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Designing an efficient algorithm for fractional partial integro-differential viscoelastic equations with weakly singular kernel

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Abstract

In this paper, the discretization method is developed by means of Mott-fractional Mott functions (MFM-Fs) for solving fractional partial integro-differential viscoelastic equations with weakly singular kernels. By taking into account the Riemann-Liouville fractional integral operator and operational matrix of integration, we convert the proposed problem to fractional partial integral equations with weakly singular kernels. It is necessary to mention that the operational matrices of integration are obtained with new numerical algorithms. These changes effectively affect the solution process and increase the accuracy of the proposed method. Besides, we investigate the error analysis of the approach. Finally, several examples are solved by applying the discretization method by combining MFM-Fs and the gained results are compared with the methods available in the literature.

Keywords. Mott-fractional Mott functions, Fractional partial integro-differential viscoelastic equations, Operational matrices of integration.
2010 Mathematics Subject Classification. 45J05, 60J60.

1. Introduction

In this work, we develop the discretization method based on MFM-Fs for solving the following fractional partial integro-differential viscoelastic equations with weakly singular kernels:

$$D_t^{\nu} u(x,t) = f(x,t) + \vartheta u_{xx}(x,t) + \int_0^t \frac{u_{xx}(x,\eta)}{(t-\eta)^{\beta}} d\eta, \quad \nu \in (0,1], \quad \beta \in (0,1), \tag{1.1}$$

subject to the initial and boundary conditions

$$\begin{aligned} u(x,0) &= g_0(x), \quad x \in [0,1], \\ u(0,t) &= h_0(t), \quad u(1,t) = h_1(t), \quad t \in [0,1], \end{aligned} \tag{1.2}$$

where $f(x,t)$, $g_0(x)$, $h_0(t)$ and $h_1(t)$ are known functions, $u(x,t)$ is unknown function and D_t^{ν} denotes the Caputo fractional derivative [16, 26]. To guarantee the existence and uniqueness of the solution $u(x,t) \in C([0,1] \times [0,1])$, it is imperative to assume that the functions $u(x,t)$ and $f(x,t)$ are sufficiently smooth. It is worth noting that this equation has appeared in physical phenomena that involve heat flow in materials with memory [14, 20], and linear viscoelastic mechanics [7, 30]. In the proposed models, the integral part denotes the viscosity part and constant-coefficient $\vartheta \geq 0$ demonstrate the Newtonian contribution in the viscosity.

In recent years, the solution of fractional partial integro-differential equations have been an attractive topic for researchers. Thus, many methods have been introduced by mathematicians. In the meantime, numerical methods have emerged as a powerful tool in solving these equations. Here are some of them [4, 6, 24, 25, 28, 32–34]. Additionally, various methods have been introduced for solving fractional partial integro-differential equations with weakly singular kernels, with a few of them being discussed here. Behera and Saha Ray [5], introduced Bernoulli wavelet and Legendre wavelet functions for nonlinear weakly singular partial integro-differential equations. Kamran et al. [1], presented a local meshless method for approximating the solution of weakly singular partial integro-differential equations. Singh

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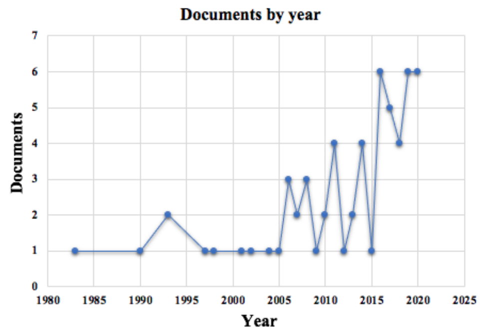

Figure 1. Publication trends of documents in "Integro-differential equations with weakly singular kernel" research field between 1980 up to 2021.

et al. [36], discussed the convergence rate of the collocation method based on Legendre wavelets for fractional partial integro-differential viscoelastic equations. Mirzaee et al. [21], explained a numerical method based on Bernstein polynomials for solving partial integro-differential equations with the weakly singular kernel. Singh et al. [35] provided two computational approaches for multi-dimensional nonlinear weakly singular fractional integro-differential equations. Maurya et al. [19] used the Lagrangian matrix approach for solving Abel integral and integro-differential equations.

In this paper, we apply Mott-fractional Mott functions together with a new scheme for computing the elements of operational matrices to calculate the approximate solution. These polynomials have a few terms in comparison to other polynomials which make less CPU time in the approximation algorithm. This feature is reflected in MFM-Fs and the numerical approach, directly. Motivated to arrive at an approximate solution with the highest degree of accuracy, we developed the discretization method based on MFM-Fs. These changes have been very effective in the solution process and have made the proposed method more powerful than other existing numerical methods. The method of calculating operational matrices is another factor that is involved in the accuracy of the proposed method. It should be noted that there are many methods for approximating the derivative of functions in equations (see [22, 29, 31]).

Recently, many researchers have introduced different fractional functions to solve problems of fractional order. Thus, these basis functions have appeared as a powerful tool in approximating the exact solution of the problems that contain the fractional degree sentence. Such as, fractional Legendre functions [2], fractional Legendre-Laguerre functions [8], fractional Bessel functions [9], fractional Genocchi functions [12], fractional Lucas functions [13], fractional Gegenbauer functions [10], Lucas-fractional Lucas functions [11] and, etc.

The main contribution of this paper is organized as follows. The next section is contains to the preliminary definitions of fractional integral and derivative. Section 3 is devoted to the definition of Mott-Fractional Mott functions and their properties. The method of calculating the operational matrices required in the problem-solving process is described in section 4. In section 5, we combine the Riemann-Liouville fractional integral operator and operational matrices of integration to convert the problem into a system of algebraic equations. In addition, the error estimation of the proposed method is discussed in section 6. In section 7, the numerical finding of the test problems is provided in the form of tables and figures. Finally, the conclusions and remarks are discussed in section 8.

1.1. **Bibliometrics.** In this section, we discuss bibliometric analysis in "Integro-differential equations with weakly singular kernel" area. To reach the aim, we extracted the data from the Scopus database, which the data was collected on 19 January 2021. The time span of bibliometric analysis is limited from 1980 up to 2021. The extracted data contain 60 documents, 58 (96.7%) Articles, and 2 (3.3%) Conference papers. The growth trend of published documents in the proposed field between 1980 up to 2021 is shown in Figure 1. Also, the cluster based on the co-occurrence of all keywords with the help of Vosviewer is plotted in Figure 2. From Figure 2, it can be understood that in recent years the operational matrices and numerical methods attracted the attention of researchers.

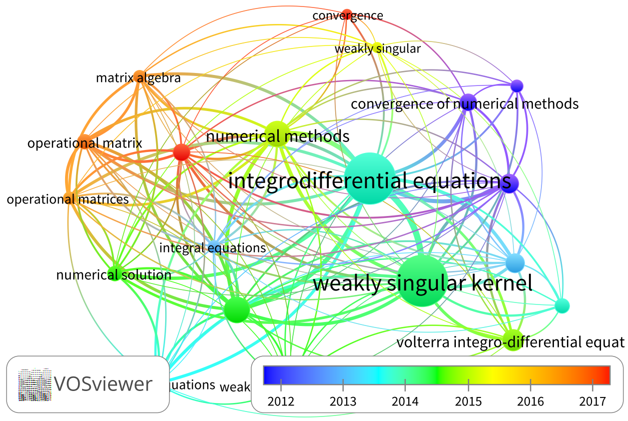

FIGURE 2. Cluster-based on the co-occurrence of all keywords in "Integro-differential equations with weakly singular kernel" research field.

2. Preliminaries

In this section, we provide the definitions of Riemann-Liouville fractional integral and Caputo fractional derivative, which are used in the methodology.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\nu \geq 0$ is defined as [8]

$$I^{\nu} f(x) = \begin{cases} \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t) dt, & \nu > 0, \\ f(x), & \nu = 0. \end{cases}$$

The properties of the operator I^{ν} can be found in [18]. Below, we consider a number of them, for $\gamma > -1$:

- $I^{\nu} I^{\gamma} f(x) = I^{\nu+\gamma} f(x)$,
- $I^{\nu} I^{\gamma} f(x) = I^{\gamma} I^{\nu} f(x)$,
- $I^{\nu} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\nu+\gamma+1)} x^{\nu+\gamma}$.

Definition 2.2. The fractional derivative of order $\nu \in (m-1, m]$, $m \in \mathbb{N}$ in the Caputo sense is defined as [8]

$$D^{\nu} f(x) = I^{m-\nu}(D^m f(x)) = \begin{cases} \frac{1}{\Gamma(m-\nu)} \int_0^x (x-t)^{m-\nu-1} f^{(m)}(t) dt, & \nu \in (m-1, m) \\ \frac{d^m}{dx^m} f(x), & \nu = m, \end{cases}$$

which has the following properties:

- $D^{\nu} C = 0, \quad (C \text{ is a constant})$,
- $D^{\nu} I^{\nu} f(x) = f(x)$,
- $I^{\nu} D^{\nu} f(x) = f(x) - \sum_{i=0}^{m-1} f^{(i)}(0) \frac{x^i}{i!}$,
- $D^{\nu} x^{\gamma} = \begin{cases} 0, & \gamma \in N_0, \quad \gamma < \nu, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\nu)} x^{\gamma-\nu}, & otherwise. \end{cases}$

3. Mott-Fractional Mott functions

In this section, we define the explicit formula of the Mott polynomials $S_m(t)$ as follows [17]:

$$S_m(t) = \left(\frac{-t}{2}\right)^m {}_3F_0\left(-m, \frac{1}{2} - \frac{m}{2}, 1 - \frac{m}{2}; ; -4t^{-2}\right). \tag{3.1}$$

Here, ${}_3F_0$ is a generalized hypergeometric function [27]

$$ {}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;t\right)=\sum_{l=0}^{\infty}\frac{(a_1)_l\ldots(a_p)_l}{(b_1)_l\ldots(b_q)_l}\frac{t^l}{l!}, $$

and $(a)_l$ denotes the Pochhammer symbol

$$ (a)_0=1,\quad (a)_l=a(a+1)(a+2)\ldots(a+l-1),\quad l\geq 1. $$

Now, fractional Mott functions are introduced with the help of Mott polynomials definitions and change of variable x to t^λ, $0<\lambda\leq 1$ as follows:

$$ S_m^\lambda(t)=\left(\frac{-t^\lambda}{2}\right)^m {}_3F_0\left(-m,\frac{1}{2}-\frac{m}{2},1-\frac{m}{2};;-4t^{-2\lambda}\right). \tag{3.2} $$

In view of the above relation, the first few functions are obtained as follows:

$$ S_0^\lambda(t)=1,\quad S_1^\lambda(t)=-\frac{1}{2}t^\lambda,\quad S_2^\lambda(t)=\frac{1}{4}t^{2\lambda},\quad S_3^\lambda(t)=-\frac{3}{4}t^\lambda-\frac{1}{8}t^{3\lambda}. $$

Therefore, from MPs and FMFs definitions in Eqs. (3.1) and (3.2), can be described MFM-Fs:

$$ S_{m_1,m_2}^\lambda(x,t)=S_{m_1}(x)S_{m_2}^\lambda(t). \tag{3.3} $$

It should be noted that each bivariate function $f\in L^2([a,b]\times[c,d])$ can be approximated in terms of MFM-Fs as follows:

$$ f(x,t)\simeq\sum_{m_1=0}^{\mathcal{M}_1}\sum_{m_2=0}^{\mathcal{M}_2} f_{m_1,m_2}S_{m_1,m_2}^\lambda(x,t), \tag{3.4} $$

where the unknown coefficients are obtained by collocation points.

4. Requirement matrices

This section provides a new algorithm for calculating the components of operational matrices of integration.

4.1. **Modified operational matrix of integration.** This section presents the modified operational matrix of integration for Mott polynomials. Hence, we consider the following relation:

$$ \int_0^x S(\xi)d\xi=\mathcal{P}S(x)+\mathcal{R}(x). \tag{4.1} $$

Here, we denote $\mathcal{P}$ and $\mathcal{R}(x)$ as modified operational matrix of integration and complement vector, respectively. To gain the desired results, we apply Eq. (3.1) as follows:

$$ \begin{aligned} \int_0^x S_m(\xi)d\xi &= \int_0^x\left(\frac{-\xi}{2}\right)^m {}_3F_0\left(-m,\frac{1}{2}-\frac{m}{2},1-\frac{m}{2};;-4\xi^{-2}\right)d\xi \\ &= \int_0^x\left(\frac{-\xi}{2}\right)^m\sum_{l=0}^{[\frac{m}{2}]}(-m)_l(\frac{1}{2}-\frac{m}{2})_l(1-\frac{m}{2})_l(-4\xi^{-2})^l d\xi \\ &= \sum_{l=0}^{[\frac{m}{2}]}(-m)_l(\frac{1}{2}-\frac{m}{2})_l(1-\frac{m}{2})_l(-1)^{m+l}2^{2l-m}\left(\int_0^x\xi^{m-2l}d\xi\right) \\ &= \sum_{l=0}^{[\frac{m}{2}]}(-m)_l(\frac{1}{2}-\frac{m}{2})_l(1-\frac{m}{2})_l(-1)^{m+l}2^{2l-m}\frac{x^{m-2l+1}}{m-2l+1}. \end{aligned} \tag{4.2} $$

In view of the above relation, for $m = 0, 1, 2, \dots, \mathcal{M}_1 - 1$, we have

$$\int_0^x S_m(\xi)d\xi = \sum_{l=0}^{[\frac{m}{2}]} \mu_{m,l} x^{m-2l+1}, \quad \mu_{m,l} = (-m)_l(\frac{1}{2} - \frac{m}{2})_l(1 - \frac{m}{2})_l \frac{(-1)^{m+l}2^{2l-m}}{m-2l+1}. \tag{4.3}$$

Next, by expanding x^{m-2l+1} with respect to Mott polynomials, we obtain

$$x^{m-2l+1} = \sum_{k=0}^{\mathcal{M}_1} a_k S_k(x).$$

As a result, we get

$$\int_0^x S_m(\xi)d\xi = \sum_{k=0}^{\mathcal{M}_1} \left(\sum_{l=0}^{[\frac{m}{2}]} a_k \mu_{m,l} \right) S_k(x) = \sum_{k=0}^{\mathcal{M}_1} \delta_{k,m,l} S_k(x). \tag{4.4}$$

Inspired by the above process, for $m = \mathcal{M}_1$, we have

$$\int_0^x S_{\mathcal{M}_1}(\xi)d\xi = \sum_{l=0}^{[\frac{\mathcal{M}_1}{2}]} \mu_{\mathcal{M}_1,l} x^{\mathcal{M}_1-2l+1} = \sum_{l=1}^{[\frac{\mathcal{M}_1}{2}]} \mu_{\mathcal{M}_1,l} x^{\mathcal{M}_1-2l+1} + \mu_{\mathcal{M}_1,0} x^{\mathcal{M}_1+1}, \tag{4.5}$$

where

$$\mu_{\mathcal{M}_1,l} = (-\mathcal{M}_1)_l(\frac{1}{2} - \frac{\mathcal{M}_1}{2})_l(1 - \frac{\mathcal{M}_1}{2})_l \frac{(-1)^{\mathcal{M}_1+l}2^{2l-\mathcal{M}_1}}{\mathcal{M}_1-2l+1}.$$

Now, we approximate $x^{\mathcal{M}_1-2l+1}$ by Mott polynomials as follows:

$$x^{\mathcal{M}_1-2l+1} = \sum_{k=0}^{\mathcal{M}_1} b_k S_k(x),$$

As a result, we get

$$\begin{aligned} \int_0^x S_{\mathcal{M}_1}(\xi)d\xi &= \sum_{k=0}^{\mathcal{M}_1} \left(\sum_{l=1}^{[\frac{\mathcal{M}_1}{2}]} b_k \mu_{\mathcal{M}_1,l} \right) S_k(x) + \mu_{\mathcal{M}_1,0} x^{\mathcal{M}_1+1} \\ &= \sum_{k=0}^{\mathcal{M}_1} \bar{\delta}_{k,m,l} S_k(x) + \mu_{\mathcal{M}_1,0} x^{\mathcal{M}_1+1}. \end{aligned} \tag{4.6}$$

Overall, the modified operational matrix of integration is achieved as follows:

$$\mathcal{P} = \begin{bmatrix} \delta_{0,0,l} & \delta_{1,0,l} & \dots & \delta_{\mathcal{M}_1,0,l} \\ \delta_{0,1,l} & \delta_{1,1,l} & \dots & \delta_{\mathcal{M}_1,1,l} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\delta}_{0,\mathcal{M}_1,l} & \bar{\delta}_{1,\mathcal{M}_1,l} & \dots & \bar{\delta}_{\mathcal{M}_1,\mathcal{M}_1,l} \end{bmatrix}, \quad \mathcal{R}(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \mu_{\mathcal{M}_1,0} x^{\mathcal{M}_1+1} \end{bmatrix}.$$

4.2. **Pseudo-operational matrix of fractional integration.** This section introduces an efficient algorithm to obtain the pseudo-operational matrix of fractional integration. Thus, we assume

$$\mathcal{I}^{\nu}\left[S^{\lambda}(t)\right] \simeq t^{\nu} \Upsilon^{\nu} S^{\lambda}(t). \tag{4.7}$$

Here, Υ^{ν} is the pseudo-operational matrix of fractional integration. In the first step, taking the fractional integration of order ν, we evaluate the components of the proposed matrix:

$$\begin{aligned}\mathcal{I}^{\nu}\left[S_m^{\lambda}(t)\right] &= \mathcal{I}^{\nu}\left[\left(\frac{-t^{\lambda}}{2}\right)^m {}_3F_0\left(-m,\frac{1}{2}-\frac{m}{2},1-\frac{m}{2};;-4t^{-2\lambda}\right)\right] \\ &= \mathcal{I}^{\nu}\left[\left(\frac{-t^{\lambda}}{2}\right)^m \sum_{l=0}^{\left[\frac{\mathcal{M}_2}{2}\right]}(-m)_l(\frac{1}{2}-\frac{m}{2})_l(1-\frac{m}{2})_l(-4t^{-2\lambda})^l\right] \\ &= \sum_{l=0}^{\left[\frac{\mathcal{M}_2}{2}\right]}(-m)_l(\frac{1}{2}-\frac{m}{2})_l(1-\frac{m}{2})_l(-1)^{m+l}2^{2l-m}\mathcal{I}^{\nu}\left[t^{(m-2l)\lambda}\right] \\ &= \sum_{l=0}^{\left[\frac{\mathcal{M}_2}{2}\right]}(-m)_l(\frac{1}{2}-\frac{m}{2})_l(1-\frac{m}{2})_l(-1)^{m+l}2^{2l-m}\frac{\Gamma((m-2l)\lambda+1)}{\Gamma((m-2l)\lambda+1+\nu)}t^{(m-2l)\lambda+\nu} \\ &= t^{\nu}\sum_{l=0}^{\left[\frac{\mathcal{M}_2}{2}\right]} r_{m,l}^{\nu,\lambda}t^{(m-2l)\lambda},\end{aligned} \tag{4.8}$$

where

$$r_{m,l}^{\nu,\lambda}=(-m)_l(\frac{1}{2}-\frac{m}{2})_l(1-\frac{m}{2})_l(-1)^{m+l}2^{2l-m}\frac{\Gamma((m-2l)\lambda+1)}{\Gamma((m-2l)\lambda+1+\nu)}.$$

To continue the computational process, we approximate $t^{(m-2l)\lambda}$ with respect to FMFs:

$$t^{(m-2l)\lambda}\simeq\sum_{k=0}^{\mathcal{M}_2}d_kS_k^{\lambda}(t).$$

Therefore, we can derive the following result:

$$\begin{aligned}\mathcal{I}^{\nu}\left[S_m^{\lambda}(t)\right] &\simeq t^{\nu}\sum_{k=0}^{\mathcal{M}_2}\sum_{l=0}^{\left[\frac{\mathcal{M}_2}{2}\right]}d_kr_{m,l}^{\nu,\lambda}S_k^{\lambda}(t) \\ &= t^{\nu}\sum_{k=0}^{\mathcal{M}_2}\gamma_{k,m,l}^{\nu,\lambda}S_k^{\lambda}(t),\quad \gamma_{k,m,l}^{\nu,\lambda}=\sum_{l=0}^{\left[\frac{\mathcal{M}_2}{2}\right]}d_kr_{m,l}^{\nu,\lambda}.\end{aligned} \tag{4.9}$$

For generally, the proposed matrix is deduced as follows:

$$\Upsilon^{\nu}=\begin{bmatrix}\gamma_{0,0,l}^{\nu,\lambda} & \gamma_{1,0,l}^{\nu,\lambda} & \cdots & \gamma_{\mathcal{M}_2,0,l}^{\nu,\lambda} \\ \gamma_{0,1,l}^{\nu,\lambda} & \gamma_{1,1,l}^{\nu,\lambda} & \cdots & \gamma_{\mathcal{M}_2,1,l}^{\nu,\lambda} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{0,\mathcal{M}_2,l}^{\nu,\lambda} & \gamma_{1,\mathcal{M}_2,l}^{\nu,\lambda} & \cdots & \gamma_{\mathcal{M}_2,\mathcal{M}_2,l}^{\nu,\lambda}\end{bmatrix}.$$

5. Description of methodology

The main objective of this section is to provide a new numerical scheme for solving the fractional partial integro-differential viscoelastic equations with weakly singular kernels.

Step 1: In this step, the proposed problem defined in Eq. (1.1) is converted to fractional partial integral equations with weakly singular kernels with the help of the Riemann-Liouville fractional integral operator [18]. Hence, we obtain

$$u(x,t)=\mathcal{I}_t^{\nu}f(x,t)+g_0(x)+\vartheta\mathbf{V}^{\nu}(x,t)+\Gamma(1-\beta)\mathbf{W}^{\nu,\beta}(x,t), \tag{5.1}$$

where

$$\begin{aligned}\mathbf{V}^{\nu}(x,t) &= \frac{1}{\Gamma(\nu)}\int_0^t (t-\eta)^{\nu-1}u_{xx}(x,\eta)d\eta = \mathcal{I}_t^{\nu}u_{xx}(x,t), \\ \mathbf{W}^{\nu,\beta}(x,t) &= \frac{1}{\Gamma(\nu-\beta+1)}\int_0^t (t-\eta)^{\nu-\beta}u_{xx}(x,\eta)d\eta = \mathcal{I}_t^{\nu-\beta}u_{xx}(x,t).\end{aligned} \tag{5.2}$$

Step 2: In this section, we intend to introduce the numerical process of obtaining the approximate solution. To this end, we assume

$$u_{xx}(x,t) \simeq S^T(x)US^{\lambda}(t), \tag{5.3}$$

where

$$U = [u_{ij}], \quad i = 0,1,\dots,\mathcal{M}_1, \quad j = 0,1,\dots,\mathcal{M}_2.$$

Next, we obtain the approximation of other functions by integrating Eq. (5.3) with respect to variable x as follows:

$$u_x(x,t) \simeq \left(S^T(x)\mathcal{P}^T + \mathcal{R}^T(x)\right)US^{\lambda}(t) + u_x(0,t), \tag{5.4}$$

In the aforementioned relation, $u_x(0,t)$ is unknown function. To evaluate this function, we take an integral from Eq. (5.4) with respect to x from 0 to 1 as follows:

$$u_x(0,t) = u(1,t) - u(0,t) - \left(\int_0^1 S^T(x)dx\mathcal{P}^T + \int_0^1 \mathcal{R}^T(x)dx\right)US^{\lambda}(t). \tag{5.5}$$

Therefore, from Eqs. (5.4) and (5.5), we have

$$\begin{aligned}u_x(x,t) &\simeq \left(S^T(x)\mathcal{P}^T + \mathcal{R}^T(x)\right)US^{\lambda}(t) + u(1,t) - u(0,t) \\ &- \left(\int_0^1 S^T(x)dx\mathcal{P}^T + \int_0^1 \mathcal{R}^T(x)dx\right)US^{\lambda}(t).\end{aligned} \tag{5.6}$$

As the integration process continues, we will have:

$$\begin{aligned}u(x,t) &\simeq \left(S^T(x)\mathcal{P}^{2T} + \mathcal{R}^T(x)\mathcal{P}^T + \int_0^x \mathcal{R}^T(\xi)d\xi\right)US^{\lambda}(t) + xu_x(0,t) + u(0,t) \\ &= \left[\begin{array}{c}\left(S^T(x)\mathcal{P}^{2T} + \mathcal{R}^T(x)\mathcal{P}^T + \int_0^x \mathcal{R}^T(\xi)d\xi\right)US^{\lambda}(t) \\ +x\left(u(1,t) - u(0,t) - \left(\int_0^1 S^T(x)dx\mathcal{P}^T + \int_0^1 \mathcal{R}^T(x)dx\right)US^{\lambda}(t)\right) \\ +u(0,t)\end{array}\right].\end{aligned} \tag{5.7}$$

Now, by combining Eq. (5.3) and the pseudo-operational matrix of fractional integration, we conclude

$$\mathcal{I}_t^{\nu}u_{xx}(x,t) \simeq S^T(x)U\mathcal{I}_t^{\nu}[S^{\lambda}(t)] = t^{\nu}S^T(x)U\Upsilon^{\nu}S^{\lambda}(t), \tag{5.8}$$

and

$$\mathcal{I}_t^{\nu-\beta}u_{xx}(x,t) \simeq S^T(x)U\mathcal{I}_t^{\nu-\beta}[S^{\lambda}(t)] = t^{\nu-\beta}S^T(x)U\Upsilon^{\nu-\beta}S^{\lambda}(t). \tag{5.9}$$

Step 3: In last step, by substituting approximation relations described in previous step in Eq. (5.1), we deduce

$$R(x,t) = \left[\begin{array}{c}\left(S^T(x)\mathcal{P}^{2T} + \mathcal{R}^T(x)\mathcal{P}^T + \int_0^x \mathcal{R}^T(\xi)d\xi\right)US^{\lambda}(t) \\ +x\left(u(1,t) - u(0,t) - \left(\int_0^1 S^T(x)dx\mathcal{P}^T + \int_0^1 \mathcal{R}^T(x)dx\right)US^{\lambda}(t)\right) + u(0,t) \\ -\mathcal{I}_t^{\nu}f(x,t) - g_0(x) - \vartheta\left(t^{\nu}S^T(x)U\Upsilon^{\nu}S^{\lambda}(t)\right) - \Gamma(1-\beta)\left(t^{\nu-\beta}S^T(x)U\Upsilon^{\nu-\beta}S^{\lambda}(t)\right)\end{array}\right]. \tag{5.10}$$

Then, by replacing the nodal points [8], we get the following system of equations:

$$R(x_i,t_j) = 0, \quad i = 1,2,\dots,\mathcal{M}_1+1, \quad j = 1,2,\dots,\mathcal{M}_2+1. \tag{5.11}$$

Finally, we achieve the elements of the unknown coefficients matrix, by using the mathematical software's.

6. Error estimation

In this section, we investigate the error of the approximate solution and residual error function through the following lemmas and theorems in Sobolev space.

Lemma 6.1. *Assume that $u \in H^m(a,b)$ and $u_{\mathcal{M}_1}(x) = \sum_{i=0}^{\mathcal{M}_1} a_i S_i(x)$ is the best approximation of $u(x)$. Then, we have [15]*

$$\|u - u_{\mathcal{M}_1}\|_{L^\infty(a,b)} \leq C\mathcal{M}_1^{\frac{1}{2}-m}\sqrt{\frac{2}{b-a}}|||u|||_{H^{m,\mathcal{M}_1}(a,b)}, \tag{6.1}$$

where C is a positive constant so that independent of $\mathcal{M}_1$ and dependent on m.

Lemma 6.2. *Let $u \in H^m(c,d)$ and $u_{\mathcal{M}_2}(t) = \sum_{j=0}^{\mathcal{M}_2} b_j S_j^\lambda(t)$ is the best approximation of $u(t)$. Then, we obtain*

$$\|u - u_{\mathcal{M}_2}\|_{L^\infty(c,d)} \leq C\kappa^{\frac{1}{2}-m}\mathcal{M}_2^{\frac{1}{2}-m}\sqrt{\frac{2}{d-c}}|||u|||_{H^{m,\lambda\mathcal{M}_2}(c,d)}. \tag{6.2}$$

Proof. In view of Eq. (6.1) and variable change of $\mathcal{M}_1 = \kappa\mathcal{M}_2$, we get the desired result, directly. □

Theorem 6.3. *Assume that the functions*

$$u_{\mathcal{M}_1}(x,t) = \sum_{m_2=0}^{\infty} f_{\mathcal{M}_1,m_2} S_{\mathcal{M}_1}(x) S_{m_2}^\lambda(t), \quad u_{\mathcal{M}_2}(x,t) = \sum_{m_1=0}^{\infty} f_{m_1,\mathcal{M}_2} S_{m_1}(x) S_{\mathcal{M}_2}^\lambda(t),$$

are approximated by MFM-Fs as follows:

$$u_{\mathcal{M}_1,m_2}(x,t) = \sum_{m_2=0}^{\mathcal{M}_2} f_{\mathcal{M}_1,m_2} S_{\mathcal{M}_1}(x) S_{m_2}^\lambda(t), \quad u_{m_1,\mathcal{M}_2}(x,t) = \sum_{m_1=0}^{\mathcal{M}_1} f_{m_1,\mathcal{M}_2} S_{m_1}(x) S_{\mathcal{M}_2}^\lambda(t).$$

Then, the upper bound of error on the interval $\Omega = (a,b) \times (c,d)$ can be expressed as follows:

$$\begin{aligned}
\|u_{\mathcal{M}_1} - u_{\mathcal{M}_1,m_2}\|_{L^\infty(\Omega)} &\leq C\kappa^{\frac{1}{2}-m}\mathcal{M}_2^{\frac{1}{2}-m}\sqrt{\frac{2}{d-c}}|||u|||_{H^{m,\lambda\mathcal{M}_2}(c,d)}\|S_{\mathcal{M}_1}\|_{L^\infty(a,b)}, \\
\|u_{\mathcal{M}_2} - u_{m_1,\mathcal{M}_2}\|_{L^\infty(\Omega)} &\leq C\mathcal{M}_1^{\frac{1}{2}-m}\sqrt{\frac{2}{b-a}}|||u|||_{H^{m,\mathcal{M}_1}(a,b)}\|S_{\mathcal{M}_2}\|_{L^\infty(c,d)}.
\end{aligned} \tag{6.3}$$

Proof. According to the assumptions and Eqs. (6.1) and (6.2), we formulate the error as follows:

$$\begin{aligned}
\|u_{\mathcal{M}_1} - u_{\mathcal{M}_1,m_2}\|_{L^\infty(\Omega)} &= \|\sum_{m_2=0}^{\infty} f_{\mathcal{M}_1,m_2} S_{\mathcal{M}_1} S_{m_2}^\lambda - \sum_{m_2=0}^{\mathcal{M}_2} f_{\mathcal{M}_1,m_2} S_{\mathcal{M}_1} S_{m_2}^\lambda\|_{L^\infty(\Omega)} \\
&\leq \|S_{\mathcal{M}_1}\|_{L^\infty(a,b)}\|\sum_{m_2=0}^{\infty} f_{\mathcal{M}_1,m_2} S_{m_2}^\lambda - \sum_{m_2=0}^{\mathcal{M}_2} f_{\mathcal{M}_1,m_2} S_{m_2}^\lambda\|_{L^\infty(c,d)} \\
&\leq C\kappa^{\frac{1}{2}-m}\mathcal{M}_2^{\frac{1}{2}-m}\sqrt{\frac{2}{d-c}}|||u|||_{H^{m,\lambda\mathcal{M}_2}(c,d)}\|S_{\mathcal{M}_1}\|_{L^\infty(a,b)},
\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}
\|u_{\mathcal{M}_2} - u_{m_1,\mathcal{M}_2}\|_{L^\infty(\Omega)} &= \|\sum_{m_1=0}^{\infty} f_{m_1,\mathcal{M}_2} S_{m_1} S_{\mathcal{M}_2}^\lambda - \sum_{m_1=0}^{\mathcal{M}_1} f_{m_1,\mathcal{M}_2} S_{m_1} S_{\mathcal{M}_2}^\lambda\|_{L^\infty(\Omega)} \\
&\leq \|\sum_{m_1=0}^{\infty} f_{m_1,\mathcal{M}_2} S_{m_1} - \sum_{m_1=0}^{\mathcal{M}_1} f_{m_1,\mathcal{M}_2} S_{m_1}\|_{L^\infty(a,b)}\|S_{\mathcal{M}_2}^\lambda\|_{L^\infty(c,d)} \\
&\leq C\mathcal{M}_1^{\frac{1}{2}-m}\sqrt{\frac{2}{b-a}}|||u|||_{H^{m,\mathcal{M}_1}(a,b)}\|S_{\mathcal{M}_2}^\lambda\|_{L^\infty(c,d)}.
\end{aligned} \tag{6.5}$$

Thus, the theorem is proved. □

Corollary 6.4. *Given that in the first and second inequalities in Eq. (6.3), components $\mathcal{M}_1$ and $\mathcal{M}_2$ are fixed, respectively. It can be concluded*

$$\begin{aligned} \mathcal{M}_2 \to \infty, \quad &\Rightarrow \quad \|u_{\mathcal{M}_1} - u_{\mathcal{M}_1,m_2}\|_{L^\infty(\Omega)} \to 0, \quad \textit{with } \mathcal{O}(\mathcal{M}_2^{\frac{1}{2}-m}), \\ \mathcal{M}_1 \to \infty, \quad &\Rightarrow \quad \|u_{\mathcal{M}_1} - u_{\mathcal{M}_1,m_2}\|_{L^\infty(\Omega)} \to 0, \quad \textit{with } \mathcal{O}(\mathcal{M}_1^{\frac{1}{2}-m}). \end{aligned}$$

Also, according to Eq. (5.1) and the approximation method process, can be defined the following relations:

$$\mathcal{T}u(x,t) = \mathcal{I}_t^\nu f(x,t) + g_0(x) + \vartheta \mathbf{V}^\nu(x,t) + \Gamma(1-\beta)\mathbf{W}^{\nu,\beta}(x,t), \tag{6.6}$$

and

$$\mathcal{T}u_{\mathcal{M}_1,\mathcal{M}_2}(x,t) = \mathcal{I}_t^\nu f(x,t) + g_0(x) + \vartheta \mathbf{V}^\nu_{\mathcal{M}_1,\mathcal{M}_2}(x,t) + \Gamma(1-\beta)\mathbf{W}^{\nu,\beta}_{\mathcal{M}_1,\mathcal{M}_2}(x,t). \tag{6.7}$$

Therefore, given the last two relations, we will have:

$$\begin{aligned} \|\mathcal{T}u - \mathcal{T}u_{\mathcal{M}_1,\mathcal{M}_2}\|_{L^\infty(\Omega)} \quad &\leq \quad \vartheta \|\mathbf{V}^\nu - \mathbf{V}^\nu_{\mathcal{M}_1,\mathcal{M}_2}\|_{L^\infty(\Omega)} \\ &+ \quad \Gamma(1-\beta)\|\mathbf{W}^{\nu,\beta} - \mathbf{W}^{\nu,\beta}_{\mathcal{M}_1,\mathcal{M}_2}\|_{L^\infty(\Omega)}. \end{aligned} \tag{6.8}$$

Besides, from Eq. (5.2), we conclude

$$\begin{aligned} \|\mathbf{V}^\nu - \mathbf{V}^\nu_{\mathcal{M}_1,\mathcal{M}_2}\|_{L^\infty(\Omega)} \quad &= \quad \frac{1}{\Gamma(\nu)}\| \int_0^t (t-\eta)^{\nu-1} \left[u_{xx}(x,\eta) - (u_{\mathcal{M}_1,\mathcal{M}_2})_{xx}(x,\eta)\right] d\eta \|_{L^\infty(\Omega)} \\ &\leq \quad \frac{1}{\Gamma(\nu)}\| \int_0^t (t-\eta)^{\nu-1} d\eta \|_{L^\infty(\Omega)} \|u_{xx}(x,\eta) - (u_{\mathcal{M}_1,\mathcal{M}_2})_{xx}(x,\eta)\|_{L^\infty(\Omega)} \\ &\leq \quad \frac{d^\nu}{\nu\Gamma(\nu)}\|u_{xx} - (u_{\mathcal{M}_1,\mathcal{M}_2})_{xx}\|_{L^\infty(\Omega)}, \end{aligned} \tag{6.9}$$

and

$$\begin{aligned} \|\mathbf{W}^{\nu,\beta} - \mathbf{W}^{\nu,\beta}_{\mathcal{M}_1,\mathcal{M}_2}\|_{L^\infty(\Omega)} \quad &= \quad \frac{1}{\Gamma(\nu-\beta+1)}\| \int_0^t (t-\eta)^{\nu-\beta} \left[u_{xx}(x,\eta) - (u_{\mathcal{M}_1,\mathcal{M}_2})_{xx}(x,\eta)\right] d\eta \|_{L^\infty(\Omega)} \\ &\leq \quad \frac{1}{\Gamma(\nu-\beta+1)}\| \int_0^t (t-\eta)^{\nu-\beta} d\eta \|_{L^\infty(\Omega)} \|u_{xx}(x,\eta) - (u_{\mathcal{M}_1,\mathcal{M}_2})_{xx}(x,\eta)\|_{L^\infty(\Omega)} \\ &\leq \quad \frac{d^{\nu-\beta+1}}{(\nu-\beta+1)\Gamma(\nu-\beta+1)}\|u_{xx} - (u_{\mathcal{M}_1,\mathcal{M}_2})_{xx}\|_{L^\infty(\Omega)}. \end{aligned} \tag{6.10}$$

As a result, we obtain

$$\begin{aligned} \|\mathcal{T}u - \mathcal{T}u_{\mathcal{M}_1,\mathcal{M}_2}\|_{L^\infty(\Omega)} \quad &\leq \quad \left(\frac{\vartheta d^\nu}{\nu\Gamma(\nu)} + \frac{d^{\nu-\beta+1}\Gamma(1-\beta)}{(\nu-\beta+1)\Gamma(\nu-\beta+1)} \right) \\ &\times \quad \|u_{xx} - (u_{\mathcal{M}_1,\mathcal{M}_2})_{xx}\|_{L^\infty(\Omega)}. \end{aligned} \tag{6.11}$$

From this result, it can be understood that the error of integral parts in Eq. (1.1) tends to zero.

7. Numerical examples

In the current section, we indicate the behavior and performance of the proposed approach with tables and graphical plots. The computations in numerical experiments were performed on a personal computer and codes were written in

MATLAB 2016. In some examples of this section, we calculate the maximum absolute error E_∞ and root-mean-square error E_2, which are defined as follows:

$$\begin{aligned} E_2 &= \left(\sum_{i=1}^{\mathcal{M}_1} \sum_{j=1}^{\mathcal{M}_2} |u(x_i, t_j) - u_{\mathcal{M}_1,\mathcal{M}_2}(x_i, t_j)|^2 \right)^{\frac{1}{2}}, \\ E_\infty &= \max_{1 \le i \le \mathcal{M}_1, 1 \le j \le \mathcal{M}_2} |u(x_i, t_j) - u_{\mathcal{M}_1,\mathcal{M}_2}(x_i, t_j)|, \end{aligned}$$

where $u_{\mathcal{M}_1,\mathcal{M}_2}$ is the approximation of u. Furthermore, in order to evaluate the stability of the proposed technique numerically, we perform the numerical technique by adding a disturbance or noise in the initial data [19, 37]. In this process, the initial condition with the added noise δ is denoted as $u_\delta(x, t)$. Therefore, we have

$$u_\delta(x_i, t) = u(x_i, t) + \delta\omega,$$

where ω is the uniform random variable defined in interval $[-1, 1]$ and

$$\max_{i \in [0, \mathcal{N}]} (u(x_i, t) - u_\delta(x_i, t)) \le \delta.$$

In our examples, we have selected the noise level δ to be the maximum absolute error, and the number of grid points $\mathcal{N}$ to be 100.

Example 7.1. Consider the following fractional partial integro-differential equation with weakly singular kernel [1, 38]:

$$D_t^\nu u(x, t) - \int_0^t (t - \eta)^{\frac{-1}{2}} u_{xx}(x, \eta) d\eta = f(x, t),$$

where

$$f(x, t) = \frac{3}{2} t^{\frac{1}{2}} \left((1 - x^2) + \frac{\sqrt{\pi}}{2} t^{\frac{3}{2}} \right),$$

with the initial condition $u(x, 0) = t^{\frac{3}{2}}$ and the boundary conditions $u(0, t) = t^{\frac{3}{2}}$, $u(1, t) = 0$. The exact solution to this problem, when $\nu = 1$ is $u(x, t) = t^{\frac{3}{2}} - x^2 t^{\frac{3}{2}}$. By implementing the described method in this example for $\mathcal{M}_1 = 2, \mathcal{M}_2 = 3$ and $\lambda = \frac{1}{2}, \nu = 1$, we get the elements of coefficient matrix as follows:

$$\begin{aligned} &u_{00} = -1.979591 \times 10^{-13}, \quad u_{01} = -23.9999999999, \quad u_{02} = -1.431847 \times 10^{-11}, \quad u_{03} = 15.9999999999, \\ &u_{10} = -2.374310 \times 10^{-12}, \quad u_{11} = 2.599921 \times 10^{-10}, \quad u_{12} = -1.555818 \times 10^{-10}, \quad u_{13} = -1.976429 \times 10^{-10}, \\ &u_{20} = -4.725873 \times 10^{-12}, \quad u_{21} = 5.183790 \times 10^{-10}, \quad u_{22} = -3.098335 \times 10^{-10}, \quad u_{23} = -3.939805 \times 10^{-10}. \end{aligned}$$

Then, from the proposed method, we gain

$$\begin{aligned} u(x, t) &= 1.978592 \times 10^{-13} x^3 - 9.897953 \times 10^{-14} x^2 - 9.845569 \times 10^{-14} x^4 \\ &- x(4.361933 \times 10^{-14} t^{\frac{1}{2}} - 1.622381 \times 10^{-13} t + 2.819795 \times 10^{-12} t^{\frac{3}{2}} + 4.240113 \times 10^{-16}) \\ &- t(1.789809 \times 10^{-12} x^2 - 3.241288 \times 10^{-12} x^3 + 1.613716 \times 10^{-12} x^4) \\ &- t^{\frac{3}{2}}(0.999999999996 x^2 + 2.0587807 \times 10^{-12} x^3 - 1.0259909 \times 10^{-12} x^4) \\ &+ t^{\frac{1}{2}}(8.07132138 \times 10^{-12} x^2 - 1.519677 \times 10^{-12} x^3 + 7.561644 \times 10^{-13} x^4) + t^{\frac{3}{2}}. \end{aligned}$$

Further to show the effect of values $\lambda, \mathcal{M}_1$ and $\mathcal{M}_2$ on the approximate solution of the problem, we consider $\mathcal{M}_1 = 2, \mathcal{M}_2 = 6$ and $\lambda = \frac{1}{4}, \nu = 1$:

$$u_{00} = -1.5255320097291032 \times 10^{-7}, \quad u_{01} = -0.0017279968902905987, \quad u_{02} = -11931.43831739695,$$
$$u_{03} = 0.0036421854538236475, \quad u_{04} = 4388.575263477915, \quad u_{05} = -0.0002492180596667261,$$
$$u_{06} = -128.00011810697552, \quad u_{10} = -5.236951878173316 \times 10^{-6}, \quad u_{11} = -0.040333470267107506,$$
$$u_{12} = -0.21590219548868025, \quad u_{13} = 0.08553701803325363, \quad u_{14} = 0.08558929919890762,$$
$$u_{15} = -0.005870935885140051, \quad u_{16} = -0.002651538995498436, \quad u_{20} = -0.000010224405357276077,$$
$$u_{21} = -0.07677885382244676, \quad u_{22} = -0.409790537138569, \quad u_{23} = 0.16289619978067094,$$
$$u_{24} = 0.16252700965988026, \quad u_{25} = -0.011182919469824443, \quad u_{26} = -0.005036910573449091.$$

Then, we obtain

$$\begin{aligned} u(x,t) &= 4.364126 \times 10^{-7}x^3 - 7.627660 \times 10^{-6}x^2 - x(7.770777 \times 10^{-6}t + 4.227549 \times 10^{-6}t^{\frac{1}{2}} \\ &- 1.227388 \times 10^{-6}t^{\frac{1}{4}} + 8.901946 \times 10^{-7}t^{\frac{3}{2}} - 7.693849 \times 10^{-6}t^{\frac{3}{4}} - 4.1143168 \times 10^{-6}t^{\frac{5}{4}} \\ &+ 1.4712761 \times 10^{-7}) - 2.130084 \times 10^{-7}x^4 - t(6.702360 \times 10^{-6}x^2 - 2.771150 \times 10^{-5}x^3 \\ &+ 1.32383 \times 10^{-5}x^3) + t^{\frac{5}{4}}(3.894032 \times 10^{-6}x^2 - 1.528889 \times 10^{-6}x^3 + 7.2805465 \times 10^{-6}x^4) \\ &+ t^{\frac{1}{4}}(7.474011 \times 10^{-7}x^2 - 3.832562 \times 10^{-6}x^3 + 1.857772 \times 10^{-6}x^4) \\ &- t^{\frac{3}{2}}(1.00000092271x^2 - 3.452524 \times 10^{-6}x^3 + 1.639619 \times 10^{-6}x^4) \\ &+ t^{\frac{3}{4}}(6.005340 \times 10^{-6}x^2 - 2.632312 \times 10^{-5}x^3 + 1.262393 \times 10^{-5}x^4) \\ &- t^{\frac{1}{2}}(2.94552 \times 10^{-6}x^2 - 1.384452 \times 10^{-5}x^3 + 6.671451 \times 10^{-6}x^4) + t^{\frac{3}{2}}. \end{aligned}$$

Moreover, the error for different choices of parameters defined in the method are exhibited in Figure 3. The exact solution for $\nu \neq 1$ does not exist. Therefore, to demonstrate the efficiency of the method, we use the following residual error formula:

$$\|Res_{\mathcal{M}_1,\mathcal{M}_2}\|^2 = \int_0^1 \int_0^1 Res^2_{\mathcal{M}_1,\mathcal{M}_2}(x,t)dxdt,$$

where

$$Res_{\mathcal{M}_1,\mathcal{M}_2}(x,t) = D_t^{\nu} u_{\mathcal{M}_1,\mathcal{M}_2}(x,t) - \int_0^t (t-\eta)^{\frac{-1}{2}} (u_{\mathcal{M}_1,\mathcal{M}_2})_{xx}(x,\eta)d\eta - f(x,t).$$

According to the above formula, for $\nu = 0.7$ and $\nu = 0.2$, the residual errors are 9.4672×10^{-21} and 6.8304×10^{-19}, respectively.

Example 7.2. Consider the following fractional partial integro-differential equation with weakly singular kernel [39]:

$$D_t^{\nu} u(x,t) = u_{xx}(x,t) + \int_0^t (t-\eta)^{\frac{-1}{2}} u_{xx}(x,\eta)d\eta + f(x,t),$$

with the initial condition $u(x,0) = 0$ and the boundary conditions $u(0,t) = 0,\ u(1,t) = t^{\nu}$. Here, we obtain the function $f(x,t)$ based on the exact solution $u(x,t) = x^2t^{\nu}$. Given the presented approach, for $\nu = \lambda = 1$ and $\mathcal{M}_1 = 2, \mathcal{M}_2 = 1$, we get the following results:

$$u_{00} = -2.4466520 \times 10^{-16}, \quad u_{01} = -4.000000000000002,$$
$$u_{10} = -2.2741255 \times 10^{-15}, \quad u_{11} = -1.8254715 \times 10^{-14},$$
$$u_{20} = -4.54825 \times 10^{-15}, \quad u_{21} = -3.650943 \times 10^{-14},$$

and

$$\begin{aligned} u(x,t) &= 1.895104 \times 10^{-16}x^3 - 1.223326 \times 10^{-16}x^2 - 9.475522 \times 10^{-17}x^4 \\ &+ t(x^2 - 7.606131 \times 10^{-16}x^3 + 3.8030658 \times 10^{-16}x^4) \\ &+ x(3.803065 \times 10^{-16}t + 2.757737 \times 10^{-17}). \end{aligned}$$

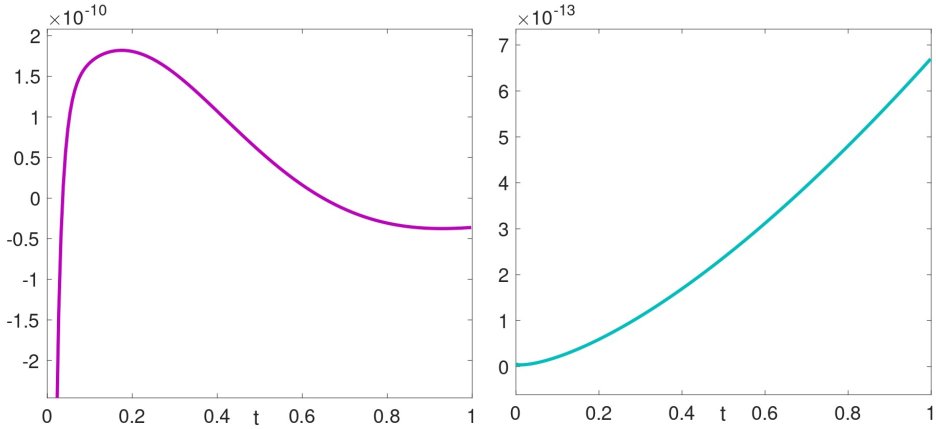

FIGURE 3. The error for $\mathcal{M}_1 = 2, \mathcal{M}_2 = 6$ with $\lambda = \frac{1}{6}$ (left) and $\mathcal{M}_1 = 2, \mathcal{M}_2 = 3$ with $\lambda = \frac{1}{2}$ (right) and $\nu = 1, x = 0.5$ of Example 7.1.

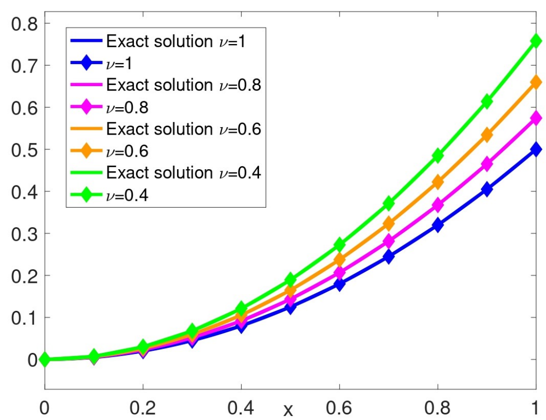

FIGURE 4. Exact and approximate solutions for different choices of $\nu = \lambda$ with $\mathcal{M}_1 = \mathcal{M}_2 = 2$ of Example 7.2.

Also, the outcomes of this example for different choices of ν, λ with $\mathcal{M}_1 = \mathcal{M}_2 = 2$ are illustrated in Figures 4 and 5. From the results, it can be understood that the proposed method is powerful in approximating the solution of problems with fractional power. Furthermore, we also assess the numerical stability of the technique. To accomplish this, we implement the proposed method by introducing a slight disturbance in the initial condition while considering various values of ν. Figures 5 and 6 showcase the error for different values of ν. Figure 5 represents the error without noise, while Figure 6 illustrates the error with noise incorporated. We can understand that the difference in numerical results between those with and without noise is negligible, which demonstrates the stability of our numerical scheme.

Example 7.3. Consider the following fractional partial integro-differential equation with weakly singular kernel [3]:

$$D_t^{\nu} u(x,t) = f(x,t) + u_{xx}(x,t) + \int_0^t \frac{u_{xx}(x,\eta)}{(t-\eta)^{\beta}} d\eta,$$

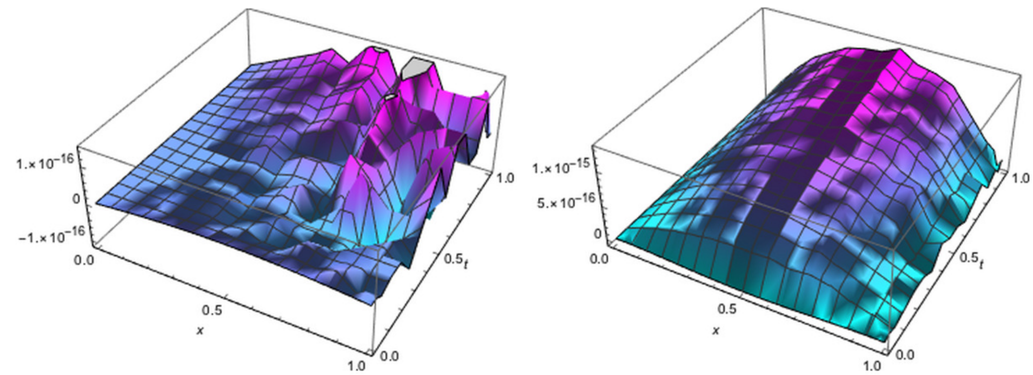

FIGURE 5. The error without noise for $\nu = \lambda = \frac{5}{6}$ (left) and $\nu = \lambda = \frac{3}{7}$ (right) with $\mathcal{M}_1 = \mathcal{M}_2 = 2$ of Example 7.2.

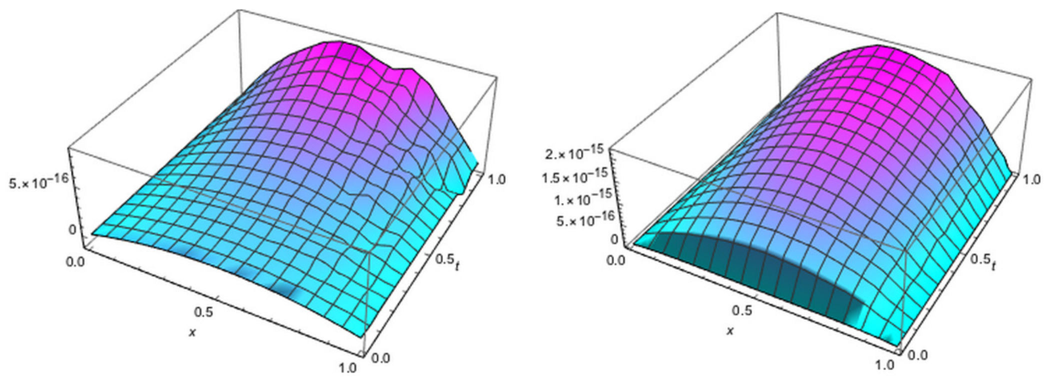

FIGURE 6. The error with noise for $\nu = \lambda = \frac{5}{6}$ (left) and $\nu = \lambda = \frac{3}{7}$ (right) with $\mathcal{M}_1 = \mathcal{M}_2 = 2$ of Example 7.2.

where

$$f(x,t) = \left(\frac{6}{\Gamma(4-\nu)}t^{3-\nu} + \frac{6\pi^2\Gamma(1-\beta)}{\Gamma(5-\beta)}t^{4+\beta} + \pi^2 t^3\right)\sin(\pi x),$$

with the initial condition $u(x,0) = 0$ and the boundary conditions $u(0,t) = u(1,t) = 0$. The exact solution to this problem is $u(x,t) = t^3 \sin(\pi x)$. Comparison of the absolute errors obtained by the present method and method in [3] are listed in Table 1. From these results it is obvious that our algorithm is more accurate the Legendre wavelet method [3]. Also, the maximum absolute error E_∞, root-mean-square error E_2 and CPU-time (in second) for different values of $\mathcal{M}_1$, $\mathcal{M}_2$ and λ are reported in Table 2. This table demonstrates that by increasing the number of basis functions, the error tends to zero. Additionally, we present a comprehensive analysis of error behavior in the entire domain, both with and without noise, as depicted in Figures 7 and 8. We can observe that the difference in numerical results between those with and without noise is negligible, which indicates the stability of our numerical scheme.

Example 7.4. Consider the following fractional partial integro-differential equation with weakly singular kernel [3]:

$$D_t^\nu u(x,t) = f(x,t) + \frac{1}{2}u_{xx}(x,t) + \int_0^t \frac{u_{xx}(x,\eta)}{(t-\eta)^\beta}d\eta,$$

where

$$f(x,t) = \left(\frac{1}{2\exp(t)} - \frac{E_{1,2-\nu}(-t)}{t^{\nu-1}} + \Gamma(1-\beta)\frac{E_{1,2-\beta}(-t)}{t^{\beta-1}}\right)\sin(x),$$

Table 1. The absolute errors for diverse values of $\mathcal{M}_1$ with $\mathcal{M}_2 = 4$ and $\nu = \frac{1}{2}, \beta = \frac{3}{4}, \lambda = 1$ of Example 7.3.

	Present method		Method in [3]	
$x = t$	$\mathcal{M}_1 = 4$	$\mathcal{M}_1 = 6$	$M = 4, k = 1$	$M = 6, k = 1$
0	0	0	–	–
0.1	1.0660×10^{-8}	2.2468×10^{-12}	1.6043×10^{-7}	3.6468×10^{-10}
0.2	1.6800×10^{-7}	1.1081×10^{-9}	1.9927×10^{-6}	1.4781×10^{-8}
0.3	5.0680×10^{-7}	1.9447×10^{-9}	1.1207×10^{-5}	6.4816×10^{-8}
0.4	8.5816×10^{-7}	5.6725×10^{-9}	4.9544×10^{-5}	1.7434×10^{-7}
0.5	1.6183×10^{-6}	1.1713×10^{-8}	7.7875×10^{-5}	4.4912×10^{-7}
0.6	2.9653×10^{-6}	1.9559×10^{-8}	2.2667×10^{-4}	7.9102×10^{-7}
0.7	6.6442×10^{-6}	2.5946×10^{-8}	2.6152×10^{-4}	1.3117×10^{-6}
0.8	1.1132×10^{-6}	7.3295×10^{-8}	3.8675×10^{-4}	1.9099×10^{-6}
0.9	7.3850×10^{-6}	7.3165×10^{-10}	9.7579×10^{-4}	1.8544×10^{-6}
1.0	7.3467×10^{-40}	7.3468×10^{-40}	–	–

Table 2. The errors for diverse values of $\mathcal{M}_1, \mathcal{M}_2$ and λ with $\nu = \frac{4}{5}, \beta = \frac{3}{4}$ of Example 7.3.

$\mathcal{M}_1$	$\mathcal{M}_2$	λ	E_2	E_∞	CPU
4	4	1	1.5062×10^{-5}	1.0916×10^{-5}	6.2265×10^{-1}
		1	4.5255×10^{-4}	3.2087×10^{-4}	6.1909×10^{-1}
		$\frac{1}{2}$	2.3976×10^{-3}	1.8657×10^{-3}	6.1681×10^{-1}
	8	$\frac{1}{2}$	1.5063×10^{-5}	1.0916×10^{-5}	1.11796
8	4	1	2.5451×10^{-10}	2.2047×10^{-10}	1.11031

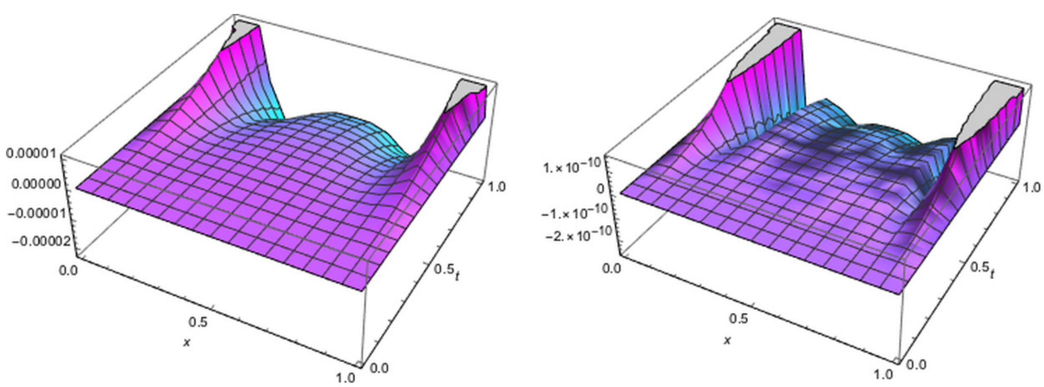

Figure 7. The error without noise for $\mathcal{M}_1 = 4$ (left) and $\mathcal{M}_1 = 6$ (right) with $\mathcal{M}_2 = 4$ and $\nu = \lambda = 1, \beta = \frac{1}{2}$ of Example 7.3.

with the initial condition $u(x, 0) = \sin(x)$ and the boundary conditions $u(0, t) = 0,\ u(1, t) = \sin(1)\exp(-t)$. The exact solution to this problem is $u(x, t) = \exp(-t)\sin(x)$. Here, $E_{\kappa,\vartheta}(t)$ denotes the Mittag-Leffler function [26]. By considering $\nu = \frac{2}{3}, \beta = \frac{1}{2}$ and $\lambda = 1$, we compare the absolute error gained by our approach with the Legendre wavelet method [3] in Table 3. Also, the error in the whole area of the domain for $\mathcal{M}_1 = 3, 6$ with $\nu = \beta = \frac{1}{2}$ and $\lambda = 1$ are plotted in Figure 9. The results indicate that the approximate solutions for different values of parameters defined in method have a good agreement with the exact solution.

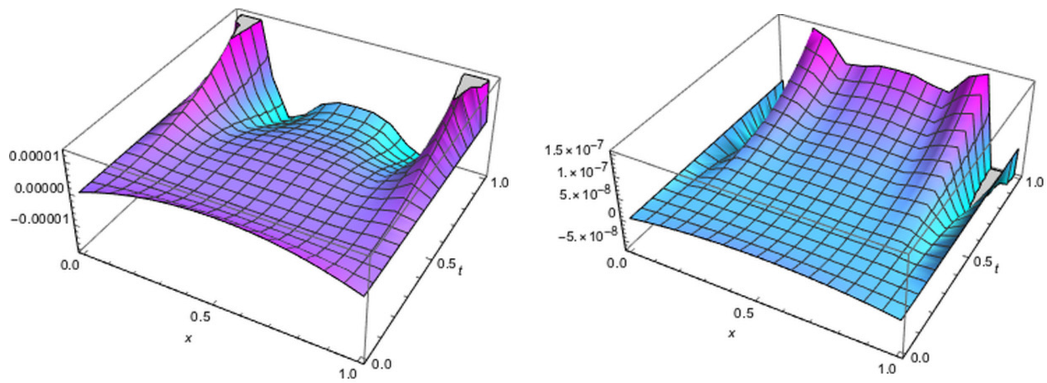

FIGURE 8. The error with noise for $\mathcal{M}_1 = 4$ (left) and $\mathcal{M}_1 = 6$ (right) with $\mathcal{M}_2 = 4$ and $\nu = \lambda = 1, \beta = \frac{1}{2}$ of Example 7.3.

TABLE 3. The absolute errors for diverse values of $\mathcal{M}_1$ and $\mathcal{M}_2$ with $\nu = \frac{2}{3}, \beta = \frac{1}{2}$ and $\lambda = 1$ of Example 7.4.

	Present method			Legendre wavelet method [3]	
$x=t$	$\mathcal{M}_1=\mathcal{M}_2=3$	$\mathcal{M}_1=\mathcal{M}_2=5$	$\mathcal{M}_1=\mathcal{M}_2=8$	$M=6,k=1$	$M=8,k=1$
0	0	0	0	–	–
0.1	9.2176×10^{-7}	4.9634×10^{-9}	2.6243×10^{-13}	1.4084×10^{-8}	4.5365×10^{-14}
0.2	3.0848×10^{-6}	1.2428×10^{-11}	9.9717×10^{-14}	2.8248×10^{-8}	6.5386×10^{-13}
0.3	5.3257×10^{-7}	4.8612×10^{-9}	6.5803×10^{-14}	4.2203×10^{-8}	1.1021×10^{-12}
0.4	3.0003×10^{-6}	1.2756×10^{-9}	3.8815×10^{-14}	2.8709×10^{-8}	2.4148×10^{-12}
0.5	2.4302×10^{-6}	2.9080×10^{-9}	3.2866×10^{-14}	5.4420×10^{-8}	2.1058×10^{-11}
0.6	2.7776×10^{-6}	5.1735×10^{-9}	2.5351×10^{-14}	1.8983×10^{-8}	1.2845×10^{-11}
0.7	6.9643×10^{-6}	2.6815×10^{-9}	3.3989×10^{-15}	3.9528×10^{-8}	1.4914×10^{-11}
0.8	2.3927×10^{-6}	1.3878×10^{-8}	1.8566×10^{-13}	4.1993×10^{-8}	1.3716×10^{-11}
0.9	9.6704×10^{-6}	6.4646×10^{-9}	6.2603×10^{-13}	2.2726×10^{-8}	9.6485×10^{-11}
1.0	6.5366×10^{-19}	6.5366×10^{-19}	6.5366×10^{-19}	–	–

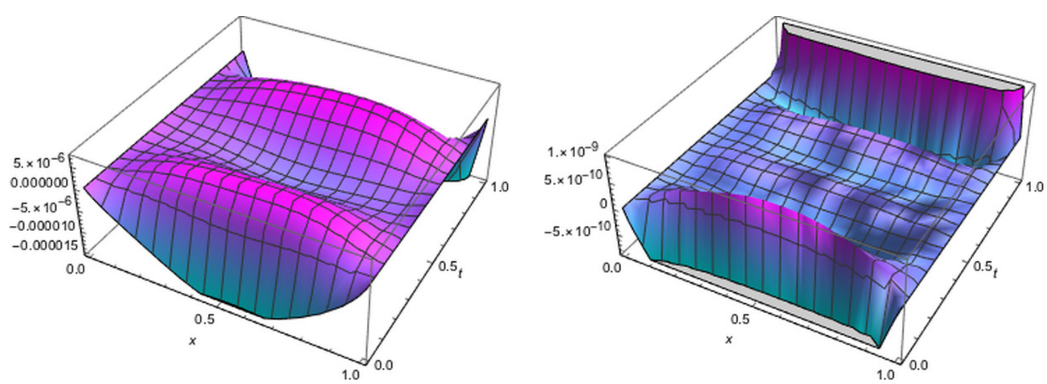

FIGURE 9. The error for $\mathcal{M}_1 = 3$ (left) and $\mathcal{M}_1 = 6$ (right) with $\nu = \beta = \frac{1}{2}$ and $\lambda = 1$ of Example 7.4.

Table 4. The absolute errors for diverse values of $\mathcal{M}_1$ and $\mathcal{M}_2$ with $\nu = \frac{3}{4}, \beta = \frac{2}{3}$ and $\lambda = 1$ of Example 7.5.

	Present method			Legendre wavelet method [3]	
$x = t$	$\mathcal{M}_1 = \mathcal{M}_2 = 3$	$\mathcal{M}_1 = \mathcal{M}_2 = 6$	$\mathcal{M}_1 = \mathcal{M}_2 = 8$	$M = 6, k = 1$	$M = 8, k = 1$
0.1	1.2521×10^{-6}	5.3351×10^{-10}	4.4006×10^{-13}	1.2642×10^{-10}	4.2675×10^{-12}
0.3	4.9041×10^{-7}	6.7423×10^{-11}	2.1693×10^{-13}	1.0828×10^{-10}	1.4221×10^{-11}
0.5	2.3560×10^{-6}	2.6231×10^{-10}	5.4807×10^{-14}	2.0648×10^{-8}	4.8033×10^{-11}
0.7	8.8738×10^{-6}	5.3648×10^{-10}	9.4766×10^{-14}	8.7613×10^{-9}	2.8281×10^{-11}
0.9	1.1595×10^{-5}	1.2304×10^{-10}	1.2264×10^{-12}	1.0682×10^{-8}	1.5557×10^{-11}
CPU	4.1341×10^{-1}	7.6758×10^{-1}	1.253916	–	–

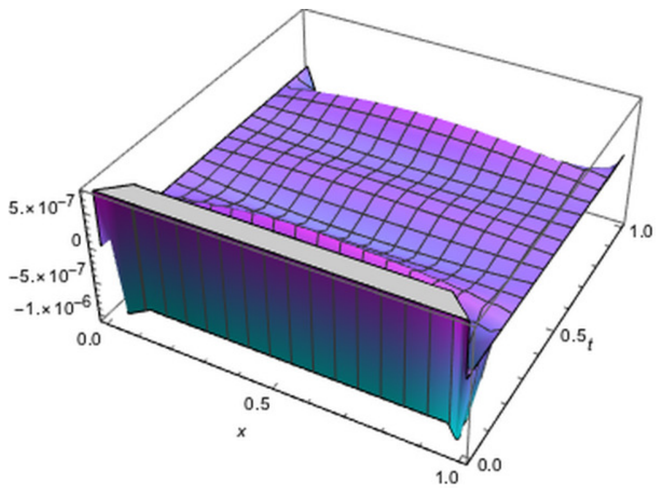

Figure 10. The error for $\mathcal{M}_1 = \mathcal{M}_1 = 6$ with $\nu = \frac{3}{4}, \beta = \frac{2}{3}$ and $\lambda = \frac{1}{2}$ of Example 7.5.

Example 7.5. Consider the following fractional partial integro-differential equation with weakly singular kernel [3]:

$$D_t^{\nu} u(x,t) = f(x,t) + u_{xx}(x,t) + \int_0^t \frac{u_{xx}(x,\eta)}{(t-\eta)^{\beta}} d\eta,$$

where

$$f(x,t) = \left(\sin(t) + \frac{E_{2,2-\nu}(-t^2)}{t^{1-\nu}} + \frac{\Gamma(\beta-1)E_{2,3-\beta}(-t^2)}{t^{\beta-2}} \right) \cos(x),$$

with the initial condition $u(x,0) = 0$ and the boundary conditions $u(0,t) = \sin(t),\ u(1,t) = \cos(1)\sin(t)$. The exact solution to this problem is $u(x,t) = \cos(x)\sin(t)$. The results of this problem are provided in Table 4 and Figure 10. The results show that the algorithm introduced in the previous section is very powerful and effective for solving these problems.

Example 7.6. Consider the following fractional partial integro-differential equation with weakly singular kernel [23]:

$$D_t^{\nu} u(x,t) = f(x,t) + u_{xx}(x,t) + \int_0^t \frac{u_{xx}(x,\eta)}{(t-\eta)^{\frac{1}{2}}} d\eta,$$

where

$$f(x,t) = \sin(\pi x) \left(\frac{1}{\Gamma(2-\nu)} t^{1-\nu} + \pi^2 t + \frac{4}{3}\pi^2 t^{\frac{3}{2}} \right),$$

TABLE 5. Errors for diverse values of ν and $\mathcal{M}_1$ with $\mathcal{M}_2 = 1$ and $\lambda = 1$ of Example 7.6.

Present method			
ν	$\mathcal{M}_1$	L_∞-error	L_2-error
0.1	4	1.0891×10^{-5}	2.4916×10^{-5}
	6	7.1699×10^{-8}	1.4053×10^{-7}
	8	1.5045×10^{-10}	3.3069×10^{-10}
0.25	4	1.0861×10^{-5}	2.4817×10^{-5}
	6	7.1507×10^{-8}	1.3987×10^{-7}
	8	1.6101×10^{-10}	3.6849×10^{-10}
0.75	4	1.0757×10^{-5}	2.4491×10^{-5}
	6	7.0871×10^{-8}	1.3772×10^{-7}
	8	1.5335×10^{-10}	3.4311×10^{-10}
Compact finite difference method [23]			
ν	h	τ	L_∞-error
0.1	$\frac{1}{100}$	$\frac{1}{10}$	1.3427×10^{-4}
		$\frac{1}{160}$	7.7091×10^{-6}
		$\frac{1}{1280}$	9.6054×10^{-7}
0.25	$\frac{1}{4}$	$\frac{1}{4}$	1.9843×10^{-3}
	$\frac{1}{64}$	$\frac{1}{8}$	1.3145×10^{-4}
	$\frac{1}{1024}$	$\frac{1}{16}$	8.2032×10^{-6}
0.75	$\frac{1}{4}$	$\frac{1}{4}$	1.9099×10^{-3}
	$\frac{1}{64}$	$\frac{1}{8}$	9.2984×10^{-5}
	$\frac{1}{1024}$	$\frac{1}{16}$	5.0958×10^{-6}

with the initial condition $u(x, 0) = 0$ and the boundary conditions $u(0, t) = 0,\ u(1, t) = t\sin(\pi)$. The exact solution to this problem is $u(x, t) = t\sin(\pi x)$. Comparison of the errors obtained by the present method and compact finite difference method [23] are listed in Table 5. This table demonstrates that by increasing the number of basis functions, the error tend to zero.

8. Conclusions

This paper provides a numerical framework for solving fractional partial integro-differential viscoelastic equations with weakly singular kernels. In this approach, the problem is converted to fractional partial integral equations with weakly singular kernels. Then, we estimate the unknown functions with the help of MFM-Fs. In addition, we introduce a novel approach to obtain the operational matrix, which directly affected the accuracy of the method. This evaluation creates good conditions to get approximate solutions with high accuracy. Also, the error analysis according to the numerical scheme is discussed in Sobolev space. Moreover, to reinforce the explanation of the methodology, we examine several numerical examples.

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