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A numerical approach for solving Caputo-Prabhakar distributed-order time-fractional partial differential equation

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Abstract

In this paper, we proposed a numerical method based on the shifted fractional order Jacobi and trapezoid methods to solve a type of distributed partial differential equations. The fractional derivatives are considered in the Caputo-Prabhakar type. By shifted fractional-order Jacobi polynomials our proposed method can provide highly accurate approximate solutions by reducing the problem under study to a set of algebraic equations which is technically simpler to handle. In order to demonstrate the error estimates, several lemmas are provided. Finally, numerical results are provided to demonstrate the validity of the theoretical analysis.

Keywords. Distributed order, Caputo-Prabhakar fractional derivative, Shifted Jacobi polynomials, Trapezoid, Numerical method.
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1. Introduction

In this paper, we consider a time-fractional partial differential equation with distributed order(TFPDE-DO) as:

$$\int_0^1 c(\mu)\,{}^{CP}\mathbf{D}_t^{\mu} u(x,t)d\mu = u_{xx}(x,t) + z(x,t,u),\ x \in [0,L],\ t \in [0,T], \tag{1.1}$$

and for Eq. (1.1), the initial and boundary conditions is defined as:

$$\begin{aligned} &u(x,0) = f(x),\\ &u(0,t) = g_0(t),\ u(L,t) = g_L(t),\ (x,t) \in (0,L) \times (0,T], \end{aligned} \tag{1.2}$$

where in Eq. (1.1), the ${}^{CP}\mathbf{D}_t^{\mu}$ is a Caputo-Prabhakar fractional derivatives of order $0 < \mu \leq 1$ and $c(\mu)$ is a weight function of fractional order, that $c(\mu) \geq 0$. We know that the Prabhakar operator is obtained by modifying the Riemann–Liouville integral operator by extending its kernel with a three-parameter Mittag–Leffler function, a function which extends the well-known two-parameter Mittag–Leffler function.

Due to the abundant application of the Prabhakar generalized Mittag-Leffler function in fractional calculus, a reason was to select this type of the Caputo-Prabhakar fractional derivative of order μ. Applications of the three-parameter Mittag-Leffler function can be described in mathematical fields such as physics and stochastic processes, electromagnetic, viscosity, various materials, and different media [3, 12, 19, 35, 41]. For the first time in the 1960s Caputo is studied the distributed-order differential equation [10] to expand the stress-strain equation of inelastic media. Later in [6], the multi-term viscoelastic equation of fractional order as a model of the distributed-order equation is developed. The differential equation of distributed-order is considered as an extension of the differential equation of multi-term fractional order.

Recently, due to the comprehensive utilization of the differential equations of distributed-order in modeling diverse fields as physics [6], engineering [28, 47] and mathematical sciences [20, 37], their have attracted much consideration

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of various authors. One of the reasons that the numerical approach is used for solving this type of equations is that there is no precise analytical method to solve them. The different numerical methods to solve this type of differential equations of fractional order have been used. Fei [14] proposed a numerical method based on the Galerkin-Legendre spectral method for solving the two-dimensional time fractional fourth-order partial differential equation of distributed-order. Zaky [48] derived a solution for the distributed-order fractional initial value problems by using the Legendre spectral-collocation method.

Bonyadi [8] studied a solution for the space-time fractional PDEs with variable coefficients by using the spectral shifted Jacobi collocation method in conjunction with the shifted Jacobi operational matrix of fractional derivatives. Zhang [49] derived a solution for the two-dimensional Riesz space distributed-order advection-diffusion equation by using Crank-Nicolson ADI Galerkin-Legendre spectral method. The nonlinear fractional differential equations of distributed-order are solved by using the Legendre-Gauss collocation method by Xu [44]. Dehghan [11] derived a numerical method for solving a fractional damped diffusion-wave equation of distributed-order by using the spectral element method.

Guo [16] derived a solution for the two-dimensional distributed-order time-space fractional reaction-diffusion equation by using the Legendre spectral element method. Morgado [32] derived a solution for the distributed order time-fractional diffusion equation by using the Chebyshev collocation method. Mashayekhi [33] applied the synthetic of block-pulse functions and Bernoulli polynomials, Gorenflo [17] proposed the Fourier and Laplace transforms for solving the one-dimensional distributed order diffusion-wave equation, Li [25] proposed a classical numerical quadrature method.

Aminikhah [1] used a combined method based on the Laplace transform and new homotopy perturbation method to solve a particular class of the distributed order fractional Riccati equation. Mashoo [30] proposed the stability of two classes of distributed-order Hilfer-Prabhakar differential equations. They [31] also proposed the stability of distributed order differential equations form of Hilfer-Prabhakar. Aminikhah [2] proposed two numerical methods to solve the distributed-order fractional Bagley-Torvik equation by the fractional differential transform and Grunwald-Letnikov method, Ye [46] used a compact difference method.

Mashoof [29] proposed an operational matrix for solving the fractional differential equations of distributed order, Yuttanan [45] studied a numerical method based on the upon Legendre wavelets polynomials for solving linear and nonlinear distributed fractional differential equations, the existence and uniqueness for differential equations of distributed order proposed by Ford [13], the uniqueness of solutions for time-fractional diffusion equations of distributed order on bounded domains proposed by Luchko [26], Bhrawy [7] proposed a numerical method based on the Jacobi-Gauss-Lobatto collocation method to solve Schrödinger equations of distributed order and Kharazmi [22] studied a solution for the fractional partial differential equations of distributed order by using pseudo-spectral method.

Besides them, other numerical methods can be mentioned in this field, as the piecewise functions together with the classical Jacobi polynomials and the Gauss–Legendre quadrature rule [21], operational matrix approach based on the Müntz–Legendre polynomials [38], computational algorithms based on Legendre wavelet, Bernstein wavelet, and standard tau approach [24], Laplacian operator in axisymmetric cylindrical geometry [4], fourth-order compact difference scheme [36], operational matrix based on shifted Legendre polynomials [27], fractional-order Fibonacci functions [43], fractional integral operator of fractional-order Bernoulli-Legendre functions and the collocation scheme [40], and homotopy analysis technique and sumudu transform [42].

The main contribution of this paper is to make a numerical scheme to enable approximation, and rigorously perform its convergence and error analyses, which is seldom studied in the current literature. Our main purpose in this work is to study a hybrid approach based on the spectral collocation method and the trapezoid formula to obtain a numerical solution of the time-fractional partial differential equation with distributed order, that this type of equation is introduced in Eq. (1.1). For this purpose, the structure of this paper is ordered as below. In section 2, we describe some mathematical preliminaries and fundamentals which are used later. In section 3, we introduce the spectral collocation method and the trapezoid formula to solve time-fractional partial differential equations of distributed order, also in this section, the convergence and error for the approximate solutions of this type of equations are studied. Some numerical examples are shown in section 4.

2. Mathematical preliminaries and fundamentals

This section recalls some main definitions and lemmas which are used in the next section.

Definition 2.1. [19]. Let $m-1<\Re(\mu)\leq m$ and $u\in L^1[0,b]$, $0<t<b\leq\infty$. Then the left-sided and the right-sided Prabhakar fractional integrals are defined by

$$(\mathbf{E}^{\gamma}_{\rho,\mu,\omega,0^+}u)(t)=\int_0^t(t-\tau)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-\tau)^{\rho})u(\tau)d\tau,$$
$$(\mathbf{E}^{\gamma}_{\rho,\mu,\omega,b^-}u)(t)=\int_t^b(\tau-t)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(\tau-t)^{\rho})u(\tau)d\tau, \tag{2.1}$$

in which $E^{\gamma}_{\rho,\mu}(\omega t^{\rho})$ is the generalized Mittag-Leffler function and displayed by

$$E^{\gamma}_{\rho,\mu}(t)=\sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!\Gamma(\rho n+\mu)}t^n.$$

Definition 2.2. [19]. Let $u\in L^1[0,b]$. Then for $\Re(\mu)\in(m-1,m]$, $m\in\mathbb{N}$. Then, the left-sided and the right-sided Prabhakar fractional derivatives are defined by

$$(D^{\gamma}_{\rho,\mu,\omega,0^+}u)(t)=\frac{d^m}{dt^m}\mathbf{E}^{-\gamma}_{\rho,m-\mu,\omega,a^+}u(t),$$
$$(D^{\gamma}_{\rho,\mu,\omega,b^-}u)(t)=(-1)^m\frac{d^m}{dt^m}\mathbf{E}^{-\gamma}_{\rho,m-\mu,\omega,b^-}u(t). \tag{2.2}$$

Moreover, for the known absolutely continuous function u, the Caputo-Prabhakar fractional derivatives is defined by

$${}^{CP}\mathbf{D}^{\mu}_t u(t)=\mathbf{E}^{-\gamma}_{\rho,m-\mu,\omega,0^+}\frac{d^m}{dt^m}u(t). \tag{2.3}$$

Lemma 2.3. *[23] Let $\rho,\mu,\gamma,\nu,\omega\in\mathbb{C}$ such that $\Re(\rho),\Re(\mu),\Re(\nu)>0$. Then*

$$\int_0^t(t-\tau)^{\mu-1}E^{\gamma}_{\rho,\mu}(\omega(t-\tau)^{\rho})\tau^{\nu-1}d\tau=\Gamma(\nu)t^{\mu+\nu-1}E^{\gamma}_{\rho,\mu+\nu}(\omega t^{\rho}).$$

Definition 2.4. [5] Let $\alpha,\beta>-1$ and $-1\leq x\leq 1$. Then the Jacobi polynomials of degree n are given by:

$$\Theta^{(\alpha,\beta)}_0(x)=1,$$
$$\Theta^{(\alpha,\beta)}_{n+1}(x)=\frac{(\alpha+\beta+2)x+\alpha-\beta}{2}, \tag{2.4}$$
$$\Theta^{(\alpha,\beta)}_{n+1}(x)=(a^{(\alpha,\beta)}_n-b^{(\alpha,\beta)}_n)\Theta^{(\alpha,\beta)}_n(x)-c^{(\alpha,\beta)}_n\Theta^{(\alpha,\beta)}_{n-1}(x),$$

where in the above relation $a^{(\alpha,\beta)}_n, b^{(\alpha,\beta)}_n$ and $c^{(\alpha,\beta)}_n$ are defined by:

$$a^{(\alpha,\beta)}_n=\frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)},$$
$$b^{(\alpha,\beta)}_n=\frac{(\beta^2-\alpha^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}, \tag{2.5}$$
$$c^{(\alpha,\beta)}_n=\frac{(n+\beta)(n+\alpha)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}.$$

Also, for the Jacobi polynomials $\Theta^{(\alpha,\beta)}_n(x)$, the following relations is given by:

$$\Theta^{(\alpha,\beta)}_n(-x)=(-1)^n\Theta^{(\alpha,\beta)}_n(x),\ \Theta^{(\alpha,\beta)}_n(-1)=(-1)^n\frac{\Gamma(n+\beta+1)}{n!\Gamma(\beta+1)},$$
$$\mathrm{D}^l\Theta^{(\alpha,\beta)}_n(x)=\frac{\Gamma(n+\alpha+\beta+l+1)}{2^n\Gamma(n+\alpha+\beta+1)}\Theta^{(\alpha+l,\beta+l)}_{n-l}(x), \tag{2.6}$$

where $\mathrm{D}^l = \frac{d^l}{dx^l}$.

Definition 2.5. [5, 9] Let $\Theta_n^{(\alpha,\beta)}(x)$ be the Jacobi polynomials of degree n. Then the shifted fractional Jacobi polynomial(SFJP) is defined as follows:

$$\begin{aligned}\Theta_{L,n}^{(\xi,\alpha,\beta)}(x) &= \Theta_n^{(\alpha,\beta)}\Big(2(\frac{x}{L})^{\xi} - 1\Big)\\ &= \sum_{k=0}^{n} \Omega_k^{(\alpha,\beta,n)}\Big(\frac{x}{L}\Big)^{\xi k},\ \xi \in (0,1),\ L > 0,\ n = 0,1,2,\dots,\end{aligned} \tag{2.7}$$

where $\Omega_k^{(\alpha,\beta,n)} = (-1)^{n-k}\frac{\Gamma(n+\beta+1)\Gamma(n+k+\alpha+\beta+1)}{\Gamma(k+\beta+1)\Gamma(n+\alpha+\beta+1)(n-k)!k!}$. Also, the orthogonal conditions for this shifted fractional Jacobi polynomial is expressed as follows:

$$\begin{aligned}&\langle\Theta_{L,i}^{(\xi,\alpha,\beta)}(x), \Theta_{L,i'}^{(\xi,\alpha,\beta)}(x)\rangle_{\omega_L^{(\xi,\alpha,\beta)}(x)}\\ &= \int_{\mathcal{L}^2_{\omega_L^{(\xi,\alpha,\beta)}}[0,L]} \Theta_{L,i}^{(\xi,\alpha,\beta)}(x)\Theta_{L,j}^{(\xi,\alpha,\beta)}(x)\omega_L^{(\xi,\alpha,\beta)}(x)dx = f_{L,j}^{(\alpha,\beta)}\delta_{ij},\end{aligned} \tag{2.8}$$

where $f_{L,j}^{(\alpha,\beta)} = \frac{L^{\xi(\alpha+\beta+1)}\Gamma(j+\alpha+1)\Gamma(j+\beta+1)}{(2j+\alpha+\beta+1)\Gamma(j+1)\Gamma(j+\alpha+\beta+1)}$ and $\omega_L^{(\xi,\alpha,\beta)}(x) = \xi(L^{\xi} - x^{\xi})^{\alpha}x^{\xi\beta+\xi-1}$. Suppose $\mathbb{F}_N = \{\Theta_{L,i}^{(\xi,\alpha,\beta)}(x) : i = 0,1,2,\dots,N\}$ be the set of SFJPs space. Then any $y(x) \in \mathcal{L}^2_{\omega_L^{(\xi,\alpha,\beta)}}[0,L]$, according to the orthogonal feature (2.8) can be presented as below:

$$y(x) = \sum_{i=0}^{\infty} a_i\Theta_{L,i}^{(\xi,\alpha,\beta)}(x), \tag{2.9}$$

where the coefficients a_i in the Eq. (2.9) are calculated as:

$$a_i = \Big(f_{L,i}^{(\alpha,\beta)}\Big)^{-1}\langle y(x), \Theta_{L,i}^{(\xi,\alpha,\beta)}(x)\rangle_{\omega_L^{(\xi,\alpha,\beta)}(x)}. \tag{2.10}$$

Lemma 2.6. *The Caputo-Prabhakar fractional derivative of order $0 < \mu \leq 1$ for the SFJP is obtained as:*

$$^{CP}\boldsymbol{D}_t^{\mu}\Theta_{L,n}^{(\xi,\alpha,\beta)}(x) = \sum_{k=0}^{n}\Omega_k^{(\alpha,\beta,n)}\frac{\Gamma(\xi k)}{L^{\xi k}}t^{\xi k-\mu}E^{-\gamma}_{\rho,\xi k-\mu+1}(\omega t^{\rho}). \tag{2.11}$$

Proof. Using Eq. (2.3) and Lemma 2.3, we obtain:

$$\begin{aligned}^{CP}\mathbf{D}_t^{\mu}\Theta_{L,n}^{(\xi,\alpha,\beta)}(x) &= {}^{CP}\mathbf{D}_t^{\mu}\Big(\sum_{k=0}^{n}\Omega_k^{(\alpha,\beta,n)}\Big(\frac{x}{L}\Big)^{\xi k}\Big)\\ &= \sum_{k=0}^{n}\Omega_k^{(\alpha,\beta,n)}\,{}^{CP}\mathbf{D}_t^{\mu}\Big(\frac{x}{L}\Big)^{\xi k}\\ &= \sum_{k=0}^{n}\Omega_k^{(\alpha,\beta,n)}\frac{\Gamma(\xi k)}{L^{\xi k}}t^{\xi k-\mu}E^{-\gamma}_{\rho,\xi k-\mu+1}(\omega t^{\rho}).\end{aligned} \tag{2.12}$$

□

Lemma 2.7. *[5] Let $x_{\rho,q}^{(\alpha,\beta)}$, $0 \leq q \leq \rho$ be the nodes of the standard Jacobi-Gauss(SJG), Jacobi-Gauss-Radau(JGR) and Jacobi Gauss Lobatto(JGL) interpolations in $[-1,1]$ and $\zeta_{\rho,q}^{(\alpha,\beta)}$, $0 \leq q \leq \rho$ be the Christoffel numbers of the SJG, JGR and JGL interpolations in $[-1,1]$. Then the Christoffel numbers of the fractional SJG, fractional JGR and fractional JGL interpolations and their nodes in $[0,L]$ are as:*

$$x_{L,\rho,q}^{(\xi,\alpha,\beta)} = L\Big[\frac{x_{\rho,q}^{(\alpha,\beta)}+1}{2}\Big]^{\frac{1}{\xi}}, \zeta_{L,\rho,q}^{(\xi,\alpha,\beta)} = \Big[\frac{L^{\xi}}{2}\Big]^{\alpha+\beta+1}\zeta_{\rho,q}^{(\alpha,\beta)}. \tag{2.13}$$

3. The SFJP and trapezoid methods for solving TFPDE-DO

In this section, we describe a numerical method for obtaining approximate solution for the TFPDE-DO. Suppose the approximate solutions of the TFPDE-DO can be approximated as:

$$\begin{aligned} u(x,t) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \upsilon_{ij}\Theta_{L,i}^{(\xi,\alpha_1,\beta_1)}(x)\Theta_{L,j}^{(\xi,\alpha_2,\beta_2)}(t) \\ &\simeq u_{n,m}(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\Delta_{ij}(x,t), \end{aligned} \tag{3.1}$$

where

$$\Delta_{ij}(x,t) = \Theta_{L,i}^{(\xi,\alpha_1,\beta_1)}(x)\Theta_{L,j}^{(\xi,\alpha_2,\beta_2)}(t).$$

Now to solve TFPDE-DO by using collocation method and trapezoid method, we need to calculate the first and second partial derivatives, we list the following relations , so we from Eq. (3.1) have:

$$\begin{aligned} \frac{\partial u_{n,m}(x,t)}{\partial x} &= \sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\frac{\partial\Theta_{L,i}^{(\xi,\alpha_1,\beta_1)}(x)}{\partial x}\Theta_{L,j}^{(\xi,\alpha_2,\beta_2)}(t) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\Delta_{ij}^{1}(x,t), \end{aligned} \tag{3.2}$$

$$\begin{aligned} \frac{\partial^2 u_{n,m}(x,t)}{\partial x^2} &= \sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\frac{\partial^2\Theta_{L,i}^{(\xi,\alpha_1,\beta_1)}(x)}{\partial x^2}\Theta_{L,j}^{(\xi,\alpha_2,\beta_2)}(t) \\ &= \sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\Delta_{ij}^{2}(x,t), \end{aligned} \tag{3.3}$$

$$\begin{aligned} \frac{\partial u_{n,m}(x,t)}{\partial t} &= \sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\Theta_{L,i}^{(\xi,\alpha_1,\beta_1)}(x)\frac{\partial\Theta_{L,j}^{(\xi,\alpha_2,\beta_2)}(t)}{\partial t} \\ &= \sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\Delta_{ij}^{3}(x,t), \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} \Delta_{ij}^{1}(x,t) &= \frac{\partial\Theta_{L,i}^{(\xi,\alpha_1,\beta_1)}(x)}{\partial x}\Theta_{L,j}^{(\xi,\alpha_2,\beta_2)}(t), \\ \Delta_{ij}^{2}(x,t) &= \frac{\partial^2\Theta_{L,i}^{(\xi,\alpha_1,\beta_1)}(x)}{\partial x^2}\Theta_{L,j}^{(\xi,\alpha_2,\beta_2)}(t) \\ \Delta_{ij}^{3}(x,t) &= \Theta_{L,i}^{(\xi,\alpha_1,\beta_1)}(x)\frac{\partial\Theta_{L,j}^{(\xi,\alpha_2,\beta_2)}(t)}{\partial t}. \end{aligned}$$

Using lemma 2.6, a analogous way can be used to obtain the Caputo-Prabhakar fractional derivative, so

$$\begin{aligned}
{}^{CP}\mathbf{D}_t^{\mu} u_{n,m}(x,t) &= \sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\Theta_{L,i}^{(\xi,\alpha_1,\beta_1)}(x)\,{}^{CP}\mathbf{D}_t^{\mu}\Theta_{L,j}^{(\xi,\alpha_2,\beta_2)}(t) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{j} \upsilon_{ij}\Theta_{L,i}^{(\xi,\alpha_1,\beta_1)}(x)\Omega_k^{(\alpha_2,\beta_2,j)} \\
&\times \frac{\Gamma(\xi k)}{L^{\xi k}} t^{\xi k-\mu} E_{\rho,\xi k-\mu+1}^{-\gamma}(\omega t^{\rho}) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\Theta_{L,i}^{(\xi,\alpha_1,\beta_1)}(x)\Delta_{ij}^{4}(x,t),
\end{aligned} \tag{3.5}$$

where

$$\Delta_{ij}^{4}(x,t) = \sum_{k=0}^{j} \Omega_k^{(\alpha_2,\beta_2,j)} \frac{\Gamma(\xi k)}{L^{\xi k}} t^{\xi k-\mu} E_{\rho,\xi k-\mu+1}^{-\gamma}(\omega t^{\rho}).$$

To calculate the left-sided integral in Eq. (1.1), we use the trapezoid method. For this aim, we divide the interval $[0,1]$ into P equal subintervals with $h = \frac{1}{P}$, so we have:

$$\begin{aligned}
\int_0^1 c(\mu)\,{}^{CP}\mathbf{D}_t^{\mu} u(x,t)d\mu &\simeq \frac{hc(w_0)\,{}^{CP}\mathbf{D}_t^{w_0} u(x,t)}{2} \\
&+ h\sum_{s=1}^{P-1} c(w_s)\,{}^{CP}\mathbf{D}_t^{w_s} u(x,t) + \frac{hc(w_P)}{2}\,{}^{CP}\mathbf{D}_t^{w_P} u(x,t) \\
&\simeq \frac{hc(0)u(x,t)}{2} \\
&+ h\sum_{s=1}^{P-1} c(w_s)\,{}^{CP}\mathbf{D}_t^{w_s} u(x,t) + \frac{hc(1)}{2}\frac{\partial u(x,t)}{\partial t}.
\end{aligned} \tag{3.6}$$

By substituting Eqs. (3.1), (3.4), and (3.5) into Eq. (3.6), the left-side integral in Eq. (1.1) is approximated as:

$$\begin{aligned}
\int_0^1 c(\mu)\,{}^{CP}\mathbf{D}_t^{\mu} u(x,t)d\mu &\simeq \frac{hc(0)}{2}\sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\Delta_{ij}(x,t) \\
&+ h\sum_{s=1}^{P-1}\sum_{i=0}^{n}\sum_{j=0}^{m} c(w_s)\upsilon_{ij} \underbrace{\Delta_{ij}^{4}(x,t)}_{\mu=w_s} + \frac{hc(1)}{2}\sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\Delta_{ij}^{3}(x,t).
\end{aligned} \tag{3.7}$$

By substituting Eqs. (3.1), (3.3), and (3.7) into Eq. (1.1), we can obtain the approximate solution of Eq. (1.1) as:

$$\begin{aligned}
\frac{hc(0)}{2}\sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\Delta_{ij}(x,t) &+ h\sum_{s=1}^{P-1}\sum_{i=0}^{n}\sum_{j=0}^{m} c(w_s)\upsilon_{ij} \underbrace{\Delta_{ij}^{4}(x,t)}_{\mu=w_s} \\
&+ \frac{hc(1)}{2}\sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\Delta_{ij}^{3}(x,t) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\Delta_{ij}^{2}(x,t) + z(x,t,\sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\Delta_{ij}^{2}(x,t)).
\end{aligned} \tag{3.8}$$

To solve Eq. (3.8), we calculate the residual function at $m+n+2$ of the given nodes $x^{(\xi,\alpha_1,\beta_1)}_{L,n,q_1}$, $t^{(\xi,\alpha_2,\beta_2)}_{L,m,q_2}$, $q_1 = 0,1,2,\dots,n$, $q_2 = 0,1,2,\dots,m$ as below:

$$\begin{aligned}\mathbb{R}(x^{(\xi,\alpha_1,\beta_1)}_{L,n,q_1},\ t^{(\xi,\alpha_2,\beta_2)}_{L,m,q_2}) &= \frac{hc(0)}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\upsilon_{ij}\Delta_{ij}(x^{(\xi,\alpha_1,\beta_1)}_{L,n,q_1},\ t^{(\xi,\alpha_2,\beta_2)}_{L,m,q_2}) + h\sum_{s=1}^{P-1}\sum_{i=0}^{n}\sum_{j=0}^{m}c(w_s)\upsilon_{ij}\underbrace{\Delta^4_{ij}(x^{(\xi,\alpha_1,\beta_1)}_{L,n,q_1},\ t^{(\xi,\alpha_2,\beta_2)}_{L,m,q_2})}_{\mu=w_s}\\ &+\frac{hc(1)}{2}\sum_{i=0}^{n}\sum_{j=0}^{m}\upsilon_{ij}\Delta^3_{ij}(x^{(\xi,\alpha_1,\beta_1)}_{L,n,q_1},\ t^{(\xi,\alpha_2,\beta_2)}_{L,m,q_2}) - \sum_{i=0}^{n}\sum_{j=0}^{m}\upsilon_{ij}\Delta^2_{ij}(x^{(\xi,\alpha_1,\beta_1)}_{L,n,q_1},\ t^{(\xi,\alpha_2,\beta_2)}_{L,m,q_2})\\ &- z(x^{(\xi,\alpha_1,\beta_1)}_{L,n,q_1},\ t^{(\xi,\alpha_2,\beta_2)}_{L,m,q_2},\sum_{i=0}^{n}\sum_{j=0}^{m}\upsilon_{ij}\Delta^2_{ij}(x^{(\xi,\alpha_1,\beta_1)}_{L,n,q_1},\ t^{(\xi,\alpha_2,\beta_2)}_{L,m,q_2})) = 0. \end{aligned}\tag{3.9}$$

By solving Eq. (3.9), the coefficients υ_{ij} are calculated.

4. Error analysis

In this section, we prove the error analysis for the solution of Eq. (1.1) by using the proposed method.

Theorem 4.1. *Let* ${}^{CP}\boldsymbol{D}^{i\mu}_x\big({}^{CP}\boldsymbol{D}^{j\mu'}_t u(x,t)\big) \in C\Big([0,1]\times[0,1]\Big)$ *that* $i = 0,1,2,\dots,n$, $j = 0,1,2,\dots,m$ *and the sequence* $P_{n,m}(x,t) \in \mathbb{F}_N$ *be the best approximation solution of* $u(x,t)$ *that* $u(x,t)$ *is the exact solution of Eq. (1.1). Then the error bound is obtained as:*

$$\| u(x,t) - P_{n,m}(x,t) \|_2 \leq \frac{\Upsilon^{\mu,\mu'}_{n,m}\Gamma(1+\alpha)}{\Gamma(n\mu+1)\Gamma(m\mu'+1)}\sqrt{\frac{\Gamma(3+2(n-1)+\beta)\Gamma(3+2(m-1)+\beta)}{\Gamma(4+2(n-1)+\alpha+\beta)\Gamma(4+2(m-1)+\alpha+\beta)}},\tag{4.1}$$

where $\Upsilon^{\mu,\mu'}_{n,m} = \sup_{(x,t)\in[0,1]\times[0,1]}\Big|{}^{CP}\boldsymbol{D}^{i\mu}_x\big({}^{CP}\boldsymbol{D}^{j\mu'}_t u(x,t)\big)u(x,t)\Big|$.

Proof. We use the Taylor's relation [39] using the Caputo-Prabhakar fractional derivative for multi-variable. So, we have:

$$\Big|u(x,t) - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\frac{x^{i\mu}t^{j\mu'}}{\Gamma(i\mu+1)\Gamma(j\mu'+1)}{}^{CP}\mathbf{D}^{i\mu}_x\big({}^{CP}\mathbf{D}^{j\mu'}_t u(x,t)\Big|_{(0,0)}\big)\Big| \leq \frac{x^{n\mu}t^{m\mu'}}{\Gamma(n\mu+1)\Gamma(m\mu'+1)}\Upsilon^{\mu,\mu'}_{n,m},\tag{4.2}$$

where $\Upsilon^{\mu,\mu'}_{n,m} = \sup_{(x,t)\in[0,1]\times[0,1]}\Big|{}^{CP}\mathbf{D}^{i\mu}_x\big({}^{CP}\mathbf{D}^{j\mu'}_t u(x,t)\big)u(x,t)\Big|$. Since $P_{n,m}(x,t) \in \mathbb{F}_N$ is the best approximation solution of $u(x,t)$, then for any $\Psi_{n,m}(x,t) \in \mathbb{F}_N$ the following relation hold:

$$\| u(x,t) - P_{n,m}(x,t) \|_2 \leq \| u(x,t) - \Psi_{n,m}(x,t) \|_2,\tag{4.3}$$

where in the above relation $\Psi_{n,m}(x,t) = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\frac{x^{i\mu}t^{j\mu'}}{\Gamma(i\mu+1)\Gamma(j\mu'+1)}{}^{CP}\mathbf{D}^{i\mu}_x\big({}^{CP}\mathbf{D}^{j\mu'}_t u(x,t)\Big|_{(0,0)}\big)$ is considered. Using Eqs. (4.2) and (4.3), so

$$\begin{aligned}\| u(x,t) - P_{n,m}(x,t) \|_2^2 &\leq \| u(x,t) - \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\frac{x^{i\mu}t^{j\mu'}}{\Gamma(i\mu+1)\Gamma(j\mu'+1)}{}^{CP}\mathbf{D}^{i\mu}_x\big({}^{CP}\mathbf{D}^{j\mu'}_t u(x,t)\Big|_{(0,0)}\big) \|_2^2\\ &\leq \frac{(\Upsilon^{\mu,\mu'}_{n,m})^2}{(\Gamma(n\mu+1)\Gamma(m\mu'+1))^2}\int_0^1 x^{2n\mu}\omega^{(\mu,\alpha,\beta)}_L(x)dx\int_0^1 t^{2m\mu'}\omega^{(\mu',\alpha,\beta)}_L(t)dt\\ &= \frac{(\Upsilon^{\mu,\mu'}_{n,m})^2(\Gamma(1+\alpha))^2\Gamma(3+2(n-1)+\beta)\Gamma(3+2(m-1)+\beta)}{(\Gamma(n\mu+1)\Gamma(m\mu'+1))^2\Gamma(4+2(n-1)+\alpha+\beta)\Gamma(4+2(m-1)+\alpha+\beta)},\end{aligned}\tag{4.4}$$

where $\int_0^1 x^{2n\mu}\omega^{(\mu,\alpha,\beta)}_L(x)dx$ and $\int_0^1 t^{2m\mu'}\omega^{(\mu',\alpha,\beta)}_L(t)dt$ in [9] calculated. By taking the square roots of Eq. (4.4), the proof is completed. □

Theorem 4.2. *Let $u(x,t)$ is the exact solution of Eq. (1.1), $u_{n,m}(x,t)$ is its approximate solution which is obtained by the suggested method and $P_{n,m}(x,t) \in \mathbb{F}_N$ be the best approximation solution of $u(x,t)$. Then*

$$\| u(x,t) - u_{n,m}(x,t) \|_2 \leq \frac{\Upsilon^{\mu,\mu'}_{n,m}\Gamma(1+\alpha)}{\Gamma(n\mu+1)\Gamma(m\mu'+1)}\sqrt{\frac{\Gamma(3+2(n-1)+\beta)\Gamma(3+2(m-1)+\beta)}{\Gamma(4+2(n-1)+\alpha+\beta)\Gamma(4+2(m-1)+\alpha+\beta)}} + \| V_{n,m} - \bar{V}_{n,m} \|_2 \sqrt{\sum_{i=0}^{n}\sum_{j=0}^{m} f^{(\alpha,\beta)}_{L,i} f^{(\alpha,\beta)}_{L,j}}, \tag{4.5}$$

where $V_{n,m} = [\upsilon_{00}, \upsilon_{01}, \ldots, \upsilon_{0m}, \ldots, \upsilon_{nm}]^T$ and $\bar{V}_{n,m} = [\Lambda_{00}, \Lambda_{01}, \ldots, \Lambda_{0m}, \ldots, \Lambda_{nm}]^T$.

Proof. We consider $u_{n,m}(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\Delta_{ij}(x,t)$ and $P_{n,m}(x,t) = \sum_{i=0}^{n}\sum_{j=0}^{m} \Lambda_{ij}\Delta_{ij}(x,t)$, then we have:

$$\| u(x,t) - u_{n,m}(x,t) \|_2 \leq \| u(x,t) - P_{n,m}(x,t) \|_2 + \| P_{n,m}(x,t) - u_{n,m} \|_2, \tag{4.6}$$

using Eq. (4.1), we obtain:

$$\| u(x,t) - u_{n,m}(x,t) \|_2 \leq \frac{\Upsilon^{\mu,\mu'}_{n,m}\Gamma(1+\alpha)}{\Gamma(n\mu+1)\Gamma(m\mu'+1)}\sqrt{\frac{\Gamma(3+2(n-1)+\beta)\Gamma(3+2(m-1)+\beta)}{\Gamma(4+2(n-1)+\alpha+\beta)\Gamma(4+2(m-1)+\alpha+\beta)}} + \| \sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\Delta_{ij}(x,t) - \sum_{i=0}^{n}\sum_{j=0}^{m} \Lambda_{ij}\Delta_{ij}(x,t) \|_2 . \tag{4.7}$$

We calculate $\| \sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\Delta_{ij}(x,t) - \sum_{i=0}^{n}\sum_{j=0}^{m} \Lambda_{ij}\Delta_{ij}(x,t) \|_2$ in the equation (4.7), so

$$\begin{aligned}
\| \sum_{i=0}^{n}\sum_{j=0}^{m} \upsilon_{ij}\Delta_{ij}(x,t) - \sum_{i=0}^{n}\sum_{j=0}^{m} \Lambda_{ij}\Delta_{ij}(x,t) \|_2^2 &= \int_0^T\int_0^L \Big| \sum_{i=0}^{n}\sum_{j=0}^{m} \Big(\upsilon_{ij} - \Lambda_{ij}\Big)\Delta_{ij}(x,t)\Big| \omega_L^{(\xi,\alpha,\beta)}(x)\omega_L^{(\xi,\alpha,\beta)}(t)dxdt \\
&\leq \int_0^T\int_0^L \Big(\sum_{i=0}^{n}\sum_{j=0}^{m}\Big|\upsilon_{ij} - \Lambda_{ij}\Big|^2\Big)\Big(\sum_{i=0}^{n}\sum_{j=0}^{m}\Big|\Delta_{ij}(x,t)\Big|^2\Big)\omega_L^{(\xi,\alpha,\beta)}(x)\omega_L^{(\xi,\alpha,\beta)}(t)dxdt \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m}\Big|\upsilon_{ij} - \Lambda_{ij}\Big|^2 \sum_{i=0}^{n}\sum_{j=0}^{m}\int_0^T\int_0^L \Big|\Delta_{ij}(x,t)\Big|^2\Big)\omega_L^{(\xi,\alpha,\beta)}(x)\omega_L^{(\xi,\alpha,\beta)}(t)dxdt \\
&= \| V_{n,m} - \bar{V}_{n,m} \|_2^2 \sum_{i=0}^{n}\sum_{j=0}^{m} \| \Delta_{ij}(x,t) \|_2^2 \\
&= \| V_{n,m} - \bar{V}_{n,m} \|_2^2 \sum_{i=0}^{n}\sum_{j=0}^{m} f^{(\alpha,\beta)}_{L,i} f^{(\alpha,\beta)}_{L,j},
\end{aligned} \tag{4.8}$$

where $V_{n,m} = [\upsilon_{00}, \upsilon_{01}, \ldots, \upsilon_{0m}, \ldots, \upsilon_{nm}]^T$ and $\bar{V}_{n,m} = [\Lambda_{00}, \Lambda_{01}, \ldots, \Lambda_{0m}, \ldots, \Lambda_{nm}]^T$. By substituting Eq. (4.8) in the Eq. (4.7), we obtain:

$$\| u(x,t) - u_{n,m}(x,t) \|_2 \leq \frac{\Upsilon^{\mu,\mu'}_{n,m}\Gamma(1+\alpha)}{\Gamma(n\mu+1)\Gamma(m\mu'+1)}\sqrt{\frac{\Gamma(3+2(n-1)+\beta)\Gamma(3+2(m-1)+\beta)}{\Gamma(4+2(n-1)+\alpha+\beta)\Gamma(4+2(m-1)+\alpha+\beta)}} + \| V_{n,m} - \bar{V}_{n,m} \|_2 \sqrt{\sum_{i=0}^{n}\sum_{j=0}^{m} f^{(\alpha,\beta)}_{L,i} f^{(\alpha,\beta)}_{L,j}}. \tag{4.9}$$

Then, the proof is proven. □

5. Numerical results

This section provides two numerical examples that by the suggested method solved. For accuracy and precision of our numerical experiences, we have employed the Matlab 2018b software with a PC of 3 GHz CPU and 6 GB memory. We consider the absolute error as:

$$e_{Error}(x,t) = |u(x,t) - u_{n,m}(x,t)|, \tag{5.1}$$

where $u(x,t)$ is the exact solution of Eq. (1.1) and $u_{n,m}(x,t)$ is its approximate solution.

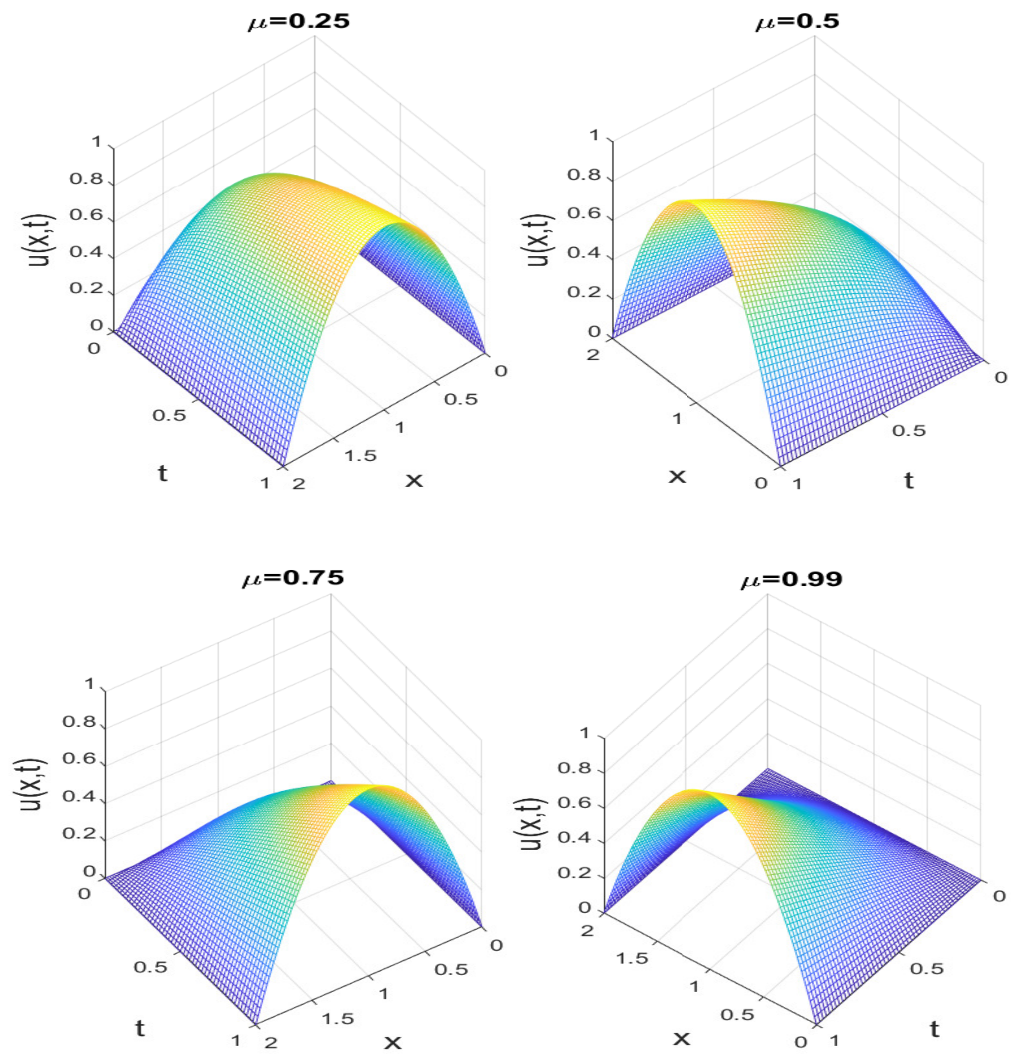

Figure 1. Diagram of the approximate solution for Example 5.1 with different values of $\alpha_1, \beta_1, \alpha_2, \beta_2, \xi$ and μ.

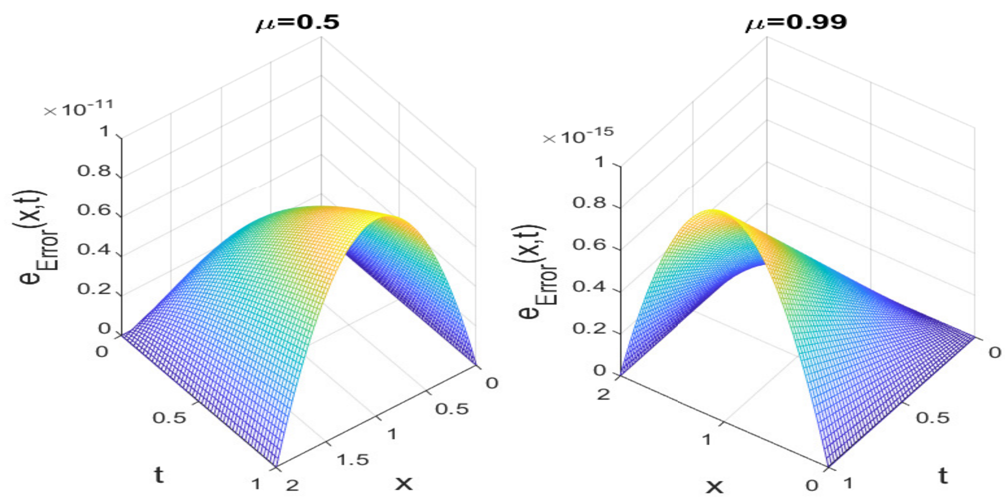

FIGURE 2. Diagram of the absolute error for Example 5.1 with different values of $\alpha_1, \beta_1, \alpha_2, \beta_2, \xi, \mu$ and $n = 4, m = 8$.

TABLE 1. The absolute error between the exact and approximate solutions for different values of μ for Example 5.1.

ξ	(n, m)	*the suggested method for* $\mu = 0.5$ *at* $(\alpha_1 = -0.5, \beta_1 = 0, \alpha_2 = 0.5, \beta_2 = 0)$	*the suggested method for* $\mu = 0.99$ *at* $(\alpha_1 = -0.5, \beta_1 = 0, \alpha_2 = 0.5, \beta_2 = 0)$
$\frac{1}{2}$	$(4, 4)$	$1.164e-11$	$7.840e-12$
	$(6, 6)$	$1.900e-13$	$1.536e-13$
	$(8, 8)$	$2.604e-14$	$1.536e-14$
	$(10, 10)$	$9.651e-15$	$9.423e-15$
	$(12, 12)$	$2.604e-16$	$7.840e-17$
$\frac{1}{3}$	$(4, 4)$	$5.099e-16$	$4.524e-16$
	$(6, 6)$	$9.743e-17$	$9.324e-17$
	$(8, 8)$	$9.900e-18$	$9.804e-18$
	$(10, 10)$	$8.399e-18$	$8.063e-19$
	$(12, 12)$	$4.815e-19$	$6.156e-20$

Example 5.1. We consider the Eq. (1.1) with

$$c(\mu) = \Gamma(3-\mu), \ z(x, t, u(x, t)) = \frac{t^{\mu} E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho})}{\Gamma(3-\mu)} + x(2-x).$$

For this example we consider the initial and boundary conditions with $u(0, t) = u(2, t) = u(x, 0) = 0, \ x \in (0, 2), \ t \in [0, 1]$. The exact solution for this problem given by $u(x, t) = \frac{x(2-x)t^{\mu} E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho})}{\Gamma(3-\mu)}$. This Ex. 5.1, by the proposed method is solved and the numerical results for different values of $\alpha_1, \beta_1, \alpha_2, \beta_2, \xi$ and μ in Fig. 1 are drawn. Plot of the absolute error for different choices of $\alpha_1, \beta_1, \alpha_2, \beta_2, \xi$ and μ in Fig. 2 are shown. Table 1 show that the our proposed method can gain a well approximation of the exact solution.

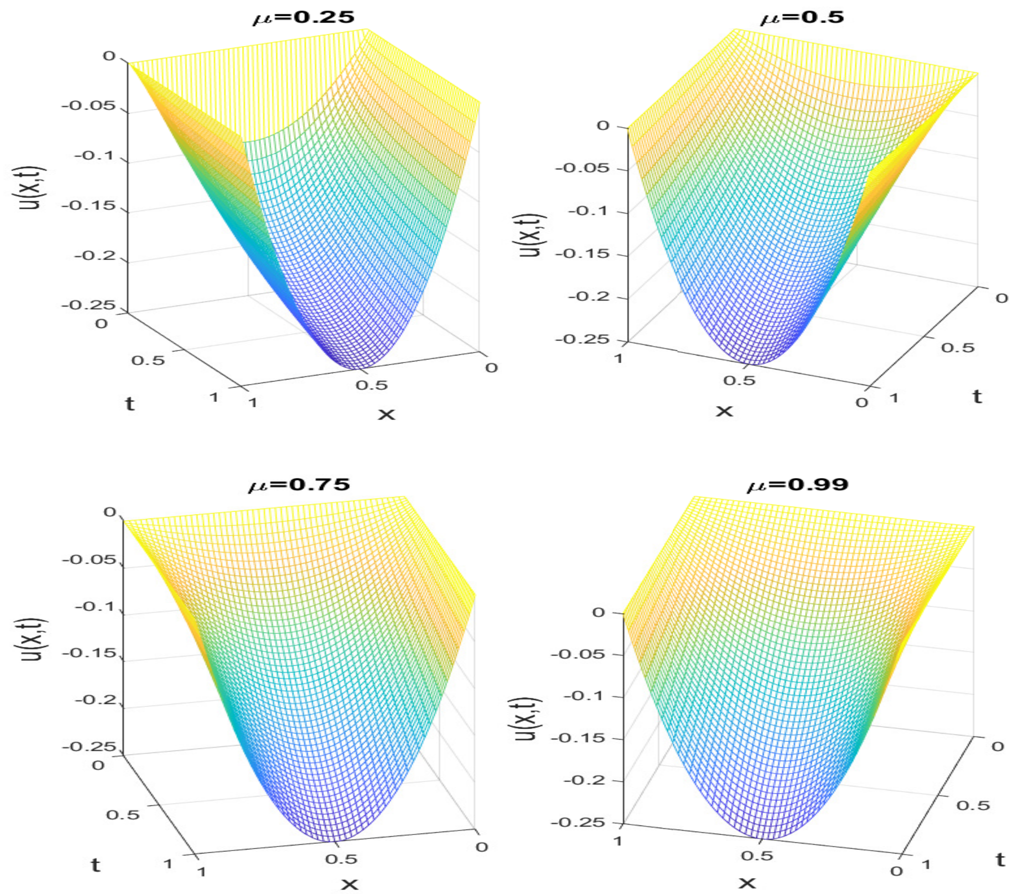

FIGURE 3. Diagram of the approximate solution for Example 5.2 with different values of $\alpha_1, \beta_1, \alpha_2, \beta_2, \xi$ and μ.

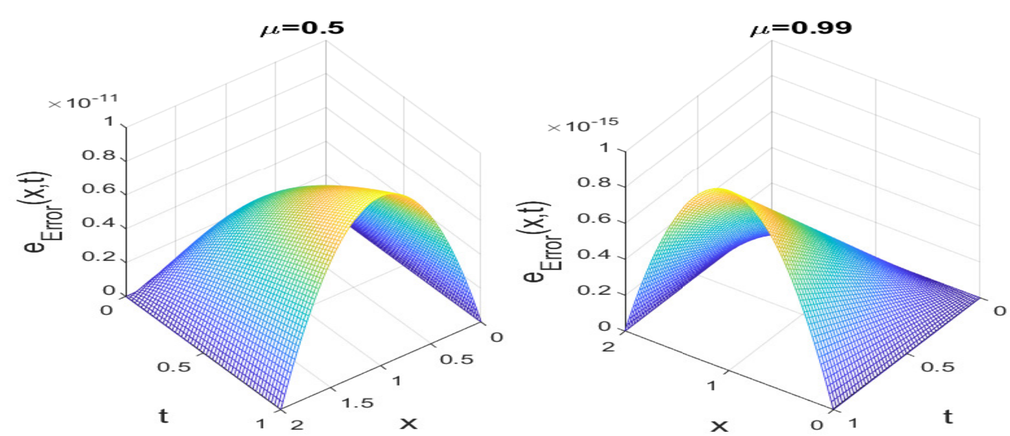

FIGURE 4. Diagram of the absolute error for Ex. 5.2 with different values of $\alpha_1, \beta_1, \alpha_2, \beta_2, \xi, \mu$ and $n = 4, m = 8$.

TABLE 2. The absolute error between the exact and approximate solutions for different values of μ for Example 5.2.

ξ	(n,m)	*the suggested method for* $\mu=0.5$ *at* $(\alpha_1=-0.5,\beta_1=0,\alpha_2=0.5,\beta_2=0)$	*the suggested method for* $\mu=0.99$ *at* $(\alpha_1=-0.5,\beta_1=0,\alpha_2=0.5,\beta_2=0)$
$\frac{1}{2}$	$(4,4)$	$1.204e-12$	$5.640e-14$
	$(6,6)$	$2.463e-13$	$2.356e-15$
	$(8,8)$	$2.484e-14$	$2.244e-16$
	$(10,10)$	$2.496e-15$	$1.716e-17$
	$(12,12)$	$2.435e-16$	$1.344e-18$
$\frac{1}{3}$	$(4,4)$	$3.840e-12$	$7.360e-13$
	$(6,6)$	$1.056e-14$	$1.344e-15$
	$(8,8)$	$2.304e-16$	$2.400e-17$
	$(10,10)$	$2.500e-18$	$2.484e-19$
	$(12,12)$	$2.176e-20$	$1.716e-21$

Example 5.2. Consider the Eq. (1.1) with

$$c(\mu)=1,\ z(x,t,u(x,t))=x(x-1)-2t^{\mu}E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho}).$$

The initial and boundary given by $u(0,t)=u(1,t)=u(x,0)=0,\ x\in(0,1),\ t\in(0,1]$. The exact solution is given by $u(x,t)=x(x-1)t^{\mu}E^{\gamma}_{\rho,\mu+1}(\omega t^{\rho})$. Fig. 3 presents the numerical results of the approximate solution of this example for various values of $\alpha_1,\beta_1,\alpha_2,\beta_2,\xi$ and $\mu\in(0,1]$. Plot of the absolute error for different choices of $\alpha_1,\beta_1,\alpha_2,\beta_2,\xi$ and μ with $\rho=\omega=1$ in Fig. 4 are shown. This example by the introduced method in this paper solved and its numerical results are shown in Table 2. Table 2 presents that we can get a good approximation of the exact solution by using the proposed method in this paper.

6. CONCLUSIONS

In this paper, we provided a numerical technique based on the fractional Jacobi polynomials for solving the distributed order time-fractional partial differential equation. In the developed technique, the time derivative has been approximated by a combined method based on the shifted fractional-order Jacobi and trapezoid method. Error analysis of the numerical technique was demonstrated. We expressed two numerical examples to propound the success of the technique. All numerical calculations were accomplished in sensible accuracy and with proportionately small number of degrees of independence. The numerical results show that this method is more accurate and efficient than other methods.

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