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Quality theorems on the solutions of quasilinear second-order parabolic equations with discontinuous coefficients

Sarvan T. Huseynov
Baku State University, Baku, Azerbaijan.

Abstract

A class of quasilinear second-order parabolic equations with discontinuous coefficients is considered in this work. The analog of Harnack inequality is proved for the non-negative solutions of these equations.

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1. Introduction

Let R^{n+1} be an $(n+1)$-dimensional Euclidean space of points $(x,t) = (x_1, x_2, ..., x_n, t)$ and D be some domain in R^{n+1}. Consider in D a quasilinear parabolic equation of the form

$$Lu = \sum_{i,k=1}^{n} a_{ik}(x,t,u,\nabla u)\frac{\partial^2 u}{\partial k_i \partial x_k} + b(x,t,u,\nabla u) - \frac{\partial u}{\partial t} = 0, \tag{1.1}$$

assuming that its coefficients satisfy the following conditions:

$$\sup_{\substack{(t,x)\in D, |\nu|\le 1 \\ -\infty<\eta_i<\infty}} \sum_{i=1}^{n} a_{ii}(t,x,\nu,\eta) = M < \infty, \tag{1.2}$$

$$\inf_{\substack{(t,x)\in D, |\nu|\le 1 \\ -\infty<\eta_i<\infty}} \min_{|\xi|=1} \sum_{i,k=1}^{n} a_{ik}(t,x,\nu,\eta)\xi_i\xi_k = \alpha > 0, \tag{1.3}$$

$$|b(t,x,v,\eta)| \le B_0\left(1+|\eta|^2\right). \tag{1.4}$$

We consider the solutions whose modules are bounded by the prescribed constant which, for simplicity, is assumed to be equal to 1. In this work, for such solutions of the Dirichlet problem we obtain Harnack-type theorems and the theorem on the regularity of boundary points.

For linear equations of parabolic type with a "small" spread of the spectrum of the higher coefficients matrix (Cordes condition), the corresponding theorems have been proved by E. M. Landis [3]. For such equations, the validity of the above theorems without Cordes condition has been established by N. V. Krylov and M. V. Safonov [2]. The same theorems without Cordes condition for quasilinear elliptic equations have been proved by A. A. Novruzov [6], O. A. Ladyzhenskaya and N. N. Uraltseva [4], N. Trudinger [7].

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Corresponding author. Email: sarvanhuseynov@rambler.ru .

For second-order parabolic equations in divergence form with non-uniform degeneration, the Harnack inequality has been proved in [1].

2. Solutions of quasilinear second-order parabolic equations

We will use the following notations:

$$\coprod_1^R = \coprod_{x^0,R}^{t^o-bR^2,t^0} = \left\{(t,x) : t^0 - bR^2 < t < t^0,\ \left|x - x^0\right| < R\right\},$$

$$\left(t^0, x^0\right) \in \overline{D},\ \ R > 0,\ \ b = \min\left(\frac{1}{16\mathrm{M}}, 1\right);$$

$$\coprod_2^R = \coprod_{x^0,R/8}^{t^o-bR^2/4,\ t^0};\quad \coprod_3^R = \coprod_{x^0,R/8}^{t^o-bR^2,\ t^0-bR^2/2};\quad E_R = \coprod_3^R \backslash D;$$

$$L^1 = \sum_{i,k=1}^{n} A_{ik}(t,x)\frac{\partial^2}{\partial x_i \partial x_k} - \frac{\partial}{\partial t},$$
$$A_{ik}(t,x) = a_{ik}(t,x,u(t,x),\nabla u(t,x)),$$

where $u(t,x)$ is a solution of the Equation (1.1).

It is not difficult to see that the function

$$\upsilon(t,x) = \exp\left[\frac{B_0}{\alpha}u(t,x)\right] - 1 + K\left|x - x^0\right|^2,$$

is a subsolution of the linear operator L^1 for suitably chosen constant $K > 0$, $\quad (K = K(M,\alpha,B_o,n))$.

We will use the following lemma proved in [5].

Lemma 2.1. *Let the domain D lie in the cylinder $\coprod_1^{\mathrm{R}}$, let it have the limit points on the proper boundary $\Gamma\left(\coprod_1^{\mathrm{R}}\right)$ of the cylinder and intersect $\coprod_2^{\mathrm{R}}$. Also, let the positive solution $u(t,x)$ of the Equation (1.1) be defined in D, be continuous in $\overline{D}$ and vanish in the part of $\Gamma(D)$ which lies strictly inside $\coprod_1^{\mathrm{R}}$, and let the conditions (1.2)-(1.4) hold. If $mesE_R \geq h_0 R^{n+2}$, then for sufficiently small $R's$*

$$\sup_D \upsilon \geq (1+\eta_0) \sup_{D\bigcap\coprod_2^{\mathrm{R}}} \upsilon,$$

where the constant $\eta_0 > 0$ depends only on $M, \alpha, B_0, n,$ and h_0.

Let's prove the lemma below, which will be significantly used in the sequel.

Lemma 2.2. *Let the domain D and the function $u(t,x)$ be the same as in Lemma 2.1. If R is sufficiently small, then for every $N > 0$ there exists $\delta > 0$, depending only on M, α, B_o, n and N, such that*

$$\sup_D \upsilon \geq N \sup_{D\bigcap\coprod_{\mathrm{x}^0,\mathrm{R}/2}^{\mathrm{t}_0-bR^2/2,t^0}} \upsilon \tag{2.1}$$

as $mesD \leq \delta R^{n+2}$.

Proof. Let η_0 be a constant from the previous lemma corresponding to $h_0 = b\Omega_n/4\cdot 8^n$, where Ω_n is a volume of the n-dimensional unit ball. Denote by m the smallest positive integer which satisfies

$$(1+\eta_0)^m > N. \tag{2.2}$$

Consider the difference $\coprod_{x^0,R}^{t^o-bR^2,t^0} \backslash \coprod_{x^0,R|2}^{t^o-bR^2|2,t^0}$.

Let

$$\coprod^{(i)} = \coprod_{x^0,\frac{R}{2}\left(1+\frac{i}{m}\right)}^{t^0-\frac{bR^2}{2}\left(1+\frac{i}{m}\right),t_0},\quad i = \overline{0, m-1}.$$

The proper boundaries $\Gamma(\coprod^{(i)})$ divide the above difference into m parts. Denote $\sup\limits_{D\bigcap\Gamma(\coprod^{(t)})} \upsilon$ by M_i. Assume that the value M_i is achieved by the function $\upsilon(t,x)$ at the point $\left(t^{i\cdot}x^i\right) \in \Gamma(\coprod^{(i)})$. Also, let

$$\coprod\nolimits_1^{(i)} = \coprod\nolimits_{x^i,R|2m}^{t^i-bR^2|4m^2,t^i};\coprod\nolimits_2^{(i)} = \coprod\nolimits_{x^i,R|16m}^{t^i-bR^2|16m^2,t^i};$$

$$\coprod\nolimits_3^{(i)} = \coprod\nolimits_{x^i,R|16m}^{t^i-bR^2|4m^2,t^i-bR^2|8m^2} \quad i = \overline{0,m-1}.$$

Let's choose $\delta > 0$ such that $mes(\coprod_3^{(i)} \setminus D) \geq \frac{mes\coprod_3^{(i)}}{2}$.

For this, it suffices that $\delta = b\Omega_n/16^{n+1}m^{n+2}$. Now let's apply Lemma 1 to the cylinders $\coprod_1^{(i)}$ and $\coprod_2^{(i)}$. Then we obtain

$$M_{i+1} \geq (1+\eta_0)\, M_i,$$

i.e.

$$M_m \geq (1+\eta_0)^m\, M_0.$$

□

Hence, by (2.2), we get the validity of the inequality (2.1).

Theorem 2.3. *Let the non-negative solution $u(t,x)$ of the Equation (1.1) be defined in the cylinder $\coprod_{x,R}^{t,t+bR^2}$ and the conditions (1.2)-(1.4) be satisfied for the coefficients. Then there exists a constant $P > 0$, depending only on M, α, B_o and n, such that for sufficiently small R*

$$\sup_{\coprod_{x,R/16}^{t+bR^2/4,t+bR^2/2}} u \leq P \inf_{\coprod_{x,R/16}^{t+3bR^2/4,t+bR^2}} u. \tag{2.3}$$

Proof. Denote

$$\coprod\nolimits_1 = \coprod\nolimits_{x,R}^{t,t+bR^2}\coprod\nolimits_2 = \coprod\nolimits_{x,R/16}^{t+3bR^2/4,\, t+bR^2}, \tag{2.4}$$

$$\coprod\nolimits_3 = \coprod\nolimits_{x,R/16}^{t+bR^2/4,\, t+bR^2/2} = \coprod\nolimits_4 = \coprod\nolimits_{x,R/8}^{t,t+bR^2/2}. \tag{2.5}$$

Let's first prove the (2.3)-type inequality for the function $\upsilon(t,x)$. Without loss of generality, we can assume $\sup\limits_{\coprod_3} \upsilon = 2$, where υ is a subsolution of the linear operator L^1. Let D^1 denote the set of points $(t,\chi) \in \coprod_4$, where $\upsilon(t,x) > 1$. Assume $N = 2^{n+3}$ in Lemma 2.2 and let $\delta > 0$ correspond to this N. Also, let

$$\gamma = \left(\frac{1}{64}\right)^{n+2}\delta.$$

Two cases are possible:

1) $mesD^1 \geq \gamma R^{n+2}$,
2) $mesD^1 < \gamma R^{n+2}$.

Consider the case 1. Denote by $\tilde{D}$ the set of points $(t,x) \in \coprod_1$ with $\upsilon(t,x) < 1$. Obviously, $\coprod_1 \setminus \tilde{D}$ contains the set D^1. By Lemma 2.1,

$$1 - \inf_{\tilde{D}\bigcap\coprod_1} \upsilon \geq (1+\eta_0)\left(1 - \inf_{\tilde{D}\bigcap\coprod_2} \upsilon\right),$$

i.e.

$$(1+\eta_0) \inf_{\tilde{D}\bigcap\coprod_2} \upsilon \geq \eta_0 + \inf_{\tilde{D}\bigcap\coprod_1} \upsilon \geq \eta_0,$$

or

$$\inf_{\tilde{D}\bigcap \amalg_2} \upsilon \geq \frac{\eta_0}{1+\eta_0}. \tag{2.6}$$

But,

$$\upsilon/_{\amalg_1\setminus\tilde{D}} \geq 1. \tag{2.7}$$

Therefore, it follows from (2.6) and (2.7) that

$$\inf_{\amalg_2} \upsilon \geq \eta_0/\left(1+\eta_0\right) = \eta_1, \tag{2.8}$$

where the constant $\eta_1 > 0$ depends only on M, α, B_o and n. Now let's consider the case 2). Denote $\amalg_{x,R\left(\rho+\frac{1}{16}\right)}^{t+\frac{bR^2}{4}(1-\rho^2),\, t+\frac{bR^2}{2}}$ by $\amalg^{(\rho)}$, and $D^1 \bigcap \left(\amalg^{(\rho)}\setminus\amalg^{(0)}\right)$ by $D_\rho^{(1)}$, $(0<\rho<1)$. Due to our choice of γ,

$$mesD_{1/32}^{(1)} < \frac{\delta R^{n+2}}{(2\cdot 32)^{n+2}}.$$

Therefore, we can find ρ_1, $0<\rho_1<1/32$, such that

$$mesD_{\rho_1}^{(1)} = (\rho_1/2)^{n+2}\,\delta R^{n+2}.$$

Let $\amalg_{(1)} = \amalg_{x^1,p^1 R/2}^{\mathrm{t}_1 - b\left(\frac{\rho_1}{2}\right)^2 R^2, t^1}$, where $\left(t^1, x^1\right)$ is a point belonging to $\mathrm{D}^1 \bigcap \Gamma\left(\amalg^{(\rho_1/2)}\right)$, with $\upsilon\left(t^1, x^1\right) \geq 2$.

Let's introduce the function

$$\upsilon_1(t,x) = \upsilon(t,x) - 1.$$

If $D_{(2)}$ is a component of the set $\mathrm{D}^1 \bigcap \amalg_{(2)}$ which contains the point $\left(t^1, x^1\right)$, then, by Lemma 2,

$$\sup_{D_{(1)}} \upsilon \geq \sup_{D_{(1)}} \upsilon_1 \geq 2^{n+3} = 2\cdot 2^{n+2}.$$

Now let D^2 be a set of points $(t,x)\in \amalg_4$ such that $\upsilon(t,x) > 2^{n+2}$, and

$$\mathrm{D}_\rho^{(2)} = D^2 \bigcap \left(\amalg^{(\rho_1+\rho)} \setminus \amalg^{(\rho_1)}\right),$$

where $0<\rho<1/16-\rho_1$.

As $\rho < 1/32$, we have

$$mesD_{1/32}^{(2)} < \frac{\delta R^{n+2}}{(2\cdot 32)^{n+2}}.$$

Therefore, these exist ρ_2 such that

$$mesD_{\rho_2}^{(2)} = (\rho_2/2)^{n+2}\delta R^{n+2}.$$

Let $\left(t^2, x^2\right)$ be a point on $\Gamma\left(\amalg^{\left(\rho_1+\frac{\rho_2}{2}\right)}\right)$, where $u\left(t^2, x^2\right) \geq 2^{n+3}$. Denote by $\amalg_{(2)}$ the cylinder

$$\amalg_{x^2,\rho_2 R/2}^{t^2 - b\left(\frac{\rho_2}{2}\right)^2 R^2, t^2}.$$

Introduce the function

$$\upsilon_2(t,x) = \upsilon(t,x) - 2^{n+2}.$$

If $D_{(2)}$ is a component of $D^2 \bigcap \amalg_{(2)}$ which contains the point $\left(t^2, x^2\right)$, then, by Lemma 2,

$$\sup_{D_{(2)}} \upsilon \geq \sup_{D_{(2)}} \upsilon_2 \geq 2^{n+3}\cdot 2^{n+2} = 2\cdot 2^{2(n+2)}.$$

We repeat this procedure similarly until

$$\rho_1 + \rho_2 + ... + \rho_k \geq 1/32. \tag{2.9}$$

Let k be a the smallest positive integer for which (2.9) holds. Such a k certainly exists, because otherwise the function $\upsilon(t, x)$ would be unbounded.

Thus, in addition to (2.9), we also get the validity of

$$\rho_1 + \rho_2 + ... + \rho_{k-1} < 1/32. \tag{2.10}$$

For every i, $1 \leq i \leq k$, there exists a set $D^{(i)}_{\rho_i}$ such that

$$mesD^{(i)}_{\rho_i} = (\rho_i/2)^{n+2}\delta R^{n+2},$$

and, besides,

$$\upsilon/_{D^{(i)}_{\rho_i}} \geq 2^{(i-1)(n+2)}.$$

Hence, by (2.9) and (2.10), we get the existence of the number i_0 such that

$$\rho_{i_0} > 2^{-(i_0+5)},$$

with

$$mesD^{(i_0)}_{\rho_{i_o}} \geq 2^{-(i_0+6)(n+2)}\delta R^{n+2},$$

and

$$\upsilon/D^{(i_0)}_{\rho_{i_o}} \geq 2^{(i_0-1)(n+2)}.$$

Consider the function

$$\upsilon'(t, x) = 2^{-(i_0-1)(n+2)}\upsilon(t, x).$$

Let $\hat{D}$ be a set of points (t, x) $\in \coprod_1$ with $\upsilon'(t, x) < 1$. As $\coprod_4 \backslash \hat{D}$ contains the set $D^{(i_0)}_{\rho_{i_o}}$, we have

$$\upsilon' \Big|_{\hat{D}\bigcap \text{Ш}_2} \geq \eta_2, \tag{2.11}$$

where the constant $\eta_2 > 0$ depends only on M, α, B_o and n, because i_0 and δ also depend on these parameters. On the other hand,

$$\upsilon' \Big|_{\text{Ш}_2 | \hat{D}} \geq 1.$$

Therefore it follows from (2.11) that

$$\upsilon' \big|_{\text{Ш}_2} \geq \eta_2,$$

i.e.

$$\inf_{\text{Ш}_2} \upsilon \geq \eta_2 2^{(i_0-1)(n+2)} = \chi. \tag{2.12}$$

Denote $\min(\eta_1, \chi)$ by χ_0. Then from (2.8) and (2.12) it follows that

$$\inf_{\text{Ш}_2} \upsilon \geq \chi_0,$$

or

$$\sup_{\text{Ш}_3} \upsilon \leq \frac{2}{\chi_0} \inf_{\text{Ш}_2} \upsilon.$$

Further, we have

$$\exp\left[\frac{B_0}{\alpha} \sup_{\text{Ш}_3} u\right] \leq \frac{2}{\chi_0} \exp\left[\frac{B_0}{\alpha} \inf_{\text{Ш}_2} u\right],$$

i.e.

$$\frac{B_0}{\alpha}\left(\sup_{\amalg_3} u - \inf_{\amalg_2} u\right) \leq \ln\frac{2}{\chi_0},$$

$$\sup_{\amalg_3} u \leq \frac{\alpha}{B_0}\ln\frac{2}{\chi_0} + \inf_{\amalg_2} u \leq \left(\frac{\alpha}{B_0}\ln\frac{2}{\chi_0}\frac{1}{\eta_3}+1\right)\inf_{\amalg_2} u, \tag{2.13}$$

provided that $\upsilon\left|_{\amalg_2} \geq \chi_0,\ u\right|_{\amalg_2} \geq \eta_3$, where the constant $\eta_3 > 0$ depends only on M, α, B_o and n. Now it suffices to put $P = 1 + \frac{\alpha}{\eta_3 B_0}\ln\frac{2}{\chi_0}$ and the desired inequality (2.3) follows from (2.13). The theorem is proved. □

Let's assume that in every strictly internal subdomain of the domain D the coefficients of the Equation (1.1) have a smoothness of minimal degree which is enough for the equation to have a solution generalized in the sense of Wiener for the first boundary value problem.

Theorem 2.4. *Let the coefficients of the Equation (1.1) be defined in the bounded domain $D \subset R^{n+1}$ and satisfy the conditions (1.2)-(1.4). For the point $\left(t^0, x^0\right) \in \Gamma(D)$ to be regular with respect to the Dirichlet problem, it is sufficient that*

$$\lim_{R\to 0}\frac{mesE_R}{R^{n+2}} > 0. \tag{2.14}$$

Proof. Let the condition (2.14) be satisfied. Then there exists $h_0 > 0$ such that for sufficiently small $R's$ $mesE_R \geq h_0 R^{n+2}$. To prove the regularity of the boundary point (t_0, x_0) it suffices to show that for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there is $\delta > 0$ such that, whatever the subdomain $D' \subset D$ lying strictly inside the halfspace $t < t^0$ and whatever the solution $u(t,x)$ of the Equation (1.1) in D' with $|u| \leq 1$, from $u/_{\Gamma(D')\bigcap O_{\varepsilon_1}(t^0,x^0)} \leq 0$ it follows $u/_{D'\bigcap O_\delta(t^0,x^0)} < \varepsilon_2$, where $O_\varepsilon\left(t^0, x^0\right)$ is a cylindrical ε-neighborhood of the point $\left(t^0, x^0\right)$.

Let the subdomain D',$\varepsilon_1, \varepsilon_2$ and the solution $u(t,x)$ be already given. Denote by m_1 the smallest positive integer which satisfies

$$8^{-m_1} < \varepsilon_1.$$

Assume there is a point

$$(t',x') \in D', \sqrt{|t'-t^0| + |x'-x^0|^2} < 8^{-m},$$

such that $m > m_1$ and $u(t',x') \geq \varepsilon_2$, i.e. $\upsilon(t',x') \geq \varepsilon_2$.

Applying Lemma 1, we get

$$B_1 \geq M_{m_1} \geq (1+\eta_0)^{m-m_1}\varepsilon_2, \tag{2.15}$$

where $B_1 = \sup_{D'} \upsilon$, $M_{m_1} = \sup_{D'\bigcap \amalg^{t^0-b8^{-2m_1},t^0}_{\mathrm{x}^0,8^{-m_1}}} \upsilon$, and the constant $\eta_0 > 0$ depends only on the coefficients of the operator L, n and h_0.

From (2.15) we obtain

$$(m-m_1)\ln(1+\eta_0) \leq \ln B_1/\varepsilon_2,$$

i.e.

$$m \leq m_1 + \frac{\ln B_1/\varepsilon_2}{\ln(1+\eta_0)}.$$

If we choose $\delta = 8^{-\left[m_1+\frac{\ln B_1/\varepsilon_2}{\ln(1+\eta_0)}\right]-1}$, then the inequality $\upsilon(t,x) < \varepsilon_2$, i.e. $u(t,x) < \varepsilon_2$, holds at all points in $D'\bigcap O_\delta(t^0,x^0)$.

The theorem is proved.

□

3. Conclusion

In this work, a class of second-order quasilinear parabolic equations with discontinuous coefficients is studied. We consider solutions bounded in modulus by a predetermined constant which, for simplicity, we assume to be 1. In our proofs, we significantly use the analogs of so-called growth lemmas stated in Landis [3]. By means of these lemmas, we prove the Harnack inequalities for non-negative solutions of the above equations.

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