. This work was inspired by Guidotti and Merino's [14] prior work, where they discussed a two-point linear model for a stationary state of a heated bar of unit length. They placed a point sensor at one end at $x = 0$ and controlled the flow of heat at the other end as per the feedback received by the sensor (which was placed at $x = 0$). Infante and Webb [19] investigated the nonlocal heat flow problem (1.1)-(1.2) with a point sensor located at an arbitrary point η rather than 0. The authors observed that the nonlocal variant of the thermostat model provides more realistic outcomes in comparison to the idealistic model, as the sensor may now cover all the interval points and not give a uniform response (for more, see [43, 44]). Moreover, by using the nonlocal differential BVPs with deviated arguments, Cabada et al. [12] have analyzed the phenomenon of a heating or cooling system in a thermostat model. By solving nonlocal differential models, one can gain deeper insights into the underlying mechanisms and intrinsic behavior of complex physical phenomena.

In the last few decades, numerous researchers have investigated (analytically and numerically) the differential equations with a variety of nonlocal conditions. In the early 1960s, Beals [4, 5] coined the term nonlocal boundary value problems while studying elliptic differential equations with non-classical boundary values. Bitsadze [6] examined nonlocal BVPs, and subsequently, Bitsadze and Samarskii [7] employed integral equation theory to study nonlocal elliptic equations within rectangular spatial domains. The existence and uniqueness of solutions for the one-dimensional parabolic equations have been examined by authors in [10, 21], whereas a class of second-order singular hyperbolic problems with nonlocal conditions is studied by Mesloub and Bouziani [27]. Il'in and Moiseev [20] initiated the study of linear second-order nonlocal ordinary differential equations with multi-point BCs. Gupta [16, 17] has conducted an extensive investigation on the existence and uniqueness of the solutions for nonlinear second-order multi-point BVPs. Making use of the monotone iterative technique with the presence of upper-lower solutions, the regions of existence for the solutions of nonlinear non-singular/singular second-order three-point BVPs have been discussed by Singh and his co-authors [34, 35, 41, 42] (for more literature, one can see [9, 29, 32]). Moreover, a variety of numerical techniques have also been employed by several researchers to solve the second-order nonlinear BVPs with three-point BCs, like Das et al. [13] applied the homotopy perturbation method whereas a decomposition method has been applied by authors in [38] to solve three-point BVPs. Ali et al. [3] employed the optimal homotopy asymptotic method while Abd-Elhameed et al. [2] utilized the wavelets collocation method using third and fourth kinds of Chebyshev polynomials for multi-point/three-point BVPs. Zhong and his co-authors [48] solved the nonlinear three-point BVPs with variable coefficient by using the properties of the integration method and Green's function. A uniform Haar wavelet collocation method for three-point BVPs was examined by Swati et al. [36]. Some recent studies on the nonlocal boundary value problems can be found in [24–26, 31, 37, 45, 47].

This study introduces a Green's function-based computationally efficient coupled iterative technique (in the presence of convergence controller parameter h) for a kind of nonlinear second-order differential problem with nonlocal three-point BCs. The following class of second-order nonlinear three-point boundary value problems (NTPBVPs) is taken into consideration

$$u''(x) + g(x, u(x)) = 0, \quad x \in [0, 1], \tag{1.3}$$

where $g : [0, 1] \times R \longrightarrow R$ is a real-valued continuous function and $u(x)$ is the unknown to be computed. The three-point BCs are taken as one of the following two types:

$$\text{Type I:} \quad u(0) = a, \quad u(1) + \delta u(\eta) = \beta, \quad 0 < \eta < 1, \tag{1.4}$$

$$\text{Type II:} \quad u'(0) = 0, \quad u(1) + \delta u(\eta) = \beta, \quad 0 < \eta < 1, \tag{1.5}$$

where a, δ, η and β are the arbitrary constants. We establish an iterative methodology that utilizes quasilinearization and Picard's iteration techniques to approximate the solution of nonlinear BVPs (1.3) with three-point BCs (1.4) and (1.5), which has several advantages

- It has been effectively applied to solve a class of three-point BVPs that appears in different physical phenomena where traditional boundary conditions are insufficient to represent complex real-world systems.

- The technique uses the quasilinearization method, which approximates a nonlinear problem by decomposing it into a sequence of linear problems, making it applicable to wide range of scientific, engineering, and mathematical contexts.
- Constructing Green's functions for multi-point BVPs can be a complex process, but the proposed methodology has streamlined the construction of Green's function by simplifying the integral transformation of the three-point BVPs.
- The convergence controller parameter is embedded in the proposed technique, which helps to accelerate the convergence of the technique.
- The developed technique is straightforward, demands minimal computational resources for excellent precision, converges fast, and outperforms numerous methods found in existing literature [36, 48, 49] in terms of accuracy.

This paper is being drafted in the following way: In section 2, we introduce a coupled iterative technique depending on Picard's iteration and quasilinearization method (with controller parameters) for a kind of NTPBVPs (1.3) with BCs (1.4), and (1.5). Moreover, the conversion of the BVPs to the corresponding integral transformation includes the formation of Green's function as well. Section 3, examines the convergence analysis of the proposed iterative scheme. Some numerical examples are provided to demonstrate the efficacy and applicability of the iterative scheme in section 4. Additionally, this section contrasts the numerical outcomes obtained by the proposed iterative scheme with existing techniques.

2. Coupled iterative technique: NTPBVPs

This section provides an in-depth description of the suggested iterative approach by utilizing quasilinearization and Picard's method for NTPBVPs (1.3) is subject to BCs (1.4). An equivalent integral equation is established in terms of Green's function. Consider the following differential equation

$$u''(x) + g(x, u(x)) = 0, \quad 0 \le x \le 1, \tag{2.1}$$

with three-point BCs

$$u(0) = a, \quad u(1) + \delta u(\eta) = \beta, \quad \eta \in (0, 1), \tag{2.2}$$

provided $\delta\eta \neq -1$. The nonlinear term $g(x, u)$ is continuous and $\frac{\partial g}{\partial u}$ exists and is also continuous. By employing optimal quasilinearization method (with controller parameter h), we expressed the Equations (2.1)-(2.2) as follows (see [39])

$$u''_{r+1} + hg'(u_r)(u_{r+1} - u_r) + g(u_r) = 0, \tag{2.3}$$

$$u_{r+1}(0) = a, \quad u_{r+1}(1) + \delta u_{r+1}(\eta) = \beta, \quad \eta \in (0, 1), \tag{2.4}$$

where $r = 0, 1, 2 \ldots,$ signifies the rth approximation to quasilinearization and $g'(u) = \frac{\partial g}{\partial u}$. It should be observed that for $h = 1$, the above iterative scheme (2.3) is referred to as Newton's quasilinearization method [8]. To find the solution of the linear differential Eq. (2.3) with boundary conditions (2.4), we convert it to the equivalent Fredholm integral equation. Let us define a function as

$$F = h\{u''_{r+1} + hg'(u_r)(u_{r+1} - u_r) + g(u_r)\} = 0. \tag{2.5}$$

By adding and subtracting u''_{r+1} from Eq. (2.5), we obtain

$$u''_{r+1} + F - u''_{r+1} = 0, \tag{2.6}$$

or,

$$u''_{r+1} = -F + u''_{r+1}. \tag{2.7}$$

To convert Eq. (2.7) into the corresponding Fredholm integral transformation, we integrate Eq. (2.7) twice from 0 to x and using the BC at $x = 0$, which gives

$$u_{r+1}(x) = a + xu'_{r+1}(0) + \int_0^x \int_0^s \left(-F + u''_{r+1}\right) dt ds. \tag{2.8}$$

By using the properties of integration, Eq. (2.8) gives

$$u_{r+1}(x) = a + xu'_{r+1}(0) + \int_0^x (x-t)(-F + u''_{r+1})dt. \tag{2.9}$$

Making use of BC (2.4) (i.e., $u_{r+1}(1) + \delta u_{r+1}(\eta) = \beta$), the value of $u'_{r+1}(0)$ is determined and we have

$$\begin{aligned} u_{r+1}(x) = a + \frac{(\beta - a(1+\delta))x}{1+\delta\eta} - \frac{x\delta}{1+\delta\eta}\int_0^\eta (\eta - t)(-F + u''_{r+1})dt \\ - \frac{x}{1+\delta\eta}\int_0^1 (1-t)(-F+u''_{r+1})dt + \int_0^x (x-t)(-F+u''_{r+1})dt. \end{aligned} \tag{2.10}$$

2.1. **Constructing Green's function.** Based on the η position, this subsection develops Green's function, for the three-point linear BVPs (2.3)-(2.4). Now, we can divide Eq. (2.10) into two cases in the following manner:
When $x \leq \eta$

$$\begin{aligned} u_{r+1}(x) &= a + \frac{(\beta - a(1+\delta))x}{1+\delta\eta} + \int_0^x (x-t)(-F+u''_{r+1})dt \\ &- \frac{x}{1+\delta\eta}\left(\int_0^x (1-t)(-F+u''_{r+1})dt + \int_x^\eta (1-t)(-F+u''_{r+1})dt + \int_\eta^1 (1-t)(-F+u''_{r+1})dt\right) \\ &- \frac{x\delta}{1+\delta\eta}\left(\int_0^x (\eta-t)(-F+u''_{r+1})dt + \int_x^\eta (\eta-t)(-F+u''_{r+1})dt\right). \end{aligned} \tag{2.11}$$

When $x \geq \eta$

$$\begin{aligned} u_{r+1}(x) &= a + \frac{(\beta - a(1+\delta))x}{1+\delta\eta} - \frac{x\delta}{1+\delta\eta}\int_0^\eta (\eta-t)(-F+u''_{r+1})dt \\ &- \frac{x}{1+\delta\eta}\left(\int_0^\eta (1-t)(-F+u''_{r+1})dt + \int_\eta^x (1-t)(-F+u''_{r+1})dt + \int_x^1 (1-t)(-F+u''_{r+1})dt\right) \\ &+ \left(\int_0^\eta (x-t)(-F+u''_{r+1})dt + \int_\eta^x (x-t)(-F+u''_{r+1})dt\right). \end{aligned} \tag{2.12}$$

Now, Equations (2.5), (2.11), and (2.12) allow us to formulate the combined form in the term of Green's function as

$$u_{r+1}(x) = a + \frac{(\beta - a(1+\delta))x}{1+\delta\eta} + \int_0^1 G(x,t)(u''_{r+1} - h[u''_{r+1} + hg'(u_r)(u_{r+1} - u_r) + g(u_r)])dt, \tag{2.13}$$

where $1 + \delta\eta \neq 0$ and $G(x,t)$ is the following Green's function

$$G(x,t) = \begin{cases} \dfrac{-t[(1-x) + \delta(\eta - x)]}{1+\delta\eta}, & 0 \leq t \leq \min\{x,\eta\} \leq 1, \\ \dfrac{-t(1-x) + \delta\eta(x-t)}{1+\delta\eta}, & 0 < \eta \leq t \leq x \leq 1, \\ \dfrac{-x[(1-t) + \delta(\eta - t)]}{1+\delta\eta}, & 0 \leq x \leq t \leq \eta < 1, \\ \dfrac{-x(1-t)}{1+\delta\eta}, & 0 \leq \max\{x,\eta\} \leq t \leq 1. \end{cases} \tag{2.14}$$

Now, by utilizing BCs (2.4) and two times integrating the first term in the integrand of the iterative scheme (2.13), we have

$$u_{r+1}(x) = u_{r+1}(x) - h\int_0^1 G(x,t)[u''_{r+1} + hg'(u_r)(u_{r+1} - u_r) + g(u_r)]dt. \tag{2.15}$$

The aforementioned integral representation (2.15) is approximated using Picard's method to establish the proposed coupled iterative technique for solving NTPBVPs (2.1) with BCs (2.2), which gives

$$u_{r+1}^{(p+1)}(x) = u_{r+1}^{(p)}(x) - h\int_0^1 G(x,t)[u_{r+1}^{\prime\prime(p)} + hg'(u_r)(u_{r+1}^{(p)} - u_r) + g(u_r)]dt, \tag{2.16}$$

where $p = 0, 1, 2, \ldots,$ is the pth approximation of Picard's iteration method. The best possible value for the convergence controller parameter h is computed by minimizing the discrete average residual norm as

$$Res(h) \approx \frac{1}{M+1}\sum_{i=0}^{M}\left(R_r\left(\frac{i}{M};h\right)\right)^2, \tag{2.17}$$

where $M > 0$ and $R_r(x;h) \equiv u_r''(x) + g(x, u_r(x))$. For $r = 0$, we start with an initial approximation $u_1^{(0)}(x) = u_0(x)$ that meets the BCs (2.2) and solves the Picard's successive approximation for it. Afterwards, we improve $u_{r+1}^{(0)}(x) = u_r^{(p)}(x)$ for every $r = 1, 2, \ldots,$ and solve with in them for subsequent iterations. Continue the process of solving until the best possible approximate solution for BVP (2.1)-(2.2) is obtained.

Remark 2.1. Following a similar analysis, one can also establish the proposed iterative scheme for boundary conditions of Type II, which is defined as

$$u_{r+1}^{(p+1)}(x) = u_{r+1}^{(p)}(x) - h\int_0^1 G(x,t)[u_{r+1}^{\prime\prime(p)} + hg'(u_r)(u_{r+1}^{(p)} - u_r) + g(u_r)]dt. \tag{2.18}$$

Here, $G(x,t)$ for three-point BVPs (1.3) with BCs (1.5) is defined as

$$G(x,t) = \begin{cases} \dfrac{-[(1-x)+\delta(\eta-x)]}{1+\delta}, & 0 \le t \le \min\{x,\eta\} \le 1, \\ \dfrac{-[(1-x)+\delta(t-x)]}{1+\delta}, & 0 < \eta \le t \le x \le 1, \\ \dfrac{-[(1-t)+\delta(\eta-t)]}{1+\delta}, & 0 \le x \le t \le \eta < 1, \\ \dfrac{-(1-t)}{1+\delta}, & 0 \le \max\{x,\eta\} \le t \le 1, \end{cases} \tag{2.19}$$

provided $1 + \delta \neq 0$.

3. Convergence analysis: NTPBVPs

In this section, we documented the convergence of the proposed coupled iterative technique (2.16) for three-point BVPs (2.1)-(2.2). But firstly, the convergence analysis of the quasilinearization scheme for the problem (2.1) with boundary condition (2.2) is established with the support of uniform boundedness of the solution [30].

To discuss the existence of the sequence $u_{r+1}(x)$ and its uniform boundedness, we convert the non-homogeneous boundary value problem (2.1)-(2.2) to homogeneous boundary value problem by the use of transformation $u(x) = y(x) - \left[a + \frac{(\beta - a(1+\delta))x}{1+\delta\eta}\right]$. Therefore, rather than the non-homogeneous boundary value problem (2.1)-(2.2), we can take into consideration the problem (2.1)-(2.2) with $a = \beta = 0$.

Theorem 3.1. *Assume that $g(x,u) : [0,1] \times R \to R$ is continuous and has a continuous bounded partial derivative of first order w.r.t. u such that*

$$\max_u \left\{|g(x,u)|, \left|\frac{\partial g(x,u)}{\partial u}\right|\right\} = m < \infty,$$

and further, suppose

$$\int_0^1 |G(x,t)| \le \alpha,$$

where $G(x,t)$ is defined by Eq. (2.14), then for $m < \frac{1}{\alpha(1+2|h|)}$ the iterative sequence specified by (2.16) approaches to the solution of the problem (2.1)-(2.2).

Proof. Consider the optimal quasilinearization scheme (2.3)-(2.4) with ($a = \beta = 0$) for the corresponding homogeneous boundary value problem (2.1)-(2.2). The equivalent integral representation of (2.3)-(2.4) with ($a = \beta = 0$) is given by

$$u_{r+1}(x) = \int_0^1 G(x,t)\{g(t,u_r) + hg'(t,u_r)(u_{r+1}(t) - u_r(t))\}dt. \tag{3.1}$$

Define the norm

$$\|u\| = \max_{0\le x\le 1} |u(x)|,$$

and also assume that

$$|u_0(x)| \le k < \infty, \text{ for } 0 \le x \le 1.$$

For $r = 0$, Eq. (3.1) gives

$$\begin{aligned}\|u_1\| &\le \max_{0\le x\le 1}\int_0^1 |G(x,t)|\{|g(u_0)| + |h|(|u_1| + |u_0|)|g'(u_0)|\}dt,\\ &\le \alpha[m(1 + k|h|) + \|u_1\|m|h|],\end{aligned} \tag{3.2}$$

which implies

$$\|u_1\| \le \frac{(1 + k|h|)m\alpha}{(1 - \alpha m|h|)} \le k, \tag{3.3}$$

provided $m \le \frac{k}{\alpha(1+2k|h|)}$. This shows that $|u_{r+1}(x)| \le k$ for $0 \le x \le 1$. So, we have a well-defined sequence. Next, to illustrate the uniform convergence of the sequence (3.1), we have

$$\delta u_{r+1}(x) = \int_0^1 G(x,t)\{g(t,u_r) - g(t,u_{r-1}) - h\delta u_r(t)g'(t,u_{r-1}) + h\delta u_{r+1}(t)g'(t,u_r)\}dt, \tag{3.4}$$

where $\delta u_{r+1}(x) = u_{r+1}(x) - u_r(x)$. Now by using mean-value theorem, we obtain

$$\begin{aligned}\|\delta u_{r+1}\| &\le \max_{0\le x\le 1}\int_0^1 |G(x,t)|\{|\delta u_r||g'(c)| + |h||\delta u_r||g'(t,u_{r-1})| + |h||\delta u_{r+1}||g'(t,u_r)|\}dt.\\ &\le \alpha\{(1 + |h|)m\|\delta u_r\| + m|h|\|\delta u_{r+1}\|\},\\ &\le \frac{(1 + |h|)m\alpha}{(1 - \alpha m|h|)}\|\delta u_r\|,\end{aligned} \tag{3.5}$$

here $g(x,u_r) - g(x,u_{r-1}) = \delta u_r g'(c)$, $u_{r-1} < c < u_r$. From Eq. (3.5), we have

$$\|\delta u_{r+1}\| \le \gamma\|\delta u_r\|, \qquad \text{where} \qquad \gamma = \frac{(1 + |h|)m\alpha}{(1 - \alpha m|h|)}.$$

Hence, we obtain

$$\|\delta u_{r+1}\| \le \gamma\|\delta u_r\| \le \gamma^2\|\delta u_{r-1}\| \le \cdots \le \gamma^r\|\delta u_1\|.$$

Making use of the induction principle yields

$$\|\delta u_{r+1}\| \le \gamma^r\|\delta u_1\|, \tag{3.6}$$

where $\gamma = \frac{(1+|h|)m\alpha}{(1-\alpha m|h|)} < 1$, provided $m < \frac{1}{\alpha(1+2|h|)}$. Hence, $\|\delta u_{r+1}\| \to 0$ as $r \to \infty$, and

$$m = \max_{0\le x\le 1}\{|g(x,u)|, |g'(x,u)|\}. \tag{3.7}$$

3.1. **Convergence of the suggested coupled technique.** The suggested coupled iterative technique's (depending on quasilinearization and Picard's methods) convergence analysis is covered in this subsection. From Eq. (2.3), we have

$$u_{r+1}^{(p+1)}(x) = u_{r+1}^{(p)}(x) - h\int_0^1 G(x,t)\{u_{r+1}^{\prime\prime(p)} + hg'(t,u_r)(u_{r+1}^{(p)} - u_r) + g(t,u_r)\}dt. \tag{3.8}$$

Utilizing the boundary conditions and performing twice integration of the first term of the integrand of Eq. (3.8), we get

$$u_{r+1}^{(p+1)} = (1-h)u_{r+1}^{(p)} + h\left[a + \frac{(\beta - a(1+\delta))x}{1+\delta\eta}\right] - h\int_0^1 G(x,t)\{hg'(t,u_r)(u_{r+1}^{(p)} - u_r) + g(t,u_r)\}dt. \tag{3.9}$$

Now for fixed value of r, define the Picard's operator K corresponding to Eq. (3.9) as

$$K[u_{r+1}^{(p)}] = (1-h)u_{r+1}^{(p)} + h\left[a + \frac{(\beta - a(1+\delta))x}{1+\delta\eta}\right] - h\int_0^1 G(x,t)\{hg'(t,u_r)(u_{r+1}^{(p)} - u_r) + g(t,u_r)\}dt.$$

Thus Eq. (3.9) can be expressed as

$$u_{r+1}^{(p+1)} = K[u_{r+1}^{(p)}]. \tag{3.10}$$

Next, K is a contraction mapping as follows

$$\begin{aligned}\|K[u_{r+1}^{(p)}] - K[z_{r+1}^{(p)}]\| &= \max_{0\le x\le 1}\left|(1-h)(u_{r+1}^{(p)}(x) - z_{r+1}^{(p)}(x)) - h^2\int_0^1 G(x,t)(u_{r+1}^{(p)} - z_{r+1}^{(p)})g'(t,u_r)dt\right|,\\ &\le |(1-h)|\|(u_{r+1}^{(p)} - z_{r+1}^{(p)})\| + h^2 m\alpha\|(u_{r+1}^{(p)} - z_{r+1}^{(p)})\|,\\ &\le \kappa\|(u_{r+1}^{(p)} - z_{r+1}^{(p)})\|,\end{aligned} \tag{3.11}$$

where $\kappa = (|1-h| + h^2 m\alpha)$ for $m = \max_u\{|g(x,u)|, |g'(x,u)|\} < \frac{1-|(1-h)|}{\alpha h^2} < \infty$. Now using Banach fixed point theorem [23], we get

$$\Delta u_{r+1}^{(p+1)} = u_{r+1}^{(p+1)} - u_{r+1}^{(p)} = K[u_{r+1}^{(p)}] - K[u_{r+1}^{(p-1)}]. \tag{3.12}$$

From Equations (3.11) and (3.12), we have

$$\|\Delta u_{r+1}^{(p+1)}\| = \|u_{r+1}^{(p+1)} - u_{r+1}^{(p)}\| = \|K[u_{r+1}^{(p)}] - K[u_{r+1}^{(p-1)}]\| \le \kappa\|\Delta u_{r+1}^{(p)}\|, \tag{3.13}$$

by induction principle, we get

$$\|\Delta u_{r+1}^{(p+1)}\| \le \kappa^p\|\Delta u_{r+1}^{(1)}\|. \tag{3.14}$$

As $p \to \infty$, while r is fixed, it gives

$$\|u_{r+1}^{(p+1)} - u_{r+1}^{(p)}\| \to 0.$$

As a final step, the error of the suggested technique is computed as

$$\begin{aligned}\|u_{r+1}^{(p+1)} - u_r^{(p)}\| &= \|u_{r+1}^{(p+1)} - u_{r+1}^{(p)} + u_{r+1}^{(p)} - u_r^{(p)}\|,\\ &\le \|u_{r+1}^{(p+1)} - u_{r+1}^{(p)}\| + \|u_{r+1}^{(p)} - u_r^{(p)}\|,\\ &= \|\Delta u_{r+1}^{(p+1)}\| + \|\delta u_{r+1}^{(p)}\|,\end{aligned} \tag{3.15}$$

which is approaching zero for $r \to \infty$ and $p \to \infty$. □

4. Numerical illustration

This section includes a numerical illustration of the suggested iterative technique. To show the applicability and efficacy of the suggested iterative technique, we take into account a number of numerical examples. We have contrasted the numerical results produced using the proposed technique with some existing methods in order to demonstrate the superiority of the proposed method.

Definition 4.1. The absolute error is determined as follows

$$E_n(x) = |u_n(x) - u(x)|, \tag{4.1}$$

here, $u_n(x)$ is the nth approximation (derived by the suggested method) and $u(x)$ denotes the exact solution to the problem.

Definition 4.2. The residual error is determined as follows

$$R_n(x) = \left|u_n''(x) + g(x, u_n(x)\right|, \tag{4.2}$$

here, $u_n(x)$ is is the problem's nth approximation.

Definition 4.3. The maximum absolute error (MAE) for the problem is given as

$$\text{MAE} = E_{max}(x) = \max |u_n(x) - u(x)|, \tag{4.3}$$

here, $u_n(x)$ is the problem's nth approximation and $u(x)$ is the exact solution.

Example 4.4. Consider the following NTPBVP

$$\begin{cases} u''(x) + e^{-x}u^2(x) - (x^2 + x + 2)e^x = 0, & 0 \le x \le 1, \\ u(0) = 0, \quad u(1) + 2u\left(\frac{1}{2}\right) = e + e^{\frac{1}{2}}. \end{cases} \tag{4.4}$$

Here, $a = 0$, $\delta = 2$, $\eta = \frac{1}{2}$, $\beta = e + e^{\frac{1}{2}}$ and $g(x, u) = e^{-x}u^2 - (x^2 + x + 2)e^x$. The problem has exact solution as $u(x) = xe^x$. Applying the proposed iterative scheme (2.16), we have

$$u_{r+1}^{(p+1)}(x) = u_{r+1}^{(p)}(x) - h \int_0^1 G(x,t)\{u_{r+1}''^{(p)} + 2he^{-t}u_r(u_{r+1}^{(p)} - u_r) + e^{-t}u_r^2 - (t^2 + t + 2)e^t\}dt, \tag{4.5}$$

here, G(x,t) is calculated as

$$G(x,t) = \begin{cases} \dfrac{-t(2-3x)}{2}, & 0 \le t \le \min\{x, \dfrac{1}{2}\} \le 1, \\ -t + \dfrac{x(1+t)}{2}, & \dfrac{1}{2} \le t \le x \le 1, \\ \dfrac{-x(2-3t)}{2}, & 0 \le x \le t \le \dfrac{1}{2}, \\ \dfrac{-x(1-t)}{2}, & 0 \le \max\{x, \dfrac{1}{2}\} \le t \le 1. \end{cases} \tag{4.6}$$

We calculate the value of the solution by Equations (2.11)-(2.12). Often, the suggested iterative approach produces highly complicated integrals, therefore to avoid this complication, we employ the Taylor series expansion to calculate such integrals. It is noticed that the accuracy of the approximation depends on the number of terms of Taylor series expansion and its centering point, and it is observed that center 0.5 gives better results.

Choose an initial approximation, $u_0(x) = u_1^{(0)}(x) = \frac{x(e+e^{\frac{1}{2}})}{2}$, that meets the BCs. By using iterative schemes (2.11) (or (2.12)), we begin the first iterate $r = 0$ for the case $x \le \frac{1}{2}$ (or $x \ge \frac{1}{2}$) and carry out two iterations at $p = 0, 1$ and considering $u_1(x) \approx u_1^{(2)}(x)$. Here, optimal parameter h is obtained by calculating Eq. (2.17) and the values of h corresponding to $u_1^{(1)}(x)$ and $u_1^{(2)}(x)$ are 0.974242 and 1.04535, respectively. Next, we update the initial approximation $u_2^{(0)}(x)$ with $u_2^{(0)}(x) = u_1(x)$ for iteration $r = 1$, where $u_1(x) \approx u_1^{(2)}(x)$. Continue solving in the same way, we achieve

Table 1. Comparing the proposed technique with other techniques for absolute errors: Example 4.4.

x	Proposed Method			Method in Ref.[49]		Difference Method[49]	
	$r=1$	$r=2$	$r=3$	$(4,2)$	$(4,4)$	$h=0.1$	$h=0.05$
0.1	1.29715×10^{-07}	9.36085×10^{-13}	5.55112×10^{-17}	4.0629×10^{-04}	8.1435×10^{-06}	1.7128×10^{-04}	4.2829×10^{-05}
0.2	2.43306×10^{-07}	1.94333×10^{-12}	1.38778×10^{-16}	7.8165×10^{-04}	1.5896×10^{-05}	3.0443×10^{-04}	7.6114×10^{-05}
0.3	3.09488×10^{-07}	3.16352×10^{-12}	1.66533×10^{-16}	1.0209×10^{-03}	2.0120×10^{-05}	3.9356×10^{-04}	9.8358×10^{-05}
0.4	3.01128×10^{-07}	4.67049×10^{-12}	1.11022×10^{-16}	1.2069×10^{-03}	2.3084×10^{-05}	4.3201×10^{-04}	1.0798×10^{-04}
0.5	2.14370×10^{-07}	6.22602×10^{-12}	5.55112×10^{-16}	1.2227×10^{-03}	2.3171×10^{-05}	4.1224×10^{-04}	1.0301×10^{-04}
0.6	7.27610×10^{-08}	7.09233×10^{-12}	2.22045×10^{-16}	1.0183×10^{-03}	1.7900×10^{-05}	3.2672×10^{-04}	8.1720×10^{-05}
0.7	8.48283×10^{-08}	6.10534×10^{-12}	0.00000×10^{-16}	6.9688×10^{-04}	1.1472×10^{-05}	1.6693×10^{-04}	4.1672×10^{-05}
0.8	2.23693×10^{-07}	2.20601×10^{-12}	2.22045×10^{-16}	9.8850×10^{-04}	3.8926×10^{-06}	7.3872×10^{-05}	1.8509×10^{-05}
0.9	3.31129×10^{-07}	4.62341×10^{-12}	4.44089×10^{-16}	2.0438×10^{-03}	2.2064×10^{-05}	4.0280×10^{-04}	1.0080×10^{-04}
1.0	4.28741×10^{-07}	1.24523×10^{-11}	4.44089×10^{-16}	3.5652×10^{-03}	4.6343×10^{-05}	8.2447×10^{-04}	2.0603×10^{-04}

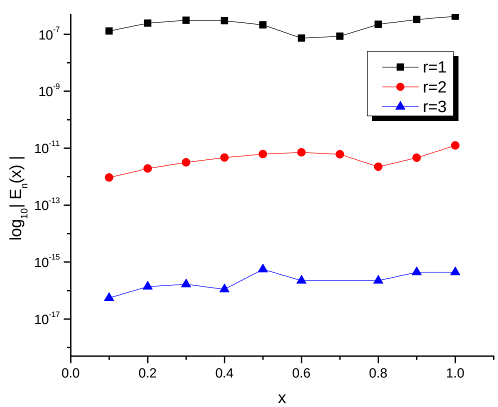

Figure 1. Plot of absolute errors at different iterations: Example 4.4.

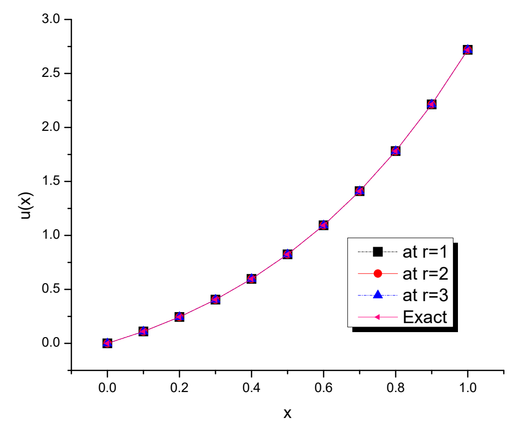

Figure 2. Plot of approximate solutions and exact solution: Example 4.4.

Table 2. Comparing the maximum absolute errors (MAE) of the proposed technique with other techniques: Example 4.4.

Proposed Method		Difference Method[49]		UHWCM [36]
$r=2$	$r=3$	$h=0.1$	$h=0.05$	$2M=16$
1.24523×10^{-11}	5.55112×10^{-16}	8.2447×10^{-04}	2.0603×10^{-04}	1.34568×10^{-04}

the successive approximations for $r = 2, 3, \ldots$, where initial approximation is updated as $u^{(0)}_{r+1}(x) = u_r(x)$. For this problem, we consider $u_1(x) \approx u^{(2)}_1(x), u_2(x) \approx u^{(3)}_2(x), u_3(x) \approx u^{(3)}_3(x)$ and $u_4(x) \approx u^{(3)}_4(x)$.

For Example 4.4, Table 1 documents the comparisons (in terms of absolute errors) of the suggested technique with other established approaches and shows supremacy over them. Additionally, Table 2 compares the maximum absolute error for the problem with difference method [49] and uniform Haar wavelet collocation method (UHWCM) [36]. Figure 1 shows a plot of the absolute errors at $r = 1, 2,$ and 3 (at logarithmic scale) against different values of x in the range of $[0, 1]$. It reveals that as the number of iterations grows, absolute errors are decreasing, and the suggested method converges to the exact solution. Furthermore, in Figure 2, we have depicted the approximate solutions obtained at

TABLE 3. Comparing the proposed technique with other techniques for absolute errors: Example 4.5.

x	Proposed Method			Method in Ref.[48]		
	$r=1$	$r=2$	$r=3$	$(4,2)$	$(4,3)$	$(4,4)$
0.1	1.20917×10^{-10}	0.00000×10^{-15}	8.88178×10^{-16}	3.5276×10^{-05}	1.7089×10^{-06}	6.9537×10^{-08}
0.2	2.21814×10^{-10}	3.33067×10^{-15}	1.55431×10^{-15}	6.8002×10^{-05}	3.3354×10^{-06}	1.3690×10^{-07}
0.3	2.83312×10^{-10}	1.11022×10^{-14}	1.99840×10^{-15}	8.7550×10^{-05}	4.2835×10^{-06}	1.7574×10^{-07}
0.4	2.91401×10^{-10}	2.33147×10^{-14}	1.99840×10^{-15}	1.0364×10^{-04}	5.0570×10^{-06}	2.0678×10^{-07}
0.5	2.42146×10^{-10}	3.70814×10^{-14}	1.77636×10^{-15}	1.0811×10^{-04}	5.3183×10^{-06}	2.1906×10^{-07}
0.6	1.44356×10^{-10}	4.66294×10^{-14}	1.11022×10^{-15}	9.6642×10^{-05}	4.6941×10^{-06}	1.9142×10^{-07}
0.7	1.88995×10^{-11}	4.57412×10^{-14}	4.44089×10^{-16}	7.9876×10^{-05}	3.9095×10^{-06}	1.5997×10^{-07}
0.8	1.05637×10^{-10}	2.93099×10^{-14}	8.88178×10^{-16}	3.2875×10^{-05}	1.5707×10^{-06}	6.3247×10^{-08}
0.9	1.99980×10^{-10}	0.00000×10^{-15}	1.77636×10^{-15}	2.3720×10^{-05}	1.2531×10^{-06}	5.4823×10^{-08}
1.0	2.42145×10^{-10}	3.73035×10^{-14}	1.77636×10^{-15}	1.0811×10^{-04}	5.3183×10^{-06}	2.1906×10^{-07}

various iterations alongside the exact solution, which shows that the obtained approximate solution is highly accurate and illustrates the strong agreement with the exact solution.

Example 4.5. Consider the following NTPBVP

$$\begin{cases} u''(x)-(1+\sin x)u(x)+e^x\sin x=0, & 0\le x\le 1,\\ u(0)=1, \quad u(1)+u\left(\frac{1}{2}\right)=e+e^{\frac{1}{2}}. \end{cases} \tag{4.7}$$

Here, $a=1$, $\delta=1$, $\eta=\frac{1}{2}$, $\beta=e+e^{\frac{1}{2}}$ and $g(x,u)=-(1+\sin x)u+e^x\sin x$. The problem has an exact solution as $u(x)=e^x$. Applying the proposed iterative scheme (2.16), we have

$$u_{r+1}^{(p+1)}(x)=u_{r+1}^{(p)}(x)-h\int_0^1 G(x,t)\{u_{r+1}^{\prime\prime(p)}-h(1+\sin(t))(u_{r+1}^{(p)}-u_r)-(1+\sin t)u_r+e^t\sin t\}dt, \tag{4.8}$$

here, G(x,t) is calculated as

$$G(x,t)=\begin{cases} \dfrac{-t(3-4x)}{3}, & 0\le t\le \min\{x,\dfrac{1}{2}\}\le 1,\\ -t+\dfrac{x(1+2t)}{3}, & \dfrac{1}{2}\le t\le x\le 1,\\ \dfrac{-x(3-4t)}{3}, & 0\le x\le t\le \dfrac{1}{2},\\ \dfrac{-2x(1-t)}{3}, & 0\le \max\{x,\dfrac{1}{2}\}\le t\le 1. \end{cases} \tag{4.9}$$

We start first iterate (i.e., for $r=0$), with initial guess $u_0(x)=u_1^{(0)}(x)=1+\frac{-4x+2x(e+e^{\frac{1}{2}})}{3}$ that also satisfies BCs and carry out three iterations at $p=0,1$ and 2, with the support of Taylor expansion (center at 0.5) and yield $u_1(x)\approx u_1^{(3)}(x)$. In the same way, the successive approximations for $r=1,2,\ldots$, with initial approximation $u_{r+1}^{(0)}(x)=u_r(x)$ (updated for every r) can be achieved. For this problem, we consider $u_1(x)\approx u_1^{(3)}(x), u_2(x)\approx u_2^{(3)}(x), u_3(x)\approx u_3^{(2)}(x)$ and $u_4(x)\approx u_4^{(1)}(x)$. Table 3 documents the comparison of the proposed technique (in terms of absolute errors) with other established approach and show the supremacy over them. Additionally, Figure 3 shows a plot of the absolute errors at $r=1,2,$ and 3 (at logarithmic scale) against different values of x in the range of $[0,1]$. It reveals that with growing iterations, absolute errors decrease, and the suggested method converges to the exact solution. We have also plotted the obtained approximate solutions at different iterations and the exact solution in Figure 4, demonstrating that the obtained approximate solution is highly accurate and illustrating the strong agreement with the exact solution.

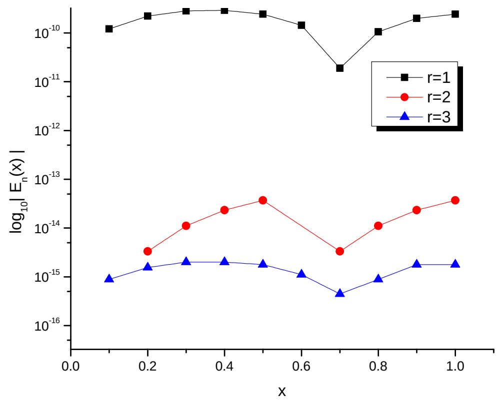

FIGURE 3. Plot of absolute errors at different iterations: Example 4.5.

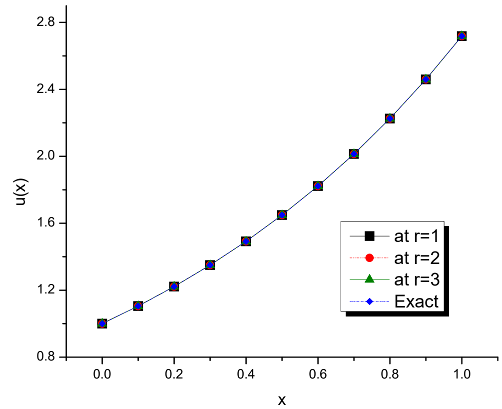

FIGURE 4. Plot of approximate solutions at different iterations and exact solution: Example 4.5.

Example 4.6. Consider the following NTPBVP

$$\begin{cases} u''(x) + \frac{e^u}{32} - \frac{1}{64} = 0, & 0 \le x \le 1, \\ u'(0) = 0, \quad u(1) = 2u\left(\frac{1}{3}\right). \end{cases} \tag{4.10}$$

Here, $\delta = -2$, $\eta = \frac{1}{3}$, $\beta = 0$ and $g(x, u) = \frac{e^u}{32} - \frac{1}{64}$. The problem doesn't have an exact solution. The approximate solution to the problem has been calculated by the proposed method. Applying the proposed iterative scheme (2.18), we have

$$u_{r+1}^{(p+1)}(x) = u_{r+1}^{(p)}(x) - h \int_0^1 G(x,t) \left\{ u_{r+1}^{\prime\prime(p)} + \frac{he^{u_r}}{32}(u_{r+1}^{(p)} - u_r) + \frac{e^{u_r}}{32} - \frac{1}{64} \right\} dt, \tag{4.11}$$

here, $G(x, t)$ is calculated as

$$G(x,t) = \begin{cases} x + \dfrac{1}{3}, & 0 \le t \le \min\{x, \dfrac{1}{3}\} \le 1, \\ x - 2t + 1, & \dfrac{1}{3} \le t \le x \le 1, \\ t + \dfrac{1}{3}, & 0 \le x \le t \le \dfrac{1}{3}, \\ -t + 1, & 0 \le \max\{x, \dfrac{1}{3}\} \le t \le 1. \end{cases} \tag{4.12}$$

We start the first iterate with an initial approximation $u_0(x) = u_1^{(0)}(x) = 0$ that satisfies the BCs and solve two iterations for p and then fix $u_1(x) \approx u_1^{(2)}(x)$. Continue to solve in the same way and obtain the successive approximations for $r = 1, 2, \ldots$, with the initial approximation (updated for every r) as $u_{r+1}^{(0)}(x) = u_r(x)$. Taking $u_1(x) \approx u_1^{(2)}(x), u_2(x) \approx u_2^{(2)}(x), u_3(x) \approx u_3^{(2)}(x)$ and $u_4(x) \approx u_4^{(1)}(x)$. To illustrate the accuracy and efficacy of the suggested method, we compute residual errors (as the exact solution is not available in the literature) and documented in Table 4. We have also an agreement with the analytical study discussed in [33], where the authors established the region of existence for the solution to the same problem. It is observed that our approximate solution lies in the same region. Additionally, Figure 5 displays a depiction of the residual errors at $r = 0, 1, 2,$ and 3 (at logarithmic

TABLE 4. Comparative analysis of different residual errors: Example 4.6.

x	$r=0$	$r=1$	$r=2$	$r=3$
0	8.96948×10^{-08}	1.00073×10^{-12}	6.28870×10^{-18}	2.76653×10^{-18}
0.1	9.94182×10^{-08}	1.25253×10^{-12}	1.05126×10^{-17}	2.54415×10^{-19}
0.2	1.26736×10^{-07}	1.93876×10^{-12}	2.31148×10^{-17}	5.64196×10^{-19}
0.3	1.66096×10^{-07}	2.86153×10^{-12}	3.78389×10^{-17}	9.50703×10^{-20}
0.4	2.08250×10^{-07}	3.72236×10^{-12}	4.58740×10^{-17}	5.18155×10^{-18}
0.5	2.40271×10^{-07}	4.16844×10^{-12}	4.73999×10^{-17}	4.76938×10^{-18}
0.6	2.45567×10^{-07}	3.85574×10^{-12}	3.76017×10^{-17}	8.54332×10^{-18}
0.7	2.03909×10^{-07}	2.52751×10^{-12}	1.05387×10^{-17}	1.01307×10^{-17}
0.8	9.14527×10^{-08}	1.06038×10^{-13}	2.29777×10^{-17}	1.26676×10^{-17}
0.9	1.19222×10^{-07}	3.20462×10^{-12}	4.70539×10^{-17}	1.61314×10^{-17}
1.0	4.59079×10^{-07}	6.80774×10^{-12}	4.42759×10^{-17}	1.37185×10^{-17}

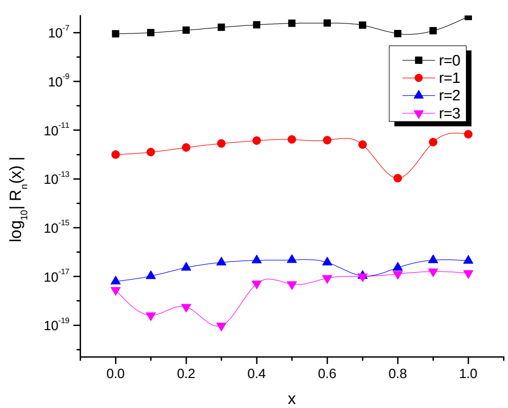

FIGURE 5. Plot of residual errors at different iterations: Example 4.6.

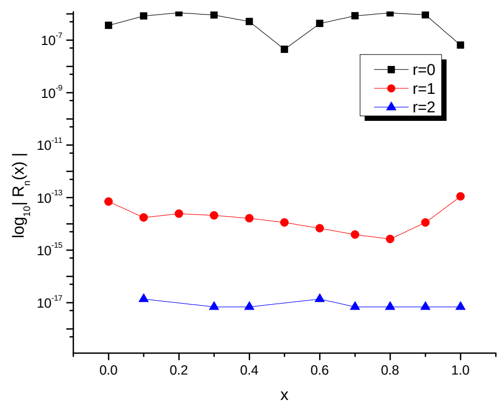

FIGURE 6. The plot of residual errors at different iterations: Example 4.7.

scale) against different values of x in the range of $[0,1]$, which shows that the method converges to the solution of the problem with an increase in the number of iterations.

Example 4.7. Consider the following NTPBVP

$$\begin{cases} u''(x)+\frac{1}{32}\left[\frac{e^2}{4}-\frac{\sin x}{4}-2u^3(x)\right]=0, \qquad 0\le x\le 1, \\ u'(0)=0, \quad u(1)=\frac{1}{3}u\left(\frac{1}{2}\right). \end{cases} \tag{4.13}$$

Here, $\delta=-\frac{1}{3}$, $\eta=\frac{1}{2}$, $\beta=0$ and $g(x,u)=\frac{1}{32}\left[\frac{e^2}{4}-\frac{\sin x}{4}-2u^3\right]$. The given problem has no exact solution. The approximate solution to the problem has been calculated by the proposed method. Applying the proposed iterative scheme (2.18), we have

$$u_{r+1}^{(p+1)}(x)=u_{r+1}^{(p)}(x)-h\int_0^1 G(x,t)\left\{u_{r+1}^{\prime\prime(p)}-\frac{3h}{16}u_r^2(u_{r+1}^{(p)}-u_r)+\frac{1}{32}\left(\frac{e^2}{4}-\frac{\sin t}{4}-2u_r^3\right)\right\}dt, \tag{4.14}$$

Table 5. Comparative analysis of different residual errors: Example 4.7.

x	$r=0$	$r=1$	$r=2$
0	3.66645×10^{-07}	7.11861×10^{-14}	0.00000×10^{-17}
0.1	8.34063×10^{-07}	1.76109×10^{-14}	1.38778×10^{-17}
0.2	1.09920×10^{-06}	2.44388×10^{-14}	0.00000×10^{-17}
0.3	9.02229×10^{-07}	2.11706×10^{-14}	6.93889×10^{-18}
0.4	5.07802×10^{-07}	1.63342×10^{-14}	6.93889×10^{-18}
0.5	4.42756×10^{-08}	1.13867×10^{-14}	0.00000×10^{-17}
0.6	4.26776×10^{-07}	6.84869×10^{-15}	1.38778×10^{-17}
0.7	8.44790×10^{-07}	3.88578×10^{-15}	6.93889×10^{-18}
0.8	1.08879×10^{-06}	2.63678×10^{-15}	6.93889×10^{-18}
0.9	9.20947×10^{-07}	1.13035×10^{-14}	6.93889×10^{-18}
1.0	6.43148×10^{-08}	1.10315×10^{-13}	6.93889×10^{-18}

here, $G(x,t)$ is calculated as

$$G(x,t) = \begin{cases} x - \dfrac{5}{4}, & 0 \le t \le \min\{x, \dfrac{1}{2}\} \le 1, \\ x + \dfrac{(t-3)}{2}, & \dfrac{1}{2} \le t \le x \le 1, \\ t - \dfrac{5}{4}, & 0 \le x \le t \le \dfrac{1}{2}, \\ \dfrac{-3(1-t)}{2}, & 0 \le \max\{x, \dfrac{1}{2}\} \le t \le 1. \end{cases} \tag{4.15}$$

We start the first iterate with an initial approximation $u_0(x) = u_1^{(0)}(x) = 0$ and carry out one iteration at $p = 0$ and then fix $u_1(x) \approx u_1^{(1)}(x)$. Continue to solve in the same way, we get the successive approximations for $r = 1, 2, \dots,$ with initial approximation $u_{r+1}^{(0)}(x) = u_r(x)$. Taking $u_1(x) \approx u_1^{(1)}(x), u_2(x) \approx u_2^{(3)}(x)$ and $u_3(x) \approx u_3^{(2)}(x)$. To show the accuracy and efficiency of the proposed method, we compute residual errors (as the exact solution is not available in the literature) and documented in Table 5. We also have an agreement with analytical study discussed in [33], where the authors established the region of existence for the solution to the same problem. It is observed that our approximate solution lies in the same region. Additionally, Figure 6 displays a depiction of the residual errors at $r = 0, 1$ and 2 (at logarithmic scale) against different values of x in the range of $[0, 1]$, which shows that the method converges to the solution of the problem with an increase in the number of iterations.

5. Conclusion

A coupled iterative technique based on the quasilinearization method (with convergence controller parameter) and Picard's iterative method is successfully proposed to find the approximate solution for a class of second-order nonlinear three-point boundary value problems (NTPBVPs). An equivalent integral representation is established in the presence of Green's function. Additionally, the proposed method's convergence analysis is also described. The proposed scheme has been tested with the help of some numerical problems, which shows its applicability and generality. Here are a few concluding remarks based on the proposed study:

- The study provides a computationally efficient iterative technique to solve a class of nonlinear nonlocal boundary value problems, which generally provides a more realistic understanding of physical phenomena.
- The nonlinearity present in the nonlocal BVPs is effectively handled by the proposed iterative method, and the application of convergence controller parameters allows for the best approximate solution in just a few iterations.
- The obtained results are also compared (in terms of absolute errors and maximum absolute error) with the existing results (see Tables 1, 2, and 3), which validate the accuracy and supremacy of the proposed technique.

In the absence of an exact solution, residual errors are calculated to show the correctness and efficiency of the technique (see Tables 4 and 5).

- Figures 1, 3, 5, and 6 illustrate the graphical representation of absolute errors and residual errors (at logarithmic scale), whereas Figures 2 and 4 represent the plot of the approximate and exact solution, which highlights the effectiveness of the technique.
- It has been observed the proposed approach shows agreement with the analytical study by [33].
- The proposed iterative technique is fast and accurate, and the desired outcomes can be enhanced with an increase in the number of iterations.

6. Conflicts of interest

All authors declare that they have no conflicts of interest.

References

[1] F. T. Akyildiz, H. Bellout, K. Vajravelu, and R. A. Van Gorder, *Existence results for third order nonlinear boundary value problems arising in nano boundary layer fluid flows over stretching surfaces*, Nonlinear Anal. Real World Appl., *12*(6) (2011), 2919–2930.

[2] W. M. Abd-Elhameed, E. H. Doha, and Y. H. Youssri, *New wavelets collocation method for solving second-order multipoint boundary value problems using Chebyshev polynomials of third and fourth kinds*, Abstr. Appl. Anal., *2013* (2013).

[3] J. Ali, S. Islam, S. Islam, and G. Zaman, *The solution of multipoint boundary value problems by the optimal homotopy asymptotic method*, Comput. Math. with Appl., *59* (2010), 2000–2006.

[4] R. Beals, *Non-local boundary value problems for elliptic partial differential operators*, Yale University, 1964.

[5] R. Beals, *Nonlocal elliptic boundary value problems*, Bull. Am. Math. Soc., *70* (1964), 693–696.

[6] A. V. Bitsadze, *On the theory of nonlocal boundary value problems*, Soviet. Math. Dock., *30* (1964), 8–10.

[7] A. V. Bitsadze and A. A. Samarskii, *Some elementary generalizations of linear elliptic boundary value problems*, Dokl. Akad. Nauk SSSR, *185*(4) (1969), 739–740.

[8] R. E. Bellman and R. E. Kalaba, *Quasilinearization and nonlinear boundary-value problems*, Elsevier Publishing Company, New York, Rand Corporation, (1965).

[9] G. Bao, X. Xu, and Y. Song, *Positive solutions for three-point boundary value problems with a non-well-ordered upper and lower solution condition*, Appl. Math. Lett., *25* (2012), 767–770.

[10] J. R. Cannon and H. M. Yin, *A class of non-linear non-classical parabolic equations*, J. Differ. Equ., *79*(2) (1989), 266–288.

[11] E. Cimen and M. Cakir, *Numerical treatment of nonlocal boundary value problem with layer behaviour*, Bull. Belg. Math. Soc. Simon Stevin, *24*(3) (2017), 339–352.

[12] A. Cabada, G. Infante, and F. A. F. Tojo, *Nonzero solutions of perturbed Hammerstein integral equations with deviate arguments and applications*, Topol. Methods Nonlinear Anal., *47*(1) (2016), 265–287.

[13] S. Das, S. Kumar, and O. P. Singh, *Solutions of nonlinear second order multi-point boundary value problems by homotopy perturbation method*, Appl. Appl. Math., *5*(10) (2010), 1592–1600.

[14] P. Guidotti and S. Merino, *Gradual loss of positivity and hidden invariant cones in a scalar heat equation*, Differ. Integral Equ., *13*(10-12) (2000), 1551–1568.

[15] F. Geng and M. Cui, *Multi-point boundary value problem for optimal bridge design*, Int. J. Comput. Math., *87*(5) (2010), 1051–1056.

[16] C. P. Gupta, *Existence theorems for a second order m-point boundary value problem at resonance*, Internat. J. Math. & Math. Sci., *18* (1995), 705–710.

[17] C. P. Gupta, *A generalized multi-point boundary value problem for second order ordinary differential equations*, Appl. Math. Comput., *89* (1998), 133–146.

[18] M. A. Hajji, *Multi-point special boundary-value problems and applications to fluid flow through porous media*, In: Proceedings of the International Multi-Conference of Engineers and Computer Scientists (IMECS-2009) vol II, Hong Kong, 2009.

[19] G. Infante and J. R. L. Webb, *Loss of positivity in a nonlinear scalar heat equation*, NODEA Nonlinear Differ. Equ. Appl., *13* (2006), 249–261.

[20] V. A. Il'in and E. I. Moiseev, *Nonlocal boundary value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects*, Differ. Equ., *23*(7) (1987), 803–810.

[21] N. I. Ionkin, *Stability of a problem in heat transfer theory with nonclassical boundary condition*, Differ. Equ., *15* (1980), 911–914.

[22] V. A. Il'in, *Optimization of the boundary control by a displacement or by an elastic force on one end of a string under a model nonlocal boundary condition*, Proc. Steklov Inst. Math., *268* (2010), 117–129.

[23] E. Kreyszig, *Introductory functional analysis with applications*, John Wiley & Sons, New York, 1989.

[24] R. Li, Z. A. B. Sinnah, Z. M. Shatouri, J. Manafian, M. F. Aghdaei, and A. Kadi, *Different forms of optical soliton solutions to the Kudryashov's quintuple self-phase modulation with dual-form of generalized nonlocal nonlinearity*, Results Phys., *46* (2023), 106293.

[25] X. Liu, B. Abd Alreda, J. Manafian, B. Eslami, M. F. Aghdaei, M. Abotaleb, and A. Kadi, *Computational modeling of wave propagation in plasma physics over the Gilson-Pickering equation*, Results Phys., (2023), 106579.

[26] R. Li, J. Manafian, H. A. Lafta, H. A. Kareem, K. F. Uktamov, and M. Abotaleb, *The nonlinear vibration and dispersive wave systems with cross-kink and solitary wave solutions*, Int. J. Geom. Methods Mod., *19*(10) (2022), 2250151.

[27] S. Mesloub and A. Bouziani, *On a class of singular hyperbolic equation with a weighted integral condition*, Internat. J. Math. & Math. Sci., *22* (1999), 511–519.

[28] R. Ma, *A survey on nonlocal boundary value problems*, Appl. Math. E-Notes, *7* (2007), 257–279.

[29] J. J. Nieto, *Existence of a solution for a three-point boundary value problem for a second-order differential equation at resonance*, Bound. Value Probl., *2013* (2013), 1–7.

[30] R. K. Pandey and S. Tomar, *An effective scheme for solving a class of nonlinear doubly singular boundary value problems through quasilinearization approach*, J. Comput. Appl. Math., *392* (2021), 113411.

[31] Y. Qian, J. Manafian, M. Asiri, K. H. Mahmoud, A. I. Alanssari, and A. S. Alsubaie, *Nonparaxial solitons and the dynamics of solitary waves for the coupled nonlinear Helmholtz systems*, Opt. Quantum Electron., *55*(11) (2023), 1022.

[32] N. Sriwastav, A. K. Barnwal, H. Ramos, R. P. Agarwal, and M. Singh, *New approach based on collection and shifted Chebyshev polynomials for a class of three-point singular BVPs*, J. Appl. Anal. Comput., *13*(4) (2023), 2162–2183.

[33] M. Singh and A. K. Verma, *Picard type iterative scheme with initial iterates in reverse order for a class of nonlinear three point BVPs*, Int. J. Differ. Equ., *2013* (2013).

[34] M. Singh, A. K. Verma, and R. P. Agarwal, *On an iterative method for a class of 2 point & 3 point nonlinear SBVPs*, J. Appl. Anal. Comput, *9*(4) (2019), 1242–1260.

[35] M. Singh, N. Urus, and A. K. Verma, *A different monotone iterative technique for a class of nonlinear three-point BVPs*, Comput. Appl. Math., *40* (2021), 1–22.

[36] Swati, K. Singh, and M. Singh, *Uniform Haar wavelet collocation method for three-point boundary value problems*, AIP Conf. Proc., 2214, (2020), 020015.

[37] R. Singh and A. M. Wazwaz, *Analytical approximations of three-point generalized Thomas-Fermi and Lane-Emden-Fowler type equations*, Eur. Phys. J. Plus, *137* (2022), 1-19.

[38] M. Tatari and M. Dehghan, *The use of the Adomian decomposition method for solving multipoint boundary value problems*, Phys. Scr., *73*(6) (2006), 672–676.

[39] S. Tomar and R. K. Pandey, *An efficient iterative method for solving Bratu-type equations*, J. Comput. Appl. Math., *357* (2019), 71–84.

[40] A. K. Verma and M. Singh, *Singular nonlinear three point BVPs arising in thermal explosion in a cylindrical reactor*, J. Math. Chem., *53* (2015), 670–684.

[41] A. K. Verma and M. Singh, *A note on existence results for a class of three-point nonlinear BVPs*, Math. Model. Anal., *20* (2015), 457–470.

[42] A. K. Verma, N. Urus, and M. Singh, *Monotone iterative technique for a class of four point BVPs with reversed ordered upper and lower solutions*, Int. J. Comput. Methods, *17* (2020), 1950066.

[43] J. R. L. Webb, *Existence of positive solutions for a thermostat model*, Nonlinear Anal. Real World Appl., *13*(2) (2012), 923–938.

[44] J. R. L. Webb and G. Infante, *Positive solutions of nonlocal boundary value problems: a unified approach*, J. London Math. Soc., *74*(3) (2006), 673–693.

[45] W. Yao, *Existence and multiplicity of solutions for three-point boundary value problems with instantaneous and noninstantaneous impulses*, Bound. Value Probl., *2023*(1) (2023), 15.

[46] Y. Zou, Q. Hu, and R. Zhang, *On numerical studies of multi-point boundary value problem and its fold bifurcation*, Appl. Math. Comput., *185*(1) (2007), 527–537.

[47] S. Zhang, J. Manafian, O. A. Ilhan, A. T. Jalil, Y. Yasin, and M. Abdulfadhil Gatea, *Nonparaxial pulse propagation to the cubic-quintic nonlinear Helmholtz equation*, Int. J. Mod. Phys. B, (2023), 2450117.

[48] X. C. Zhong and Q. A. Huang, *Approximate solution of three-point boundary value problems for second-order ordinary differential equations with variable coefficients*, Appl. Math. Comput., *247* (2014), 18–29.

[49] X. C. Zhong, Q. A. Huang, and H. M. Wei, *Existence and approximation of solution for a nonlinear second-order three-point boundary value problem*, Bound. Value Probl., *2016* (2016), 169.