Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 12, No. 4, 2024, pp. 808-826
DOI:10.22034/cmde.2024.57684.2418

Higher-order multi-step Runge-Kutta-Nyström methods with frequency-dependent coefficients for second-order initial value problem $u'' = f(x, u, u')$

Athraa Abdulsalam[1,2], Norazak Senu[2,3,*], Zanariah Abdul Majid[2,3], and Nik Mohd Asri Nik Long[2,3]
[1]Department of Mathematics and Computer Applications, College of Sciences, Al-Nahrain University, Jadriya, Baghdad, Iraq.
[2]Institute for Mathematical Research, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia.
[3]Department of Mathematics and Statistics, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia.

Abstract

In this study, for the numerical solution of general second-order ordinary differential equations (ODEs) that exhibit oscillatory or periodic behavior, fifth- and sixth-order explicit multi-step Runge-Kutta-Nyström (MSGRKN) methods, respectively, are constructed. The parameters of the proposed methods rely on the frequency ω of each problem whose solution is a linear combination of functions $\{e^{(i\omega x)},\quad e^{(-i\omega x)}\}$ or $\{\cos(\omega x),\quad \sin(\omega x)\}$. The study also includes an analysis of the linear stability of the suggested methods. The numerical results indicate the efficiency of the proposed methods in solving such problems compared to methods with similar characteristics in the literature.

Keywords. Explicit methods, Trigonometrical fitting, Multi-step Runge-Kutta-Nyström methods, Initial value problems, General second-order oscillatory differential equations.

2010 Mathematics Subject Classification. 65L05, 65L06.

1. Introduction

Many physical phenomena in the applied sciences, such as problems of orbital dynamics, control theory, chemical kinetics, and electric circuits, can be modeled as second-order differential equations. In this article, we look at a second-order initial value problem (IVP) of the model given below and its numerical solution:

$$u'' = f(x, u, u'), \quad x \in [a, b], \qquad u(a) = \alpha, \quad u'(a) = \beta, \tag{1.1}$$

for which its solution has an oscillating behavior, where $u(x) \in \mathbb{R}^m$, $f(x, u, u') : [a, b] \times \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a continuous vector function. The periodic or oscillatory solution property of Eq. (1.1) has not been taken into account by many Runge-Kutta or -Nyström (RK) or (RKN) methods in the literature, which is why they often provide unsatisfactory numerical results. Several attempts have been made to adapt RK or RKN methods by including the oscillatory structure in their formulations. The techniques of exponential (or trigonometric) fitting introduced early on by [14] and [9] are the most successful and effective attempts. Since then, researchers have introduced several exponentially/trigonometrically fitted methods and applied them in different scientific fields. It is suggested to refer to [1, 7, 19, 20] for an intriguing study on the development and analysis of exponentially fitted RK or RKN methods. Recently, in the context of RKN or RK methods, the availability of some higher derivatives of the solution prompted many researchers to utilize them to increase the accuracy and efficiency of the numerical methods after incorporating exponential or trigonometric fitting techniques (see [5, 16, 17]). Although the efficiency of these methods over the classical RK or RKN methods in reaching a higher-order with a higher order with fewer function evaluations per step have numerical experiments, unfortunately, computing the higher derivatives requires additional computational costs

Received: 24 July 2023; Accepted:27 March 2024.
∗ Corresponding author. Email: norazak@upm.edu.my .

[5]. More recently, [11] proposed trigonometrically fitted multi-step RKN (TFMSRKN) methods. These methods have the advantage of being able to reduce the number of function evaluations at each step and they do not require extra costs as in higher derivatives RKN. However, the only drawback of these methods is their inability to solve the problem stated in Eq. (1.1), as they were integrated to solve $u'' = f(x, u)$ where the equation is not explicitly dependent on the first derivative u'. This prompted [13] to derive trigonometrically fitted multi-step RKN methods able to solve the problem in Eq. (1.1), but this time the defect of the new derived methods was that they were of lower-order, which are the third and fourth-order, which was possible to obtain more accurate results by deriving higher-order methods. So, that motivated us in this study to derive higher-order methods of order five and six, respectively, to solve the problem in Eq. (1.1) in order to obtain more accurate results. In section 2, we address the description of explicit multi-step Runge-Kutta-Nyström (MSGRKN) methods and the definition of trigonometrically fitted. We devote section 3 to the construction of the new trigonometrically fitted MSGRKN methods and to the study of the linear stability of the proposed methods. Some test problems are presented to examine the numerical behavior of the suggested methods in section 4, along with a discussion of the obtained results. Lastly, we give a conclusion in section 5.

2. Fundamental concepts

2.1. **The definition of multi-step Runge-Kutta-Nyström methods.** The following formulae are used to define an explicit κ-stage $\jmath$-step Runge-Kutta-Nyström methods for the problem given in Eq. (1.1) (see [12])

$$
\begin{aligned}
Y_i &= \sum_{\ell=1}^{\jmath} r_{i\ell} u_{n-\ell+1} + h \sum_{\ell=1}^{\jmath} p_{i\ell} u'_{n-\ell+1} + h^2 \sum_{j=1}^{\kappa} a_{ij} f(t_n + c_j h, Y_j, Y'_j), \\
Y'_i &= \sum_{\ell=1}^{\jmath} \bar{r}_{i\ell} u'_{n-\ell+1} + h \sum_{j=1}^{\kappa} \bar{a}_{ij} f(t_n + c_j h, Y_j, Y'_j), \\
u_{n+1} &= \sum_{\ell=1}^{\jmath} q_\ell u_{n-\ell+1} + h \sum_{\ell=1}^{\jmath} w_\ell u'_{n-\ell+1} + h^2 \sum_{i=1}^{\kappa} b_i f(t_n + c_i h, Y_i, Y'_i), \\
u'_{n+1} &= \sum_{\ell=1}^{\jmath} \nu_\ell u'_{n-\ell+1} + h \sum_{i=1}^{\kappa} \bar{b}_i f(t_n + c_i h, Y_i, Y'_i),
\end{aligned}
\tag{2.1}
$$

where c_i, a_{ij}, $r_{i\ell}$, b_i, $p_{i\ell}$, q_ℓ, ν_ℓ, w_ℓ, $\bar{b}_i$, $\bar{a}_{ij}$, and $\bar{r}_{i\ell}$ for $\ell = 1, \dots, \jmath$ and $i, j = 1, 2, \dots, \kappa$ are real numbers, and Butcher tableau can be used to summarise Eq. (2.1) as follows:

$$
\begin{array}{c|ccc|ccc|ccc|ccc|ccc}
c_1 & r_{11} & \dots & r_{1\jmath} & p_{11} & \dots & p_{1\jmath} & a_{11} & \dots & a_{1\kappa} & \bar{r}_{11} & \dots & \bar{r}_{1\jmath} & \bar{a}_{11} & \dots & \bar{a}_{1\kappa} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
c_\kappa & r_{\kappa 1} & \dots & r_{\kappa\jmath} & p_{\kappa 1} & \dots & p_{\kappa\jmath} & a_{\kappa 1} & \dots & a_{\kappa\kappa} & \bar{r}_{\kappa 1} & \dots & \bar{r}_{\kappa\jmath} & \bar{a}_{\kappa 1} & \dots & \bar{a}_{\kappa\kappa} \\
\hline
 & q_1 & \dots & q_\jmath & w_1 & \dots & w_\jmath & b_1 & \dots & b_\kappa & \nu_1 & \dots & \nu_\jmath & \bar{b}_1 & \dots & \bar{b}_\kappa
\end{array}
$$

Consider the simplifying conditions in equations (3) and (4) from [2], which are as follows:

$$
\begin{aligned}
\sum_{j=1}^{\kappa} a_{ij} c_j^{\alpha} &= \frac{1}{(\alpha+1)(\alpha+2)} \Big(c_i^{\alpha+2} - \sum_{\ell=1}^{\jmath} r_{i\ell} (1-\ell)^{\alpha+2} \\
&- \sum_{\ell=1}^{\jmath} p_{i\ell} (1-\ell)^{\alpha+1} (\alpha+2) \Big), \quad 0 \le \alpha \le q_1 - 2,
\end{aligned}
\tag{2.2}
$$

$$
\sum_{j=1}^{\kappa} \bar{a}_{ij} c_j^{\alpha} = \frac{1}{(\alpha+1)} \Big(c_i^{\alpha+1} - \sum_{\ell=1}^{\jmath} \bar{r}_{i\ell} (1-\ell)^{\alpha+1} \Big), \quad 0 \le \alpha \le q_2 - 1,
\tag{2.3}
$$

where q_1 and q_2 are the stage order of Y_i and Y'_i respectively, by taking $\alpha = 0, 1, 2, 3, 4$ in Eq. (2.2), we will get the five simplifying conditions of Y_i and by taking $\alpha = 0, 1, 2, 3, 4, 5$ in Eq. (2.3), we will get the six simplifying conditions

of Y_i' which are utilized in the construction of MSGRKN methods.
As is common for RKN methods, under the local assumptions

$$u_{n-\ell+1} = u(x_n + (1-\ell)h), \quad u'_{n-\ell+1} = u'(x_n + (1-\ell)h), \quad \ell = 1, \dots, \jmath. \tag{2.4}$$

We can expand the local truncation errors as

$$\begin{aligned} u(x_n + h) - u_{n+1} &= \sum_{t \in NT} \frac{h^{\rho}(t)}{\rho(t)!} \alpha(t) \bar{\chi}(t) F(t)(u(x_n), u'(x_n)), \\ u'(x_n + h) - u'_{n+1} &= \frac{1}{h} \sum_{t \in NT} \frac{h^{\rho}(t)}{\rho(t)!} \alpha(t) \chi(t) F(t)(u(x_n), u'(x_n)), \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \bar{\chi}(t) &= 1 - \sum_{\ell=1}^{\jmath} q_\ell (1-\ell)^{\rho(t)} - \sum_{\ell=1}^{\jmath} w_\ell (1-\ell)^{\rho(t)-1} \rho(t) - \sum_{i=1}^{\kappa} b_i \Psi_i''(t), \\ \chi(t) &= \rho(t) - \sum_{\ell=1}^{\jmath} \nu_\ell (1-\ell)^{\rho(t)-1} \rho(t) - \sum_{i=1}^{\kappa} \bar{b}_i \Psi_i''(t). \end{aligned} \tag{2.6}$$

For the definitions of NT, $\rho(t)$ and $\Psi_i''(t)$, readers may refer to Definition 3.2. and Lemma 3.1. in [12].

Theorem 2.1. *[12]. MSGRKN methods, given in Eq. (2.1), are convergent of order ($p \geq 2$) iff*

$$\begin{aligned} 1 &= \sum_{\ell=1}^{\jmath} q_\ell (1-\ell)^{\rho(t)} + \sum_{\ell=1}^{\jmath} w_\ell (1-\ell)^{\rho(t)-1} \rho(t) + \sum_{i=1}^{\kappa} b_i \Psi_i''(t), \quad \rho(t) \leq p, \\ \rho(t) &= \sum_{\ell=1}^{\jmath} \nu_\ell (1-\ell)^{\rho(t)-1} \rho(t) + \sum_{i=1}^{\kappa} \bar{b}_i \Psi_i''(t), \quad \rho(t) \leq p+1, \end{aligned} \tag{2.7}$$

where $t \in NT$.

Under the five simplifying conditions of Y_i obtained from Eq. (2.2) and the six simplifying conditions of Y_i' obtained from Eq. (2.3), Theorem 2.1 gives the seventh-order conditions for the MSGRKN methods as follows:

The order conditions for u:

$$\sum_{\ell=1}^{\jmath} q_\ell (1-\ell)^{\alpha+2} + (\alpha+2) \sum_{\ell=1}^{\jmath} w_\ell (1-\ell)^{\alpha+1} + (\alpha+2)(\alpha+1) \sum_{i=1}^{\kappa} b_i c_i^{\alpha} = 1, \quad 0 \leq \alpha \leq 5. \tag{2.8}$$

The order conditions for u':

$$\sum_{\ell=1}^{\jmath} \nu_\ell (1-\ell)^{\alpha+1} + (\alpha+1) \sum_{i=1}^{\kappa} \bar{b}_i c_i^{\alpha} = 1, \quad 0 \leq \alpha \leq 6. \tag{2.9}$$

To get the higher-order MSGRKN methods, the following simplifying conditions are utilized to help reduce the number of equations to be solved

$$\begin{aligned} &\sum_{\ell=1}^{\jmath} r_{i\ell} = 1, \quad \sum_{\ell=1}^{\jmath} \bar{r}_{i\ell} = 1, \quad \sum_{\ell=1}^{\jmath} q_\ell = 1, \quad \sum_{\ell=1}^{\jmath} \nu_\ell = 1, \\ &\sum_{\ell=1}^{\jmath} r_{i\ell}(1-\ell) + \sum_{\ell=1}^{\jmath} p_{i\ell} = c_i, \quad \sum_{\ell=1}^{\jmath} q_\ell (1-\ell) + \sum_{\ell=1}^{\jmath} w_\ell = 1. \end{aligned} \tag{2.10}$$

2.2. **Trigonometrically fitted MSGRKN method.**

Definition 2.2. MSGRKN method shown in Eq. (2.1) is considered trigonometrically fitted if it can integrate exactly the functions $\exp(i\omega x)$ and $\exp(-i\omega x)$, where i is the imaginary unit. As a result, the equations shown below are obtained.

$$\begin{aligned}
\exp(\pm i c_i s) &= \sum_{\ell=1}^{\jmath} r_{i\ell}\exp(\pm i(1-\ell)s) \pm\, is\sum_{\ell=1}^{\jmath} p_{i\ell}\exp(\pm i(1-\ell)s)\\
&- \ s^2\sum_{j=1}^{\kappa} a_{ij}\exp(\pm i c_j s),\\
\pm i\omega\exp(\pm i c_i s) &= \pm i\omega\sum_{\ell=1}^{\jmath} \bar{r}_{i\ell}\exp(\pm i(1-\ell)s) - \ s\omega\sum_{j=1}^{\kappa} \bar{a}_{ij}\exp(\pm i c_j s),\\
\exp(\pm i s) &= \sum_{\ell=1}^{\jmath} q_\ell\exp(\pm i(1-\ell)s) \pm i\, s\sum_{\ell=1}^{\jmath} w_\ell\exp(\pm i(1-\ell)s)\\
&- \ s^2\sum_{i=1}^{\kappa} b_i\exp(\pm i c_i s),\\
\pm i\omega\exp(\pm i s) &= \pm i\omega\sum_{\ell=1}^{\jmath} \nu_\ell\exp(\pm i(1-\ell)s) - s\,\omega\sum_{i=1}^{\kappa} \bar{b}_i\exp(\pm i c_i s),
\end{aligned} \tag{2.11}$$

where $s=\omega h$. By utilizing the formula of Euler $\exp(\pm i\, s)=\cos(s)\pm i\sin(s)$ in the system of Eqs. (2.11) and comparing the real and imaginary parts, the trigonometrically fitting (TF) conditions shown below are obtained:

$$\sin(c_i\, s) = \sum_{\ell=1}^{\jmath} r_{i\ell}\sin(s(1-\ell)) + \ s\sum_{\ell=1}^{\jmath} p_{i\ell}\cos(s(1-\ell)) - s^2\sum_{j=1}^{\kappa} a_{ij}\ \sin(c_j\, s), \tag{2.12}$$

$$\cos(c_i\, s) = \sum_{\ell=1}^{\jmath} r_{i\ell}\cos(s(1-\ell)) - \ s\sum_{\ell=1}^{\jmath} p_{i\ell}\sin(s(1-\ell)) - s^2\sum_{j=1}^{\kappa} a_{ij}\ \cos(c_j\, s), \tag{2.13}$$

$$\sin(c_i s) = \sum_{\ell=1}^{\jmath} \bar{r}_{i\ell}\sin(s(1-\ell)) + \ s\sum_{j=1}^{\kappa} \bar{a}_{ij}\cos(c_j s), \tag{2.14}$$

$$\cos(c_i s) = \sum_{\ell=1}^{\jmath} \bar{r}_{i\ell}\cos((1-\ell)s) - \ s\sum_{j=1}^{\kappa} \bar{a}_{ij}\sin(c_j s), \tag{2.15}$$

$$\cos(s) = \sum_{\ell=1}^{\jmath} q_\ell\ \cos(s(1-\ell)) - \ s\sum_{\ell=1}^{\jmath} w_\ell\ \sin(s(1-\ell)) - s^2\sum_{i=1}^{\kappa} b_i\ \cos(c_i s), \tag{2.16}$$

$$\sin(s) = \sum_{\ell=1}^{\jmath} q_\ell\ \sin(s(1-\ell)) + \ s\sum_{\ell=1}^{\jmath} w_\ell\ \cos(s(1-\ell)) - s^2\sum_{i=1}^{\kappa} b_i\ \sin(c_i s), \tag{2.17}$$

$$\cos(s) = \sum_{\ell=1}^{\jmath} \nu_\ell\cos(s(1-\ell)) - s\ \sum_{i=1}^{\kappa} \bar{b}_i\ \sin(c_i s), \tag{2.18}$$

$$\sin(s) = \sum_{\ell=1}^{\jmath} \nu_\ell\sin(s(1-\ell)) + s\ \sum_{i=1}^{\kappa} \bar{b}_i\ \cos(c_i s) \tag{2.19}$$

3. Construction of the proposed methods

The development of trigonometrically fitted two-step Runge-Kutta-Nyström (TFTSGRKN) when $\jmath = 2$ will be discussed in this section. In specific, two explicit TFTSGRKN methods of orders five and six respectively, will be constructed.

3.1. **Trigonometrically fitted TSGRKN method of order five.** This method is constructed based on the fifth-order method of TSGRKN5(4) pair given in [2] as illustrated in Table 1

Table 1. The fifth-order method of TSGRKN5(4) pair in [2].

-1	0	1	0	0	0				0	1	0			
0	1	0	0	0	0	0			1	0	0	0		
$\frac{3}{5}$	$\frac{17408}{3125}$	$-\frac{14283}{3125}$	$-\frac{6144}{3125}$	$-\frac{6264}{3125}$	$-\frac{864}{3125}$	$\frac{2304}{3125}$	0		$-\frac{64}{125}$	$\frac{189}{125}$	$\frac{72}{125}$	$\frac{192}{125}$	0	
1	$\frac{41}{25}$	$-\frac{16}{25}$	$\frac{14}{25}$	$-\frac{1}{5}$	$-\frac{13}{800}$	$\frac{119}{300}$	$\frac{23}{96}$	0	$\frac{16}{17}$	$\frac{1}{17}$	$\frac{3}{68}$	$\frac{8}{51}$	$\frac{175}{204}$	0
	$\frac{41}{25}$	$-\frac{16}{25}$	$\frac{14}{25}$	$-\frac{1}{5}$	$-\frac{13}{800}$	$\frac{119}{300}$	$\frac{23}{96}$	0	$\frac{16}{17}$	$\frac{1}{17}$	$\frac{1}{68}$	$\frac{16}{51}$	$\frac{125}{204}$	$\frac{2}{17}$

For the parameters of this method, we solve the TF conditions in Eqs. (2.16)-(2.19) with the fundamental relation given in Eq. (2.10)

$$\sum_{\ell=1}^{2} q_\ell = 1, \qquad \sum_{\ell=1}^{2} \nu_\ell = 1, \qquad \sum_{\ell=1}^{2} q_\ell (1-\ell) + \sum_{\ell=1}^{2} w_\ell = 1, \tag{3.1}$$

to find the values of b_i, $\bar{b}_i$, q_ℓ, ν_ℓ, and w_ℓ, we obtain the solution:

$$\begin{aligned}
b_2 &= -\frac{(-\cos(s)\, s + \sin(s))\, b_1}{(-1)s + \sin(s)} \\
&- \frac{(\sin(c_3 s)\cos(s) + \sin(s)\cos(c_3 s) - \cos(c_3 s)\, s - \sin(c_3 s))\, b_3}{(-1)\, s + \sin(s)} \\
&+ \frac{\left((\sin(s))^2 - \sin(s)\, s + (\cos(s))^2 - 2\cos(s) + 1\right) w_2}{s\,(\sin(s) - s)} - 2\,\frac{\cos(s) - 1}{s^2},
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
\bar{b}_2 &= \frac{1}{s\sin(s)} \sin(c_3 s)\cos(s)\, s\bar{b}_3 - 2\cos(s)\, s\bar{b}_1 \sin(s) - \sin(s)\cos(c_3 s)\, s\bar{b}_3 \\
&+ (\sin(s))^2\, w_2 - (\cos(s))^2\, w_2 + (\sin(s))^2 + (\cos(s))^2 + w_2\cos(s) - \cos(s),
\end{aligned} \tag{3.3}$$

$$\bar{b}_4 = -\frac{\sin(c_3 s)\, s\bar{b}_3 - \sin(s)\, s\bar{b}_1 - 1 - w_2\cos(s) + \cos(s) + w_2}{\sin(s)\, s}, \tag{3.4}$$

$$q_1 = 2 - \frac{s\,(\cos(s) - 1)\, w_2}{\sin(s) - s} - \frac{\sin(s)\, s^2 b_1}{\sin(s) - s} + \frac{\sin(c_3 s)\, s^2 b_3}{\sin(s) - s}, \tag{3.5}$$

$$q_2 = -1 + \frac{s\,(\cos(s) - 1)\, w_2}{\sin(s) - s} + \frac{\sin(s)\, s^2 b_1}{\sin(s) - s} - \frac{\sin(c_3 s)\, s^2 b_3}{\sin(s) - s}, \tag{3.6}$$

$$w_1 = \frac{\sin(s)\, s^2 b_1}{\sin(s) - s} - \frac{\sin(c_3 s)\, s^2 b_3}{\sin(s) - s} - \frac{(-\cos(s)\, s + \sin(s))\, w_2}{\sin(s) - s}, \tag{3.7}$$

by substituting coefficients in Table 1 for the values of b_1, b_3, c_3, $\bar{b}_1$, $\bar{b}_3$, and w_2, we obtain

$$
\begin{aligned}
b_2 &= \frac{13}{800}\,\frac{-\cos(s)\,s+\sin(s)}{(-1)\,s+\sin(s)} \\
&- \frac{23}{96}\,\frac{\sin(3/5\,s)\cos(s)+\sin(s)\cos(3/5\,s)-\cos(3/5\,s)\,s-\sin(3/5\,s)}{(-1)\,s+\sin(s)} \\
&- \frac{1}{5}\,\frac{(\sin(s))^2-\sin(s)\,s+(\cos(s))^2-2\cos(s)+1}{s\,(\sin(s)-s)} - \frac{2(\cos(s)-1)}{s^2}, \qquad (3.8)\\
\bar{b}_2 &= \frac{1}{s\sin(s)}\Big(\frac{125}{204}\sin(\frac{3}{5}s)\cos(s)s - \frac{1}{34}\cos(s)s\sin(s) \\
&- \frac{125}{204}\sin(s)\cos(\frac{3}{5}s)s + \frac{18}{17}(\sin(s))^2 + \frac{16}{17}(\cos(s))^2 - \frac{16}{17}\cos(s)\Big), \qquad (3.9)\\
\bar{b}_4 &= -\frac{1}{s\sin(s)}\Big(\frac{125}{204}\sin(\frac{3}{5}s)s - \frac{1}{68}\sin(s)\,s - \frac{16}{17} + \frac{16}{17}\cos(s)\Big), \qquad (3.10)\\
q_1 &= 2 + \frac{1}{5}\,\frac{s\,(\cos(s)-1)}{(-1)\,s+\sin(s)} + \frac{13}{800}\,\frac{\sin(s)\,s^2}{(-1)\,s+\sin(s)} + \frac{23}{96}\,\frac{\sin(3/5\,s)\,s^2}{(-1)\,s+\sin(s)}, \qquad (3.11)\\
q_2 &= -1 - \frac{1}{5}\,\frac{s\,(\cos(s)-1)}{(-1)\,s+\sin(s)} - \frac{13}{800}\,\frac{\sin(s)\,s^2}{(-1)\,s+\sin(s)} - \frac{23}{96}\,\frac{\sin(3/5\,s)\,s^2}{(-1)\,s+\sin(s)}, \qquad (3.12)\\
w_1 &= -\frac{13}{800}\,\frac{\sin(s)\,s^2}{(-1)\,s+\sin(s)} - \frac{23}{96}\,\frac{\sin(3/5\,s)\,s^2}{(-1)\,s+\sin(s)} + \frac{1}{5}\,\frac{-\cos(s)\,s+\sin(s)}{(-1)\,s+\sin(s)}. \qquad (3.13)
\end{aligned}
$$

In a similar manner, by solving Eqs. (2.12) and (2.13) for the coefficients a_{ij}, $r_{\kappa\jmath}$, and $p_{\kappa\jmath}$, yield the following solution

$$
\begin{aligned}
r_{31} &= \cos(s)\,s^2 a_{31} - s p_{32}\sin(s) + s^2 a_{32} - r_{32}\cos(s) + \cos(c_3 s), \qquad (3.14)\\
p_{31} &= -\sin(s)\,s a_{31} + \frac{\sin(s)\,r_{32}}{s} - p_{32}\cos(s) + \frac{\sin(c_3 s)}{s}, \qquad (3.15)
\end{aligned}
$$

by substituting coefficients in Table 1 for the values of a_{31}, p_{32}, a_{32}, and r_{32}, we obtain

$$
\begin{aligned}
r_{31} &= -\frac{864}{3125}\cos(s)\,s^2 + \frac{6264}{3125}\sin(s)\,s + \frac{2304}{3125}s^2 + \frac{14283}{3125}\cos(s) \\
&+ \cos\left(\frac{3}{5}s\right), \qquad (3.16)\\
p_{31} &= \frac{864}{3125}\sin(s)\,s - \frac{14283}{3125}\,\frac{\sin(s)}{s} + \frac{6264}{3125}\cos(s) + \frac{\sin(3/5\,s)}{s}. \qquad (3.17)
\end{aligned}
$$

Solving Eqs. (2.14) and (2.15) for the coefficients $\bar{a}_{ij}$ and $\bar{r}_{\kappa\jmath}$, result in the following solution

$$
\begin{aligned}
\bar{r}_{31} &= \cos(3/5\,s) - \bar{r}_{32}\cos(s) - s\,\bar{a}_{31}\sin(s), \qquad (3.18)\\
\bar{a}_{32} &= \frac{-\cos(s)\,s\,\bar{a}_{31} + \bar{r}_{32}\sin(s) + \sin(3/5\,s)}{s}, \qquad (3.19)\\
\bar{a}_{41} &= \frac{\sin(3/5\,s)\,\bar{a}_{43}}{\sin(s)} + \frac{16}{17}\,\frac{\cos(s)-1}{\sin(s)\,s}, \qquad (3.20)\\
\bar{a}_{42} &= -\frac{(\cos(s)\sin(3/5\,s)+\sin(s)\cos(3/5\,s))\,\bar{a}_{43}}{\sin(s)} \\
&+ \frac{2}{17}\,\frac{9\,(\sin(s))^2 - 8\,(\cos(s))^2 + 8\cos(s)}{\sin(s)\,s}, \qquad (3.21)
\end{aligned}
$$

by substituting coefficients in Table 1, we obtain

$$\bar{r}_{31} = \cos\left(\frac{3}{5}s\right) - \frac{189}{125}\cos(s) - \frac{72}{125}s\sin(s), \tag{3.22}$$

$$\bar{a}_{32} = \frac{1}{s}\left(-\frac{72}{125}\cos(s)\,s + \frac{189}{125}\sin(s) + \sin\left(\frac{3}{5}s\right)\right), \tag{3.23}$$

$$\bar{a}_{41} = \frac{175}{204}\frac{\sin(3/5\,s)}{\sin(s)} + \frac{16}{17}\frac{\cos(s)-1}{\sin(s)\,s}, \tag{3.24}$$

$$\bar{a}_{42} = -\frac{175}{204}\frac{\cos(3/5\,s)\sin(s) + \sin(3/5\,s)\cos(s)}{\sin(s)} + \frac{2}{17}\frac{9\,(\sin(s))^2 - 8\,(\cos(s))^2 + 8\,\cos(s)}{\sin(s)\,s}. \tag{3.25}$$

For small values of s the aforementioned parameters undergo heavy cancellations, and the below Taylor series expansions must be applied

$$b_2 = \frac{119}{300} + \frac{5357}{31500000}s^4 - \frac{6329}{350000000}s^6 + \ldots, \tag{3.26}$$

$$\bar{b}_2 = \frac{16}{51} - \frac{1}{19125}s^4 + \frac{2441}{80325000}s^6 + \frac{4489}{6024375000}s^8 + \ldots, \tag{3.27}$$

$$\bar{b}_4 = \frac{2}{17} + \frac{1}{19125}s^4 + \frac{739}{80325000}s^6 + \frac{12727}{12048750000}s^8 + \ldots, \tag{3.28}$$

$$q_1 = \frac{41}{25} + \frac{461}{1312500}s^4 + \frac{803}{87500000}s^6 + \frac{708923}{6063750000000}s^8 + \ldots, \tag{3.29}$$

$$q_2 = -\frac{16}{25} - \frac{461}{1312500}s^4 - \frac{803}{87500000}s^6 - \frac{708923}{6063750000000}s^8 + \ldots, \tag{3.30}$$

$$w_1 = \frac{14}{25} - \frac{461}{1312500}s^4 - \frac{803}{87500000}s^6 - \frac{708923}{6063750000000}s^8 + \ldots, \tag{3.31}$$

$$r_{31} = \frac{17408}{3125} - \frac{96}{78125}s^6 + \frac{1368}{13671875}s^8 + \ldots, \tag{3.32}$$

$$p_{31} = -\frac{6144}{3125} + \frac{1152}{2734375}s^6 - \frac{3632}{205078125}s^8 + \ldots, \tag{3.33}$$

$$\bar{r}_{31} = -\frac{64}{125} + \frac{24}{625}s^4 - \frac{216}{78125}s^6 + \frac{2111}{27343750}s^8 + \ldots, \tag{3.34}$$

$$\bar{a}_{32} = \frac{192}{125} - \frac{168}{15625}s^4 + \frac{1352}{2734375}s^6 - \frac{4139}{410156250}s^8 + \ldots, \tag{3.35}$$

$$\bar{a}_{41} = \frac{3}{68} + \frac{4}{255}s^2 + \frac{143}{95625}s^4 + \frac{58577}{401625000}s^6 + \frac{879661}{60243750000}s^8 + \ldots, \tag{3.36}$$

$$\bar{a}_{42} = \frac{8}{51} - \frac{4}{255}s^2 + \frac{832}{95625}s^4 - \frac{58217}{401625000}s^6 + \frac{1160699}{60243750000}s^8 + \ldots. \tag{3.37}$$

The fifth-order method given by Eqs. (3.26)-(3.37) is denoted as TFTSGRKN4s5. As $s \to 0$, the coefficients b_2, $\bar{b}_2$, $\bar{b}_4$, q_1, q_2, r_{31}, $\bar{r}_{31}$, p_{31}, w_1, $\bar{a}_{32}$, $\bar{a}_{41}$, and $\bar{a}_{42}$ of the proposed method reduce to those of the original method.

3.2. **Trigonometrically fitted TSGRKN method of order six.** This method is constructed using the TSGRKN6(5) pair's sixth-order method given in [2] as shown in Table 2

Table 2. The sixth-order method of TSGRKN6(5) pair in [2].

-1	0	1	0	0	0					0	1	0				
0	1	0	0	0	0	0				1	0	0	0			
$\frac{1}{5}$	$\frac{3456}{3125}$	$-\frac{331}{3125}$	$\frac{432}{3125}$	$-\frac{138}{3125}$	$-\frac{18}{3125}$	$\frac{108}{3125}$	0			$\frac{108}{125}$	$\frac{17}{125}$	$\frac{6}{125}$	$\frac{36}{125}$	0		
$\frac{7}{10}$	$-\frac{1852201}{1100000}$	$\frac{2952201}{1100000}$	$\frac{103173}{44000}$	$\frac{285719}{275000}$	$\frac{9615319}{79200000}$	$-\frac{3129581}{4400000}$	$\frac{1685159}{3168000}$	0		$\frac{743}{3250}$	$\frac{2507}{3250}$	$\frac{333}{1000}$	$\frac{2299}{13000}$	$\frac{25}{26}$	0	
1	$\frac{121}{125}$	$\frac{4}{125}$	$\frac{11466}{11125}$	$\frac{3}{2225}$	$-\frac{4993}{4539000}$	$-\frac{19253}{934500}$	$\frac{19727}{53400}$	$\frac{109324}{794325}$	0	$\frac{144}{143}$	$-\frac{1}{143}$	$-\frac{29}{2431}$	$\frac{68}{231}$	$\frac{5}{429}$	$\frac{35680}{51051}$	0
	$\frac{121}{125}$	$\frac{4}{125}$	$\frac{11466}{11125}$	$\frac{3}{2225}$	$-\frac{4993}{4539000}$	$-\frac{19253}{934500}$	$\frac{19727}{53400}$	$\frac{109324}{794325}$	0	$\frac{144}{143}$	$-\frac{1}{143}$	$-\frac{32}{21879}$	$-\frac{32}{3003}$	$\frac{1175}{2574}$	$\frac{6400}{13923}$	$\frac{229}{2574}$

For the coefficients of this method, solve Eqs. (2.16)-(2.19) together with the following conditions

$$\sum_{\ell=1}^{2} q_\ell = 1, \qquad \sum_{\ell=1}^{2} \nu_\ell = 1, \qquad \sum_{\ell=1}^{2} q_\ell(1-\ell) + \sum_{\ell=1}^{2} w_\ell = 1, \tag{3.38}$$

for the values of b_i, $\bar{b}_i$, q_ℓ, ν_ℓ, and w_ℓ, this lead to the following solution:

$$\begin{aligned}
b_2 =& -\frac{(-\cos(s)\,s+\sin(s))\,b_1}{\sin(s)-s} \\
&-\frac{(\sin(c_3 s)\cos(s)+\sin(s)\cos(c_3 s)-\cos(c_3 s)\,s-\sin(c_3 s))\,b_3}{\sin(s)-s} \\
&-\frac{(\sin(c_4 s)\cos(s)+\sin(s)\cos(c_4 s)-\cos(c_4 s)\,s-\sin(c_4 s))\,b_4}{\sin(s)-s} \\
&+\frac{\left((\sin(s))^2-\sin(s)\,s+(\cos(s))^2-2\cos(s)+1\right)s_2}{s\,(\sin(s)-s)}-2\,\frac{\cos(s)-1}{s^2},
\end{aligned} \tag{3.39}$$

$$\begin{aligned}
\bar{b}_2 =& \frac{1}{s\sin(s)}\sin(c_3 s)\cos(s)\,s\,\bar{b}_3+\sin(c_4 s)\cos(s)\,s\,\bar{b}_4-2\cos(s)\,s\bar{b}_1\sin(s) \\
&-\sin(s)\cos(c_3 s)\,s\bar{b}_3-\sin(s)\cos(c_4 s)\,s\bar{b}_4+(\sin(s))^2\,\nu_2 \\
&-(\cos(s))^2\,\nu_2+(\sin(s))^2+(\cos(s))^2+\nu_2\cos(s)-\cos(s),
\end{aligned} \tag{3.40}$$

$$\bar{b}_5 = -\frac{\sin(c_3 s)\,s\bar{b}_3+\sin(c_4 s)\,s\bar{b}_4-\sin(s)\,s\bar{b}_1-1-\nu_2\cos(s)+\cos(s)+\nu_2}{\sin(s)\,s}, \tag{3.41}$$

$$q_1 = 2-\frac{s\,(\cos(s)-1)\,w_2}{\sin(s)-s}+\frac{\sin(c_3 s)\,s^2 b_3}{\sin(s)-s}+\frac{\sin(c_4 s)\,s^2 b_4}{\sin(s)-s}-\frac{\sin(s)\,s^2 b_1}{\sin(s)-s}, \tag{3.42}$$

$$q_2 = -1+\frac{s\,(\cos(s)-1)\,w_2}{\sin(s)-s}-\frac{\sin(c_3 s)\,s^2 b_3}{\sin(s)-s}-\frac{\sin(c_4 s)\,s^2 b_4}{\sin(s)-s}+\frac{\sin(s)\,s^2 b_1}{\sin(s)-s}, \tag{3.43}$$

$$w_1 = \frac{\sin(s)\,s^2 b_1}{\sin(s)-s}-\frac{\sin(c_3 s)\,s^2 b_3}{\sin(s)-s}-\frac{\sin(c_4 s)\,s^2 b_4}{\sin(s)-s}-\frac{(-\cos(s)\,s+\sin(s))\,w_2}{\sin(s)-s}, \tag{3.44}$$

by substituting coefficients in Table 2 for the values of b_1, b_3, b_4, c_3, c_4, $\bar{b}_1$, $\bar{b}_3$, $\bar{b}_4$, and w_2, we will get

$$\begin{aligned}
b_2 =& \frac{4993}{4539000}\,\frac{-\cos(s)\,s+\sin(s)}{\sin(s)-s} \\
&-\frac{19727}{53400}\,\frac{\sin(1/5\,s)\cos(s)+\sin(s)\cos(1/5\,s)-\cos(1/5\,s)\,s-\sin(1/5\,s)}{\sin(s)-s} \\
&-\frac{109324}{794325}\,\frac{\left(\sin\left(\frac{7}{10}\,s\right)\cos(s)+\sin(s)\cos\left(\frac{7}{10}\,s\right)-\cos\left(\frac{7}{10}\,s\right)s-\sin\left(\frac{7}{10}\,s\right)\right)}{(\sin(s)-s)} \\
&+\frac{3}{2225}\,\frac{(\sin(s))^2-\sin(s)\,s+(\cos(s))^2-2\cos(s)+1}{s\,(\sin(s)-s)}-2\,\frac{\cos(s)-1}{s^2},
\end{aligned} \tag{3.45}$$

$$\bar{b}_2 = \frac{1}{(\sin(s))\,s}\left(\frac{1175}{2574}\sin\left(\frac{1}{5}s\right)\cos(s)\,s + \frac{6400}{13923}\sin\left(\frac{7}{10}s\right)\cos(s)\,s\right.$$
$$+\frac{64}{21879}\cos(s)\,s\sin(s) - \frac{1175}{2574}\sin(s)\cos\left(\frac{1}{5}s\right)s - \frac{6400}{13923}\sin(s)\cos\left(\frac{7}{10}s\right)s$$
$$\left.+\frac{142}{143}(\sin(s))^2 + \frac{144}{143}(\cos(s))^2 - \frac{144}{143}\cos(s)\right), \tag{3.46}$$
$$\bar{b}_5 = -\frac{1}{(\sin(s))\,s}\left(\frac{1175}{2574}\sin\left(\frac{1}{5}s\right)s + \frac{6400}{13923}\sin\left(\frac{7}{10}s\right)s + \frac{32}{21879}\sin(s)\,s\right.$$
$$\left.-\frac{144}{143} + \frac{144}{143}\cos(s)\right), \tag{3.47}$$
$$q_1 = 2 - \frac{3}{2225}\,\frac{s(\cos(s)-1)}{\sin(s)-s} + \frac{19727}{53400}\,\frac{\sin\left(\frac{1}{5}s\right)s^2}{\sin(s)-s} + \frac{109324}{794325}\,\frac{\sin\left(\frac{7}{10}s\right)s^2}{(\sin(s)-s)}$$
$$+\frac{4993}{4539000}\,\frac{\sin(s)\,s^2}{\sin(s)-s}, \tag{3.48}$$
$$q_2 = -1 + \frac{3}{2225}\,\frac{s(\cos(s)-1)}{\sin(s)-s} - \frac{19727}{53400}\,\frac{\sin(1/5\,s)\,s^2}{\sin(s)-s} - \frac{109324}{794325}\,\frac{\sin\left(\frac{7}{10}s\right)s^2}{(\sin(s)-s)}$$
$$-\frac{4993}{4539000}\,\frac{\sin(s)\,s^2}{\sin(s)-s}, \tag{3.49}$$
$$w_1 = -\frac{4993}{4539000}\,\frac{\sin(s)\,s^2}{\sin(s)-s} - \frac{19727}{53400}\,\frac{\sin(1/5\,s)\,s^2}{\sin(s)-s} - \frac{109324}{794325}\,\frac{\sin\left(\frac{7}{10}s\right)s^2}{(\sin(s)-s)}$$
$$-\frac{3}{2225}\,\frac{-\cos(s)\,s + \sin(s)}{\sin(s)-s} \tag{3.50}$$

Similarly, solving the TF conditions in Eqs. (2.12) and (2.13) for the coefficients a_{ij}, $r_{\kappa j}$, and $p_{\kappa j}$ produces the following solution.

$$r_{31} = \cos(s)\,s^2 a_{31} - s p_{32}\sin(s) + s^2 a_{32} - r_{32}\cos(s) + \cos(1/5\,s), \tag{3.51}$$
$$r_{41} = \cos\left(\frac{s}{5}\right)s^2 a_{43} + \cos(s)\,s^2 a_{41} - s p_{42}\sin(s) + s^2 a_{42} - r_{42}\cos(s) + \cos\left(\frac{7\,s}{10}\right), \tag{3.52}$$
$$p_{31} = \frac{-s^2 a_{31}\sin(s) - \cos(s)\,s p_{32} + r_{32}\sin(s) + \sin\left(\frac{1}{5}s\right)}{s}, \tag{3.53}$$
$$p_{41} = -\sin(s)\,s a_{41} + s\sin\left(\frac{1}{5}s\right)a_{43} + \frac{\sin(s)\,r_{42}}{s} - p_{42}\cos(s) + \sin\left(\frac{\frac{7}{10}s}{s}\right), \tag{3.54}$$

by substituting coefficients in Table 2 for the values of a_{ij}, r_{32}, r_{42}, p_{32}, and p_{42}, we obtain

$$r_{31} = -\frac{18}{3125}\cos(s)\,s^2 + \frac{138}{3125}\sin(s)\,s + \frac{108}{3125}s^2 + \frac{331}{3125}\cos(s) + \cos\left(\frac{1}{5}s\right), \tag{3.55}$$
$$r_{41} = \frac{1685159}{3168000}\cos\left(\frac{1}{5}s\right)s^2 + \frac{9615319}{79200000}\cos(s)\,s^2 - \frac{285719}{275000}\sin(s)\,s$$
$$-\frac{3129581}{4400000}s^2 - \frac{2952201}{1100000}\cos(s) + \cos\left(\frac{7}{10}s\right), \tag{3.56}$$

$$p_{31}=\frac{1}{s}\left(\frac{18}{3125}\sin(s)\,s^2+\frac{138}{3125}\cos(s)\,s-\frac{331}{3125}\sin(s)+\sin\left(\frac{1}{5}\,s\right)\right), \tag{3.57}$$

$$p_{41}=-\frac{9615319}{79200000}\sin(s)\,s+\frac{1685159}{3168000}\,s\sin(1/5\,s)+\frac{2952201}{1100000}\frac{\sin(s)}{s}$$
$$-\frac{285719}{275000}\cos(s)+\sin\left(\frac{7}{10}\,s\right)s^{-1}. \tag{3.58}$$

Solving Eqs. (2.14) and (2.15) for the coefficients $\bar{a}_{ij}$ and $\bar{r}_{\kappa\jmath}$, result in the following solution

$$\bar{r}_{31}=\cos\left(\frac{1}{5}\,s\right)-\bar{r}_{32}\cos(s)-s\,\bar{a}_{31}\sin(s), \tag{3.59}$$

$$\bar{r}_{41}=-\sin(s)\,s\,\bar{a}_{41}+\sin\left(\frac{1}{5}\,s\right)s\,\bar{a}_{43}-\bar{r}_{42}\cos(s)+\cos\left(\frac{7}{10}\,s\right), \tag{3.60}$$

$$\bar{a}_{32}=\frac{-\cos(s)\,s\,\bar{a}_{31}+\bar{r}_{32}\sin(s)+\sin\left(\frac{1}{5}\,s\right)}{s}, \tag{3.61}$$

$$\bar{a}_{42}=\frac{1}{s}\left(-\cos(s)\,s\,\bar{a}_{41}-\cos\left(\frac{1}{5}\,s\right)s\,\bar{a}_{43}+\bar{r}_{42}\sin(s)+\sin\left(\frac{7}{10}\,s\right)\right), \tag{3.62}$$

$$\bar{a}_{51}=\frac{\sin\left(\frac{1}{5}\,s\right)\bar{a}_{53}}{\sin(s)}+\frac{\sin\left(\frac{7}{10}\,s\right)\bar{a}_{54}}{\sin(s)}+\frac{144}{143}\frac{\cos(s)-1}{\sin(s)\,s}, \tag{3.63}$$

$$\bar{a}_{52}=-\frac{\left(\sin(s)\cos\left(\frac{1}{5}\,s\right)+\cos(s)\sin\left(\frac{1}{5}\,s\right)\right)\bar{a}_{53}}{\sin(s)}$$
$$-\left(\sin(s)\cos\left(\frac{7}{10}\,s\right)+\cos(s)\sin\left(\frac{7}{10}\,s\right)\right)\bar{a}_{54}\,(\sin(s))^{-1}$$
$$+\frac{2}{143}\frac{71\,(\sin(s))^2-72\,(\cos(s))^2+72\cos(s)}{\sin(s)\,s}, \tag{3.64}$$

by substituting coefficients in Table 2 for the values of $\bar{a}_{31}$, $\bar{a}_{41}$, $\bar{a}_{43}$, $\bar{a}_{53}$, $\bar{a}_{54}$, $\bar{r}_{32}$, and $\bar{r}_{42}$, we obtain

$$\bar{r}_{31}=\cos\left(\frac{1}{5}\,s\right)-\frac{17}{125}\cos(s)-\frac{6}{125}\sin(s)\,s, \tag{3.65}$$

$$\bar{r}_{41}=-\frac{333}{1000}\sin(s)\,s+\frac{25}{26}\,s\sin\left(\frac{1}{5}\,s\right)-\frac{2507}{3250}\cos(s)+\cos\left(\frac{7}{10}\,s\right), \tag{3.66}$$

$$\bar{a}_{32}=\left(-\frac{6}{125}\cos(s)\,s+\frac{17}{125}\sin(s)+\sin\left(\frac{1}{5}\,s\right)\right)s^{-1}, \tag{3.67}$$

$$\bar{a}_{42}=\left(-\frac{333}{1000}\cos(s)\,s-\frac{25}{26}\cos\left(\frac{1}{5}\,s\right)s+\frac{2507}{3250}\sin(s)+\sin\left(\frac{7}{10}\,s\right)\right)s^{-1}, \tag{3.68}$$

$$\bar{a}_{51}=\frac{5}{429}\frac{\sin\left(\frac{1}{5}\,s\right)}{\sin(s)}+\frac{35680}{51051}\sin\left(\frac{7}{10}\,s\right)(\sin(s))^{-1}+\frac{144}{143}\frac{\cos(s)-1}{\sin(s)\,s}, \tag{3.69}$$

$$\bar{a}_{52}=-\frac{5}{429}\frac{\sin(s)\cos\left(\frac{1}{5}\,s\right)+\cos(s)\sin\left(\frac{1}{5}\,s\right)}{\sin(s)}$$
$$-\frac{35680}{51051}\left(\sin(s)\cos\left(\frac{7}{10}\,s\right)+\cos(s)\sin\left(\frac{7}{10}\,s\right)\right)(\sin(s))^{-1}$$
$$+\frac{2}{143}\frac{71\,(\sin(s))^2-72\,(\cos(s))^2+72\cos(s)}{\sin(s)\,s}. \tag{3.70}$$

For small values of s the aforementioned parameters undergo heavy cancellations, and the below Taylor series expansions must be applied

$$
\begin{aligned}
b_2 &= -\frac{19253}{934500} - \frac{251}{3115000000}s^4 - \frac{10346563}{2403000000000}s^6 + \ldots,\\
\bar{b}_2 &= -\frac{32}{3003} - \frac{277}{337837500}s^6 + \frac{21227}{231660000000}s^8 + \ldots,\\
\bar{b}_5 &= \frac{229}{2574} + \frac{3}{6256250}s^6 + \frac{10289}{180180000000}s^8 + \ldots,\\
q_1 &= \frac{121}{125} - \frac{251}{1557500000}s^4 - \frac{5781487}{2803500000000}s^6 - \frac{691344389}{11513040000000000}s^8 + \ldots,\\
q_2 &= \frac{4}{125} + \frac{251}{1557500000}s^4 + \frac{5781487}{2803500000000}s^6 + \frac{691344389}{11513040000000000}s^8 + \ldots,\\
w_1 &= \frac{11466}{11125} + \frac{251}{1557500000}s^4 + \frac{5781487}{2803500000000}s^6 + \frac{691344389}{11513040000000000}s^8 + \ldots,\\
r_{31} &= \frac{3456}{3125} - \frac{3}{156250}s^6 + \frac{51}{27343750}s^8 - \frac{8261}{164062500000}s^{10} + \ldots,\\
r_{41} &= -\frac{1852201}{1100000} - \frac{292013981}{10560000000000}s^8 + \frac{50166461219}{57024000000000000}s^{10} + \ldots,\\
p_{31} &= \frac{432}{3125} + \frac{3}{390625}s^6 - \frac{557}{1640625000}s^8 + \frac{14341}{2255859375000}s^{10} + \ldots,\\
p_{41} &= \frac{103173}{44000} - \frac{45980767}{396000000000}s^6 + \frac{16610612263}{2851200000000000}s^8 - \frac{727479047771}{62726400000000000000}s^{10} + \ldots,\\
\bar{r}_{31} &= \frac{108}{125} + \frac{3}{1250}s^4 - \frac{33}{156250}s^6 + \frac{2691}{437500000}s^8 - \frac{23329}{246093750000}s^{10} + \ldots,\\
\bar{r}_{41} &= \frac{743}{3250} + \frac{100093}{3120000}s^4 - \frac{17451437}{9360000000}s^6 + \frac{845071471}{17472000000000}s^8 - \frac{336291538237}{471744000000000000}s^{10} + \ldots,\\
\bar{a}_{32} &= \frac{36}{125} - \frac{27}{31250}s^4 + \frac{31}{781250}s^6 - \frac{5353}{6562500000}s^8 + \ldots,\\
\bar{a}_{42} &= \frac{2299}{13000} - \frac{953209}{156000000}s^4 + \frac{192099941}{655200000000}s^6 - \frac{9469563703}{1572480000000000}s^8 + \ldots,\\
\bar{a}_{51} &= -\frac{29}{2431} - \frac{229}{715000}s^4 - \frac{126703}{3003000000}s^6 - \frac{4664017}{1029600000000}s^8 + \ldots,\\
\bar{a}_{52} &= \frac{68}{231} + \frac{13511}{6435000}s^4 - \frac{5873983}{27027000000}s^6 - \frac{7005989}{5896800000000}s^8 + \ldots.
\end{aligned} \tag{3.71}
$$

We denote the sixth-order method determined by Eqs. (3.71)-(3.71) as TFTSGRKN5s6. As $s \to 0$, the proposed method's coefficients b_2, $\bar{b}_2$, $\bar{b}_5$, q_1, q_2, w_1, r_{31}, r_{41}, $\bar{r}_{31}$, $\bar{r}_{41}$, p_{31}, p_{41}, $\bar{a}_{32}$, $\bar{a}_{42}$, $\bar{a}_{51}$, and $\bar{a}_{52}$ reduce to those of the original method.

3.3. **Absolute stability of the proposed methods.** In the subsequent part, we will present the absolute stability of the trigonometrically fitted MSGRKN methods. For a full study of stability, readers may refer to [13].
Applying the trigonometrically fitted MSGRKN method in Eq. (2.1) to the given test problem

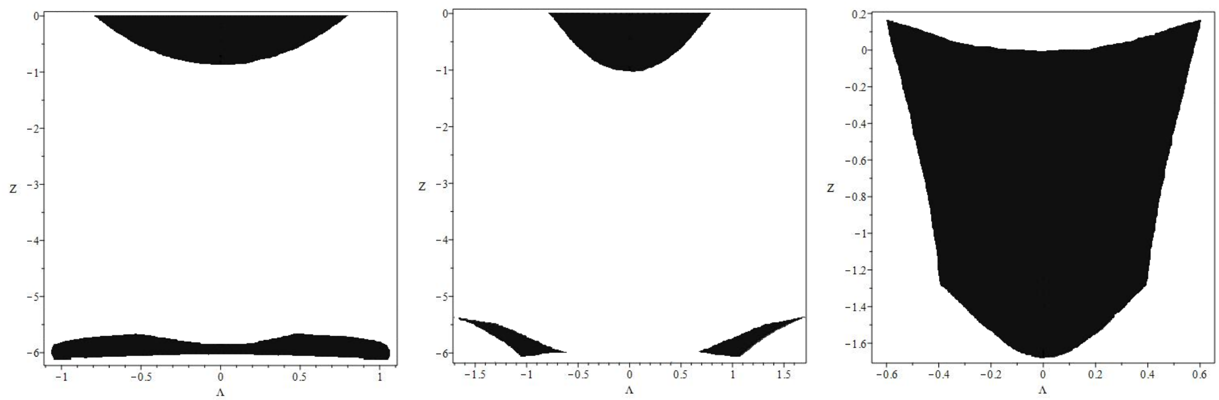
$$u''(x) = -\lambda^2 u(x) + \mu u'(x), \tag{3.72}$$

the stability matrix $M(s, \Lambda, Z)$ given in [13] will be obtained as follow

$$M(s,\Lambda,Z) = \begin{bmatrix} q^T - \Lambda^2\, b^T\, M + Z\, b^T\, M' & w^T - \Lambda^2\, b^T\, N + Z\, b^T\, N' \\ I & \mathbf{0} \\ -\Lambda^2\, \bar{b}^T\, M + Z\, \bar{b}^T\, M' & \nu^T - \Lambda^2\, \bar{b}^T\, N + Z\, \bar{b}^T\, N' \\ \mathbf{0} & I \end{bmatrix}_{2\jmath\times 2\jmath}, \tag{3.73}$$

with

$$I = \begin{bmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}_{(\jmath-1)\times \jmath},$$



(a) s=1 (b) s=2 (c) s=3

FIGURE 1. The stability regions sections of TFTSGRKN4s5 method at $s = 1, 2, 3$.

By using the following stability function for the trigonometrically fitted MSGRKN method

$$\Omega\Big(\xi, M(s,\Lambda,Z)\Big) = det\Big[\xi I - M(s,\Lambda,Z)\Big], \tag{3.74}$$

Where $\Lambda = \lambda h$ and $Z = \mu h$. Following this, we provide the stability definition for the trigonometrically fitted MSGRKN method.

Definition 3.1. For the trigonometrically fitted MSGRKN method in Eq. (2.1) with the stability matrix $M(s,\Lambda,Z)$ given by Eq. (3.73), suppose the stability matrix $M(s,\Lambda,Z)$ has eigenvalues $r_i(s,\Lambda,Z)$, $i = 1,2,\ldots,2\jmath$. The region of the three dimensional space

$$R_s = \Big\{(s,\Lambda,Z) : |r_i(s,\Lambda,Z)| < 1, i = 1,\ldots,2\jmath\Big\}$$

is referred to as the absolute stability region of the method.

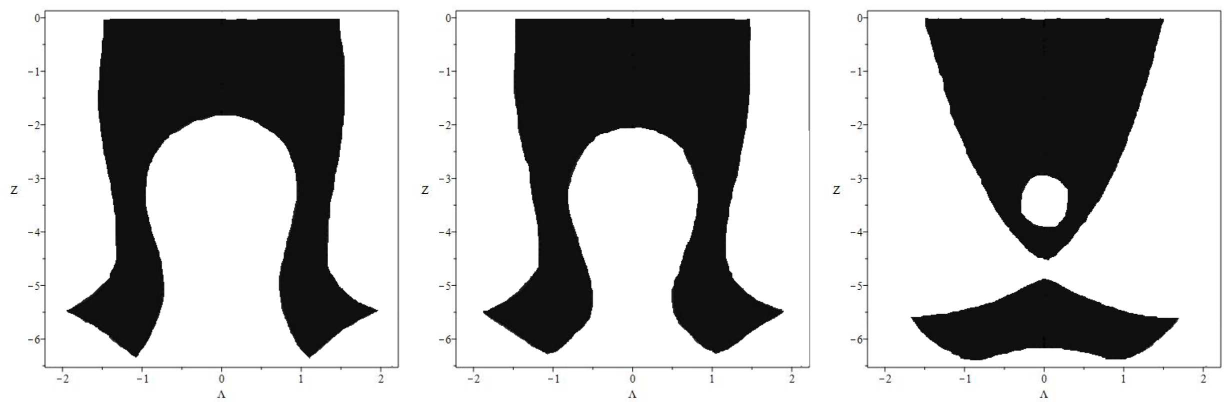
Remark 3.2. In the context of the trigonometrically fitted MSGRKN method, understanding the three-dimensional stability the region in the (s,Λ,Z) space can be challenging. This paper addresses this difficulty by presenting a selection of sections through the three-dimensional stability region by planes where the parameter s is constant.

According to Definition 3.1 and Remark 3.2, the stability regions sections of the TFTSGRKN4s5 and TFTSGRKN5s6 methods by plane $s = 1, 2, 3$, respectively are illustrated in Figures 1 and 2, which are the regions in black.

4. Numerical experiments

In this section, to assess the performance of the suggested methods, we compare the effectiveness of the proposed methods with existing methods by solving a set of test problems. The numerical comparisons here are based on two primary criteria: the computation of the maximum error and the number of function evaluations. The following abbreviations are used in our numerical results. The experiments were conducted using Code Blocks 16.01 and Lenovo PC with the following specifications: Intel(R) Core(TM) i3-5005U CPU @2.00GHz.

- **TFTSGRKN5s6:** the five-stage sixth-order trigonometrically fitted TSGRKN method constructed in this study;
- **TFTSGRKN4s5:** the four-stage fifth-order trigonometrically fitted TSGRKN method constructed in this study;
- **TSGRKN5s6:** the five-stage sixth-order TSGRKN method given in [2];
- **TSGRKN4s5:** the four-stage fifth-order TSGRKN method given in [2];
- **L2TFMSRKN4s5:** the four-stage trigonometrically fitted MSRKN method of order five derived in [11];
- **TFSTDRKN3s5:** the three-stage trigonometrically fitted TDRKN method of order five derived in [5];
- **PFAFRKN4s5:** the four-stage fifth-order phase fitted and amplification fitted RKN method given in [6];
- **MSRKN5s6:** the five-stage sixth-order two-step RKN method obtained by [10];
- **ARKNG6s5:** the six-stage fifth-order adapted RKN given in [8];
- **RK7s6:** the seven-stage sixth-order RK method given in [4];



(a) s=1 (b) s=2 (c) s=3

FIGURE 2. The stability regions sections of TFTSGRKN5s6 method at $s = 1, 2, 3$.

- **NFE:** the number of function evaluations;
- **MaxError:** Max ($|u(x_n) - u_n|$) which is the maximum between absolute errors of the computed solution and the exact solution;

Problem 1. Orbital problem of Stiefel and Bettis which was studied in [11]
Consider the nearly periodic second-order ODE

$$u'' = 0.001e^{ix} - u, \quad u(0) = 1, \quad u'(0) = 0.9995i, \quad x \in [0, 80\pi].$$

Exact solution: $u(x) = \cos(x) + 0.0005\, x \sin(x) + i\Big(\sin(x) - 0.0005\, x \cos(x)\Big)$.

The exact solution represents the motion of a perturbed circular orbit in the complex plane. The problem has the equivalent form

$$\begin{aligned} u_1'' &= 0.001 \cos(x) - u_1, \quad u_1(0) = 1, \quad u_1'(0) = 0, \\ u_2'' &= 0.001 \sin(x) - u_2, \quad u_2(0) = 0, \quad u_2'(0) = 0.9995, \end{aligned}$$

Exact solution: $u_1(x) = \cos(x) + 0.0005\, x \sin(x), \quad u_2(x) = \sin(x) - 0.0005\, x \cos(x)$.

The frequency $\omega = 1$ is chosen as the fitting parameter. For solving this problem, which is a special second-order IVP that does not include the first derivative $u'(x)$. As shown in Table 3, the sixth-order TFTSGRKN5s6 method produces the lowest maximum errors compared to the sixth-order TSGRKN5s6 and MSRKN5s6 methods that are used to solve general and special second-order IVPs, respectively. This is because, unlike the TFTSGRKN5s6 method, the TSGRKN5s6 and MSRKN5s6 methods do not use the trigonometrically fitted technique. The adaptation of the trigonometrically fitted technique improves the accuracy of the methods. The maximum errors of the TFTSGRKN5s6 method were also compared to those of the fifth-order methods (the newly derived TFTSGRKN4s5 method, the TSGRKN4s5 method that used to solve general second-order IVPs, and the L2TFMSRKN4s5 and PFAFRKN4s5 methods that are used to solve special second-order IVPs). Table 3 demonstrates that the TFTSGRKN5s6 method is more accurate at all values of h except for $h = \frac{\pi}{9}$ where the maximum errors for the TFTSGRKN5s6 and PFAFRKN4s5 methods are almost the same. For the comparison of the proposed fifth-order TFTSGRKN4s5 method, we compare the maximum errors for the TFTSGRKN4s5 method with those of the other fifth-order methods (TSGRKN4s5, L2TFMSRKN4s5, and PFAFRKN4s5). We can observe from Table 3 that the TFTSGRKN4s5 method achieves numerical solutions that are as accurate as those obtained by the PFAFRKN4s5 method and better than those of the TSGRKN4s5 and L2TFMSRKN4s5 methods at all values of h except at $h = \frac{\pi}{9}$ where PFAFRKN4s5 method is more accurate.

Problem 2. Consider the two-body problem [3, 18]

$$u_1'' = -\frac{u_1}{(u_1^2 + u_2^2)^{3/2}} - \frac{(2\,\epsilon + \epsilon^2)\, u_1}{(u_1^2 + u_2^2)^{5/2}}, \quad u_2'' = -\frac{u_2}{(u_1^2 + u_2^2)^{3/2}} - \frac{(2\,\epsilon + \epsilon^2)\, u_2}{(u_1^2 + u_2^2)^{5/2}},$$

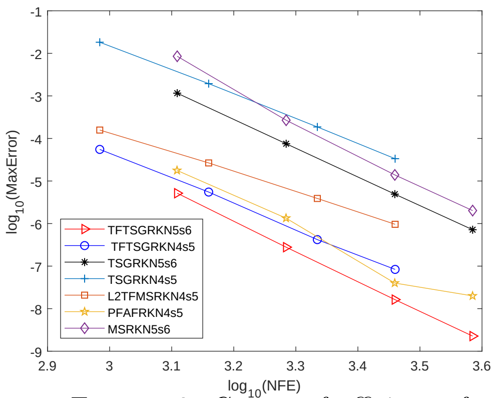


FIGURE 3. Curves of efficiency for Problem 1.

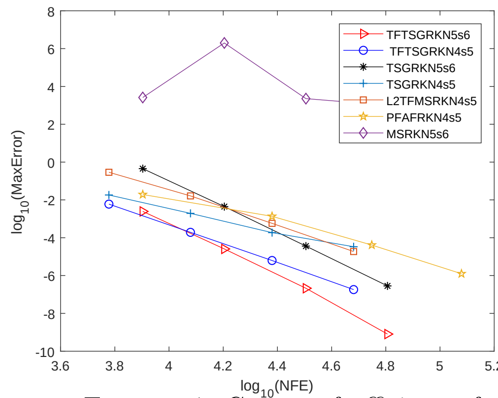


FIGURE 4. Curves of efficiency for Problem 2.

$$u_1(0) = 1, \quad u_1'(0) = 0, \quad u_2(0) = 0, \quad u_2'(0) = 1 + \epsilon,$$

with $\epsilon = 10^{-3}$, the fitting parameter $\omega = 1$ and $x \in [0, 1000]$
Exact solution: $u_1(x) = \cos(x + \epsilon\, x), \quad u_2(x) = \sin(x + \epsilon\, x)$.

This problem is also a special second-order IVP. Table 4 presents the findings obtained by the TFTSGRKN5s6 method. These results are compared with the sixth-order TSGRKN5s6 and MSRKN5s6 methods, as well as the fifth-order TFTSGRKN4s5, TSGRKN4s5, L2TFMSRKN4s5, and PFAFRKN4s5 methods. In terms of accuracy, Table 4 clearly demonstrates that TFTSGRKN5s6 outperforms the sixth- and fifth-order methods at all values of j except at $j = 1$ where the errors produced by TFTSGRKN5s6 and TFTSGRKN4s5 methods are competitive. In addition, the accuracy of the new TFTSGRKN4s5 method is compared with the accuracy of the fifth-order (TSGRKN4s5, L2TFMSRKN4s5, and PFAFRKN4s5) methods. Table 4 shows that the TFTSGRKN4s5 method performs better than those of fifth-order methods at all values of j except at $j = 4$ where TFTSGRKN4s5 and the sixth-order TSGRKN5s6 method have the same errors.

Problem 3. Consider the linear nonhomogeneous oscillatory system [16]
$u_1'' = -13u_1 + 12u_2 + u_1' + 2\,u_2' + 24\sin(5x), \quad u_1(0) = -\frac{9012}{3005}, \quad u_1'(0) = -\frac{7438}{601},$
$u_2'' = 12u_1 - 13u_2 - 2u_1' - 3\,u_2' + \cos(5x), \quad u_2(0) = \frac{8292}{3005}, \quad u_2'(0) = \frac{4583}{601}.$
Exact solution: $u_1(x) = -\frac{7438}{3005}\sin(5x) - \frac{9012}{3005}\cos(5x), \quad u_2(x) = \frac{4583}{3005}\sin(5x) + \frac{8292}{3005}\cos(5x).$

Where $x \in [0, 1000]$ and the fitting parameter $\omega = 5$. In this problem, where u' appears explicitly, we compared the results of the newly proposed TFTSGRKN5s6 and TFTSGRKN4s5 methods with the TSGRKN5s6, TSGRKN4s5, TFSTDRKN3s5, and ARKNG6s5 methods, which were derived to solve general second-order ODEs (1.1). Moreover, the findings of the newly derived methods are compared with the classical RK7s6 method. The errors we got are compared for different step sizes and illustrated in Table 5 shows that the TFTSGRKN5s6 and TFTSGRKN4s5 methods generate smaller errors.

Problem 4. Consider the linear problem studied in [15]

$$u'' = -u' + \cos(x), \quad x \in [0, 100\pi],$$
$$u(0) = -\frac{1}{2}, \quad u'(0) = \frac{1}{2}.$$

Exact solution: $u(x) = \frac{1}{2}\left(sin(x) - \cos(x)\right)$. The fitting parameter $\omega = 1$.

To solving this problem where the first derivative appears explicitly. Table 6 demonstrates that the TFSTDRKN3s5 method's findings are competitive with those of our methods. However, our methods are more accurate than the TSGRKN5s6, TSGRKN4s5, ARKNG6s5, and RK7s6 methods.

Besides the comparison in terms of the maximum errors, comparing the computational efficiency of the proposed methods by considering the number of function evaluations is an important aspect of the numerical comparison. The number of function evaluations directly impacts the computational cost and runtime of the numerical integration process. A method that requires

Table 3. Comparison of the numerical outcomes for Problem 1.

h	Methods	NFE	MaxError
$\frac{\pi}{4}$	TFTSGRKN5s6	1285	5.161572 (-6)
	TFTSGRKN4s5	964	5.523519 (-5)
	TSGRKN5s6	1285	1.158438 (-3)
	TSGRKN4s5	964	1.822003 (-2)
	L2TFMSRKN4s5	964	1.575662 (-4)
	PFAFRKN4s5	1284	1.774120 (-5)
	MSRKN5s6	1285	8.528549 (-3)
$\frac{\pi}{6}$	TFTSGRKN5s6	1925	2.781890 (-7)
	TFTSGRKN4s5	1444	5.498992 (-6)
	TSGRKN5s6	1925	7.511925 (-5)
	TSGRKN4s5	1444	1.941552 (-3)
	L2TFMSRKN4s5	1444	2.658036 (-5)
	PFAFRKN4s5	1924	1.339640 (-6)
	MSRKN5s6	1925	2.680592 (-4)
$\frac{\pi}{9}$	TFTSGRKN5s6	2881	1.633438 (-8)
	TFTSGRKN4s5	2161	5.194966 (-7)
	TSGRKN5s6	2881	4.912598 (-6)
	TSGRKN4s5	2161	1.863227 (-4)
	L2TFMSRKN4s5	2161	3.888469 (-6)
	PFAFRKN4s5	2880	3.999111 (-8)
	MSRKN5s6	2881	1.388938 (-5)
$\frac{\pi}{12}$	TFTSGRKN5s6	3845	2.266398 (-9)
	TFTSGRKN4s5	2884	9.359432 (-8)
	TSGRKN5s6	3845	7.163709 (-7)
	TSGRKN4s5	2884	3.343283 (-5)
	L2TFMSRKN4s5	2884	9.608069 (-7)
	PFAFRKN4s5	3844	1.996201 (-8)
	MSRKN5s6	3845	2.015801 (-6)

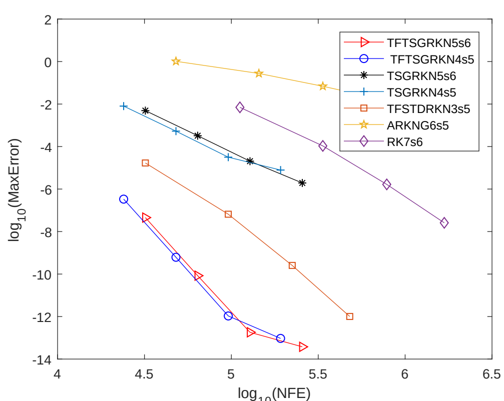


Figure 5. Curves of efficiency for Problem 3.

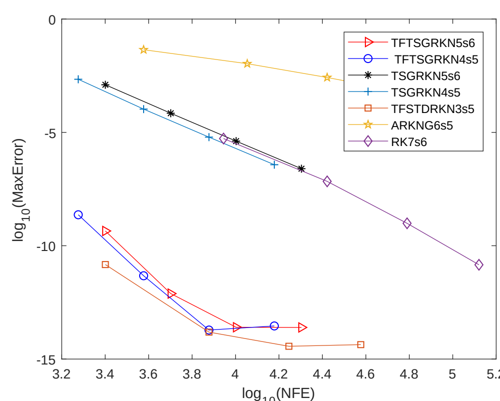


Figure 6. Curves of efficiency for Problem 4.

TABLE 4. Comparison of the numerical outcomes for Problem 2.

$h = 1/(2^j)$	Methods	NFE	MaxError
$j = 1$	TFTSGRKN5s6	8000	2.464736 (-3)
	TFTSGRKN4s5	6000	5.993949 (-3)
	TSGRKN5s6	8000	4.506096 (-1)
	TSGRKN4s5	6000	1.822003 (-2)
	L2TFMSRKN4s5	6000	2.899678 (-1)
	PFAFRKN4s5	8000	1.928877 (-2)
	MSRKN5s6	8000	2.600481 (+3)
$j = 2$	TFTSGRKN5s6	16000	2.554999 (-5)
	TFTSGRKN4s5	12000	1.929336 (-4)
	TSGRKN5s6	16000	4.421979 (-3)
	TSGRKN4s5	12000	1.941552 (-3)
	L2TFMSRKN4s5	12000	1.626069 (-2)
	PFAFRKN4s5	24000	1.357440 (-3)
	MSRKN5s6	16000	1.990184 (+6)
$j = 3$	TFTSGRKN5s6	32000	2.130471 (-7)
	TFTSGRKN4s5	24000	6.249775 (-6)
	TSGRKN5s6	32000	3.617804 (-5)
	TSGRKN4s5	24000	1.863227 (-4)
	L2TFMSRKN4s5	24000	5.777271 (-4)
	PFAFRKN4s5	56000	4.106792 (-5)
	MSRKN5s6	32000	2.266863 (+3)
$j = 4$	TFTSGRKN5s6	64000	8.058148 (-10)
	TFTSGRKN4s5	48000	1.812122 (-7)
	TSGRKN5s6	64000	2.830579 (-7)
	TSGRKN4s5	48000	3.343283 (-5)
	L2TFMSRKN4s5	48000	1.863324 (-5)
	PFAFRKN4s5	120000	1.251015 (-6)
	MSRKN5s6	64000	1.010329 (+3)

fewer function evaluations is generally more computationally efficient. Tables 3 and 4 illustrate that the TFTSGRKN5s6 method has the same number of function evaluations as the TSGRKN5s6 and MSRKN5s6 methods. This is because these methods are two-step Runge-Kutta-Nyström that only require the evaluation of $f(Y_{\jmath+1}, Y'_{\jmath+1}), \ldots, f(Y_\kappa, Y'_\kappa)$ in each step ($\kappa - \jmath$ function evaluations). Additionally, the NFE of the TFTSGRKN4s5, TSGRKN4s5, and L2TFMSRKN4s5 methods are same and fewer than those of the PFAFRKN4s5 method. This is because the TFTSGRKN4s5, TSGRKN4s5, and L2TFMSRKN4s5 methods are two-step Runge-Kutta-Nyström that only requires the evaluation of ($\kappa - \jmath$ function evaluations). While the PFAFRKN4s5 method is a one-step Runge-Kutta-Nyström that requires (κ function evaluations) in each step. Tables 5 and 6 show that the sixth-order two-step Runge-Kutta-Nyström TFTSGRKN5s6 and TSGRKN5s6 methods only need three function evaluations per step compared with the sixth-order classical RK7s6 method that requires reducing problem in Eq. (1.1) to an equivalent system of first-order implies doubling the dimension of the problem. While for the fifth-order two-step Runge-Kutta-Nyström, TFTSGRKN4s5 and TSGRKN4s5 methods require only two function evaluations per step compared with TFSTDRKN3s5 and ARKNG6s5 methods, which require three and six function evaluations per step, respectively. Moreover, from the plots of efficiency curves for test problems (1)–(4) given in Figures 3–6, it is evident that the Figures are consistent with the data of numerical results presented in the Tables. In general, these curves demonstrate that the proposed methods are accurate enough and very efficient for solving the type of ODE in Eq. (1.1).

TABLE 5. Comparison of the numerical outcomes for Problem 3.

$h = 1/(2^j)$	Methods	NFE	MaxError
$j = 3$	TFTSGRKN5s6	32000	4.547125 (-8)
	TFTSGRKN4s5	24000	3.355738 (-7)
	TSGRKN5s6	32000	4.913395 (-3)
	TSGRKN4s5	24000	8.010632 (-3)
	TFSTDRKN3s5	32000	1.690212 (-5)
	ARKNG6s5	48000	1.027010 (+0)
	RK7s6	112014	7.006685 (-3)
$j = 4$	TFTSGRKN5s6	64000	8.398282 (-11)
	TFTSGRKN4s5	48000	6.168723 (-10)
	TSGRKN5s6	64000	3.244349 (-4)
	TSGRKN4s5	48000	5.316291 (-4)
	TFSTDRKN3s5	96000	6.544592 (-8)
	ARKNG6s5	144000	2.756132 (-1)
	RK7s6	336028	1.072075 (-4)
$j = 5$	TFTSGRKN5s6	128000	1.816325 (-13)
	TFTSGRKN4s5	96000	1.067146 (-12)
	TSGRKN5s6	128000	2.047889 (-5)
	TSGRKN4s5	96000	3.163733 (-5)
	TFSTDRKN3s5	224000	2.545351 (-10)
	ARKNG6s5	336000	6.892470 (-2)
	RK7s6	784042	1.657023 (-6)
$j = 6$	TFTSGRKN5s6	256000	3.774758 (-14)
	TFTSGRKN4s5	192000	9.370282 (-14)
	TSGRKN5s6	256000	1.279162 (-6)
	TSGRKN4s5	192000	1.935512 (-6)
	TFSTDRKN3s5	480000	1.009859 (-12)
	ARKNG6s5	720000	1.718241 (-2)
	RK7s6	1680056	2.574848 (-8)

5. Conclusions

In this article, the development of the fifth and sixth-order explicit multi-step Runge-Kutta-Nyström methods with the trigonometrically fitting technique used to obtain the trigonometrically fitted multi-step Runge-Kutta-Nyström methods. In this technique, each stage formula is imposed to exactly integrate the functions $\exp(i\omega x)$ and $\exp(-i\omega x)$, where i is the imaginary. These methods are compared with the existing RK methods and their trigonometrically- and phase-fitted versions in the literature.

Acknowledgment

All authors gratefully acknowledge for the financial support by Putra Grant (project code: GP-IPS/2022/9716200) from Universiti Putra Malaysia.

Table 6. Comparison of the numerical outcomes for Problem 4.

$h = 1/(2^j)$	Methods	NFE	MaxError
$j = 1$	TFTSGRKN5s6	2516	4.477934 (-10)
	TFTSGRKN4s5	1888	2.322399 (-9)
	TSGRKN5s6	2516	1.262954 (-3)
	TSGRKN4s5	1888	2.205504 (-3)
	TFSTDRKN3s5	2516	1.476419 (-11)
	ARKNG6s5	3774	4.443020 (-2)
	RK7s6	8806	5.353845 (-6)
$j = 2$	TFTSGRKN5s6	5033	7.743112 (-13)
	TFTSGRKN4s5	3775	4.735934 (-12)
	TSGRKN5s6	5033	6.998643 (-5)
	TSGRKN4s5	3775	1.064347 (-4)
	TFSTDRKN3s5	7548	1.543210 (-14)
	ARKNG6s5	11322	1.075808 (-2)
	RK7s6	26418	6.898729 (-8)
$j = 3$	TFTSGRKN5s6	10061	2.528186 (-14)
	TFTSGRKN4s5	7546	1.905420 (-14)
	TSGRKN5s6	10061	4.123126 (-6)
	TSGRKN4s5	7546	6.173361 (-6)
	TFSTDRKN3s5	17608	3.663736 (-15)
	ARKNG6s5	26412	2.645903 (-3)
	RK7s6	61628	9.736510 (-10)
$j = 4$	TFTSGRKN5s6	20117	2.482389 (-14)
	TFTSGRKN4s5	15088	2.891090 (-14)
	TSGRKN5s6	20117	2.503142 (-7)
	TSGRKN4s5	15088	3.752632 (-7)
	TFSTDRKN3s5	37724	4.329870 (-15)
	ARKNG6s5	56586	6.561930 (-4)
	RK7s6	132034	1.444511 (-11)

References

[1] Z. A. Anastassi and T. E. Simos, *An optimized Runge–Kutta method for the solution of orbital problems*, Journal of Computational and Applied Mathematics, *1* (2005), 1–9.

[2] A. Abdulsalam, N. Senu, Z. Abdul Majid, and N. M. A. Nik Long, *Adaptive multi-step Runge–Kutta–Nyström methods for general second-order ordinary differential equations*, Journal of Computational and Applied Mathematics, *421* (2023), 1–26.

[3] A. Abdulsalam, N. Senu, Z. Abdul Majid, and N. M. A. Nik Long, *Development of high-order adaptive multi-step Runge–Kutta–Nyström method for solving special second-order ODEs*, Mathematics and Computers in Simulation, *216* (2024), 104-125.

[4] J. C. Butcher, *Numerical methods for ordinary differential equations*, 2nd ed, Chichester, England, John Wiley and Sons, 2016.

[5] Z. Chen, L. Shi, S. Liu, and X. You, *Trigonometrically fitted two-derivative Runge-Kutta-Nyström methods for second-order oscillatory differential equations*, Applied Numerical Mathematics, *142* (2019), 171–189.

[6] M. A. Demba, H. Ramos, P. Kumam, and W. Watthayu, *A phase-fitted and amplification-fitted explicit Runge–Kutta–Nyström pair for oscillating systems*, Mathematical and Computational Applications, *26* (2021), 59–71.

[7] J. M. Franco, *Exponentially fitted explicit Runge–Kutta–Nyström methods*, Journal of Computational and Applied Mathematics, *167* (2004), 1–19.

[8] J. M. Franco, *Runge-Kutta-Nyström methods adapted to the numerical integration of perturbed oscillators*, Computer Physics Communications, *147* (2002), 770787.

[9] W. Gautschi, *Numerical integration of ordinary differential equations based on trigonometric polynomials*, Numerische Mathematik, *3* (1961), 381–397.

[10] L. Jiyong and W. Xianfen, *Multi-step Runge–Kutta–Nyström methods for special second-order initial value problems*, Applied Numerical Mathematics, *113* (2017), 54–70.

[11] L. Jiyong and D. Shuo, *Trigonometrically fitted multi-step RKN methods for second-order oscillatory initial value problems*, Applied Mathematics and Computation, *1* (2018), 740–753.

[12] L. Jiyong, S. Deng, and X. Wang, *Multi-step Nyström methods for general second-order initial value problems* $y''(t) = f(t, y(t), y'(t))$, International Journal of Computer Mathematics, *96* (2019), 1254–1277.

[13] L. Jiyong and G. Yachao, *Modified multi-step Nyström methods for oscillatory general second-order initial value problems*, International Journal of Computer Mathematics, *2* (2021), 223–237.

[14] T. Lyche, *Chebyshevian multistep methods for ordinary differential equations*, Numerische Mathematik, *19* (1972), 65–75.

[15] T. S. Mohamed, N. Senu, Z. B. Ibrahim, and N. M. A. Nik Long, *Efficient two-derivative Runge-Kutta-Nyström methods for solving general second-order ordinary differential equations* $y''(x) = f(x, y(x), y'(x))$, Discrete Dynamics in Nature and Society, *2018* (2018), 1–10.

[16] T. S. Mohamed, N. Senu, Z. B. Ibrahim, and N. M. A. Nik Long, *Exponentially fitted and trigonometrically fitted two-derivative Runge-Kutta-Nyström methods for solving* $y'' = f(x, y, y')$, Mathematical Problems in Engineering, *1* (2018), 1–19.

[17] Th. Monovasilis, Z. Kalogiratou, and T. E. Simos, *Trigonometrical fitting conditions for two derivative Runge-Kutta methods*, Numerical Algorithms, *79* (2018), 787–800.

[18] F. F. Ngwane and S. N. Jator, *A trigonometrically fitted block method for solving oscillatory second-order initial value problems and hamiltonian system*, International Journal of Differential Equations, *1* (2017).

[19] B. Paternoster, *Runge-Kutta-Nyström methods for ODEs with periodic solutions based on trigonometric polynomials*, Applied Numerical Mathematics, *28* (1998), 401–412.

[20] T. E. Simos, *An exponentially-fitted Runge-Kutta method for the numerical integration of initial-value problems with periodic or oscillating solutions*, Computer Physics Communications, *115* (1998), 1–8.