Sliding Mode H_∞ Control of Discrete-time Delayed Singular Markovian Jump Systems

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Abstract
This article addresses the problem of sliding mode control (SMC) for discrete-time delayed singular Markovian jump systems (MJSs) subject to parameter uncertainties and both of matched/mismatched disturbances. The transition probabilities of the Markov chain are considered to have partially unknown entries. Firstly, a sliding surface is proposed by taking the singular matrix into account, and hence the resultant dynamics of the sliding mode system becomes a full order delayed singular MJS. Secondly, the problem of bounded real lemma for the dynamics of the sliding mode system is investigated using the Lyapunov–Krasovskii functional. A delay-dependent solvability condition for the desired sliding surface is derived as linear matrix inequalities (LMIs), which guarantees that the dynamics of the sliding mode system to be robust stochastically admissible and satisfies the H_∞ performance. The slack matrix technique is utilized together with the delay partitioning approach to diminish the conservatism of the obtained condition. Thirdly, a sliding mode controller is synthesized to ensure that the sliding surface is reachable. Finally, two numerical examples are given to show the improved conservatism and the adequacy of the proposed design method.

Keywords
Sliding mode control (SMC), Markovian jump systems (MJSs), Delayed singular systems, Lyapunov-Krasovskii functional, H_∞ control.

1. Introduction

Markovian jump systems (MJSs) have been drastically investigated for their ability of dealing with systems having random sudden changes of the system structure or parameters [1-3]. These systems belong to the category of hybrid stochastic systems that switch at random discrete-time points dominated under a Markov process. MJSs have been demonstrated by a great number of successful applications such as fault-tolerant systems [4], multi-agent systems [5], manufacturing systems [6], and networked control systems [7, 8]. Recently, considerable fundamental works have been made on stability analysis and synthesis of MJSs in the framework of H_2, H_∞, and H_2/H_∞ control problems [9], neural network problems [10], and time-delay problems [11]. From a control perspective, it is remarkable to consider the general form of MJSs with partially unknown transition probabilities.

On the other hand, differential-algebraic systems, famous as singular systems or descriptor systems, provide appropriate and natural representations for describing diverse dynamical systems, including engineering systems, social systems, economic systems, biological systems and so on [12-14]. Recently, much interest has been focused on the problems of stability, stabilization, and filtering for singular systems (see, for example, [15-18]). The authors in [19] explored the H_∞ control problem of networked control systems in the framework of singular MJSs (SMJs). The H_∞ finite-time stabilization of discrete-time singular MJSs was studied in [20].

Sliding mode control (SMC) has fascinated considerable attention from industrial and academic researchers owing to its high robustness against external disturbances and insensitivity to modeling errors and system parameter variations, fast global convergence, and simplicity of implementation [21,22]. Up to date, SMC has been extensively utilized in many complex systems, including uncertain systems [23], MJSs [24], time-delay systems [25], and singular systems [26]. More recently, the discrete-time SMC has attracted considerable attention owing to the broad application of computers in control systems. However, it cannot be synthesized easily by counterpart equivalent of the continuous systems. There have been some studies related to the discrete-time SMC, including systems with input/state delay, singular systems, asynchronous SMC, and memory-based SMC. The authors in [27] proposed a finite-time SMC scheme for the time-delay MJSs subject to both matched/mismatched disturbances. However, the model transformation introduced in [28] is not applicable to obtain the regular form of the singular MJSs. Therefore, an appropriate sliding surface have to be proposed in a way that the resulting dynamics of the sliding mode control system exists under a simply testable stability criterion. In addition, most of the existing results about the SMC problem of discrete-time singular systems are based on the assumption that the controlled-plant is only subject to matched disturbances. Consequently, the SMC problem of singular MJSs in the discrete-time domain subject to mismatched disturbances is still a challenging issue.

Based on the above discussions, this paper addresses the sliding mode controller for uncertain time-delay discrete-time singular MJSs in the presence of matched/mismatched disturbances. The design of proposed SMC system consists of two steps. Firstly, when the system trajectories reach on the sliding-mode surface, the resulting dynamics of the sliding mode system will be stochastic admissible while fulfilling a prescribed H_∞ disturbance attenuation level. Secondly, an SMC law is synthesized in a way that the system trajectories are driven to the sliding manifold and remain on it thereafter. The contributions and significance of the paper mainly lie in the following three aspects:

- A sliding-mode surface is proposed by involving the singular matrix, and therefore the resultant dynamics of the sliding mode system will be a full order (not a reduced order) time-delay singular MJS.
- By employing a Lyapunov–Krasovskii functional, delay-dependent solvability criteria for the appropriate sliding-mode surface are obtained using strict linear matrix inequality (LMI) based optimization, which guarantee the dynamics of the sliding mode system to be stochastic admissible with a prescribed H_∞ performance level. In addition to using delay partitioning method, some

intermediate free variables are introduced such that the conservativeness of the given admissibility criteria can be further reduced. This will be verified thorough numerical examples.

- An appropriate sliding mode controller is synthesized to enforce the system to achieve the sliding surface in the presence of matched/mismatched disturbances, while leading to a decreasing zigzag motion on this surface.

The remaining sections of this paper are organized as follows. Section 2 provides a description of the system dynamics and presents the problem formulation. In Section 3, a sliding surface is first proposed and an LMI-based solvability condition is given for the desired sliding surface such that the sliding mode dynamics meets the H_∞ performance requirements. Secondly, a sliding mode controller is designed to globally drive the state trajectories onto the pre-designed surface and maintain them therein for subsequent periods. Two numerical examples are provided in Section 4 to certify the feasibility of the method. Finally, Section 5 concludes this paper.

Notation: Throughout this paper, $Q<0$ indicates that Q is a symmetric negative definite matrix. Vertical bars $\|\circ\|$ denote the Euclidean norm, and $\mathrm{L}_2[0,\infty)$ consists of all vector-valued square-summable functions over $[0,\infty)$. $\lambda_{\max}(Q)$ denotes the maximum eigenvalue of symmetric matrix Q, and $Her(Q)$ reprensents $Q+Q^T$. In symmetric block matrices, asterisk $*$ is utilized for the representation of a term that is induced by symmetry.

2. Problem Formulation

Let us consider the uncertain discrete-time SMJS with time-delay and external disturbances described by

$$\begin{cases} Ex(k+1)=(A(r_k)+\Delta A(r_k))x(k) \\ \quad +(A_d(r_k)+\Delta A_d(r_k))x(k-d) \\ \quad +B(r_k)\big(u(k)+f(x(k),k)\big)+B_\omega(r_k)\omega(k) \\ z(k)=C(r_k)x(k) \\ x(k)=\Psi(k),\ \ k=-d,-d+1,\ldots,0 \end{cases} \tag{1}$$

where $x(k)\in R^n$, $u(k)\in R^{m_u}$, $f(x(k),k)\in R^{m_u}$, $\omega(k)\in R^{p_\omega}$, $z(k)\in R^{s_z}$ denote the system state, control input, matched disturbance satisfying $\|f(x(k),k)\|\le\lambda\|x(k)\|$, mismatched disturbance belonging to $\mathrm{L}_2[0,\infty)$ which satisfies $\|\omega(k)\|\le\varpi$, and controlled-output, respectively. λ and ϖ are considered to be known positive scalars. d is the constant time-delay, and $\Psi(k)$ is the vector-valued initial condition of the system. $\{r_k,k\ge 0\}$ represents the discrete-time Markov chain with state space $\mathrm{I}=\{1,2,\ldots,N\}$ and state transition matrix $\Lambda=[\lambda_{ij}]$, where each element is given by

$$\lambda_{ij}=\Pr\{r_{k+1}=j\mid r_k=i\}\quad \forall i,j\in\mathrm{I}, \tag{2}$$

with $\lambda_{ij}\ge 0\ \forall i,j\in\mathrm{I}$, and $\sum_{j=1}^{N}\lambda_{ij}=1,\ \forall i\in\mathrm{I}$. The matrix E is considered to be singular, i.e. $rank(E)=r<n$. In the sequel, the system matrices of the $i\,th$ mode are denoted by A_i, A_{di}, B_i, $B_{\omega i}$, and C_i. The parameter uncertainties are supposed to have the following structure:

$$\begin{bmatrix}\Delta A_i & \Delta A_{di}\end{bmatrix}=E_i\Delta(i,k)\begin{bmatrix}F_i & F_{di}\end{bmatrix}$$

Moreover, the transition probabilities of the jumping process are considered to be partly accessed, that is, the transition probability matrix has the following form

$$\begin{bmatrix}\lambda_{11} & \hat{\lambda}_{12} & \cdots & \lambda_{1N}\\ \hat{\lambda}_{21} & \hat{\lambda}_{22} & \cdots & \lambda_{2N}\\ \vdots & \vdots & \ddots & \vdots\\ \hat{\lambda}_{N1} & \hat{\lambda}_{N2} & \cdots & \hat{\lambda}_{NN}\end{bmatrix}$$

where $\hat{\lambda}_{ij}$ denotes the unknown element. For notational clarity, $\forall i\in\mathrm{I}$, $\mathcal{I}_{\mathcal{K}}^{(i)}=\{j,\lambda_{ij}\text{ is known}\}$ and $\mathcal{I}_{\mathcal{UK}}^{(i)}=\{j,\lambda_{ij}\text{ is unkown}\}$.

Assumption 1: The matrix B_i, $\forall i\in\mathrm{I}$ is assumed to be of full column rank.

The following definition is essential for this paper.

Definition 1 ([19]): Let γ be a given positive scalar. The open-loop form of SMJS (1) without parameter uncertainties is said to be

- Regular, if $\det(zE-A_i)\ne 0$ for each $i\in\mathrm{I}$,
- Causal, if $\deg\big(\det(zE-A_i)\big)=rank(E)$ for each $i\in\mathrm{I}$,
- Stochastically stable, if for $\omega(k)=0$, any $x_0=\Psi(k)$ and $r_0\in\mathrm{I}$, the following condition holds

$$\lim_{N\to\infty}\Xi\left[\sum_{k=0}^{N}\|x(k)\|^2\mid r_0,x_0\right]<\infty.$$

- Stochastically admissible, if the above three conditions are met.
- The system meets the H_∞ performance requirements (stocahstically admissible with an H_∞ performance level) if 1) the system is stocahstically admissible for $\omega(k)=0$, and 2) the following condition holds for $\omega(k)\in\mathrm{L}_2[0,\infty)$:

$$\Xi\left[\sum_{k=0}^{\infty}\|z(k)\|^2\right]<\gamma^2\|\omega\|_2^2.$$

This paper aims to synthesize a sliding mode controller for the SMJS (1) with partially unknown transition probabilities such that 1) the system trajectories are driven robustly to a region near the sliding surface and maintain their presence there afterward, and 2) the resulting sliding mode dynamics meets the H_∞ performance requirements. In doing so, the analysis of the sliding mode dynamics is carried out by constructing a Markovian Lyapunov–Krasovskii functional, and the solvability criterion for the desired switching surface function is obtained in terms of LMIs.

3. Main Results

This section describes the SMC design problem for the system (1) subject to matched/mismatched disturbances. At first, solvability conditions for the existence of a quasi-sliding surface are proposed in terms of LMIs to make sure that the sliding mode dynamics (when the system is constrained by the SMC law to stay onto a predefined surface) meets the H_∞ performance requirements. Then, a robust SMC law is synthesized to enforce the system to enter into the pre-designed surface and maintains their presence there afterward. The following preliminary lemmas play an important role in our later development.

Lemma 1 ([29]): Consider the matrices E, F, G, and $\Delta(k)$, and suppose $\mathrm{E}=\mathrm{E}^T$. Then,

$$\mathrm{E}+Her\big(\mathrm{F}\Delta(k)\mathrm{G}\big)<0,\quad \forall\|\Delta(k)\|\le 1,$$

if and only if

$$\mathrm{E}+\delta^{-1}\mathrm{F}\mathrm{F}^T+\delta\mathrm{G}^T\mathrm{G}<0\qquad for\ \ some\ \delta>0.$$

Lemma 2 ([30]): Let $p<q$, $R=R^T>0$, then

$$-(q-p+1)\sum_{j=p}^{q} y^T(j)Ry(j) \le -\left(\sum_{j=p}^{q} y(j)\right)^T R\left(\sum_{j=p}^{q} y(j)\right)$$

where $y:\{p, p+1,\ldots,q\} \to R^n$ is a vector function.

3.1. *Sliding surface design*

We select the quasi-sliding surface as follows

$$s(k) = G_i Ex(k) - G_i\left(A_i + B_i K_i\right)x(k-1) \tag{3}$$

where $G_i = \left(B_i^T B_i\right)^{-1} B_i^T$, and K_i is the tuning matrix to be designed for each $i \in \mathrm{I}$ in such a way that the sliding mode dynamics becomes stochastically admissible with a given disturbance attenuation level.

When the system dynamic motion is restricted to the ideal quasi-sliding surface, it follows that $s(k) = s(k+1) = \cdots = 0$. Let $s(k+1) = 0$, then the following equivalent control law is obtained:

$$\begin{aligned} u_{eq} = -G_i\big[&\Delta A_i - B_i K_i + x(k) + \left(A_{di} + \Delta A_{di}\right)x(k-d) \\ &B_{\omega i}\omega(k)\big] - f(x(k),k) \end{aligned} \tag{4}$$

Substituting (4) into (1), the sliding mode dynamics is formulated as

$$\begin{cases} Ex(k+1) = \left(\bar{A}_i + \Delta\bar{A}_i\right)x(k) + \left(\bar{A}_{di} + \Delta\bar{A}_{di}\right)x(k-d) \\ \quad + \bar{B}_{\omega i}\omega(k) \\ z(k) = C_i x(k) \\ x(k) = \Psi(k), \quad k = -d, -d+1, \ldots, 0 \end{cases} \tag{5}$$

where

$$\bar{A}_i = A_i + B_i K_i, \quad \bar{A}_{di} = \tilde{J}_i A_{di}, \quad \bar{B}_{\omega i} = \tilde{J}_i B_{\omega i}$$

$$\begin{bmatrix} \Delta\bar{A}_i & \Delta\bar{A}_{di} \end{bmatrix} = \tilde{J}_i E_i \begin{bmatrix} F_i & F_{di} \end{bmatrix}, \quad \tilde{J}_i = I - B_i G_i$$

Therefore, the problem of determining the tuning parameter K_i in (3) is transformed to the design of H_∞ state feedback controller for time-delay SMJSs.

3.2. *Sliding mode dynamics analysis*

In this subsection, we will analyze the stochastic admissibility with an H_∞ performance of the sliding mode dynamics (5), and a sufficient criterion to obtain the tuning parameter K_i will be derived in terms of LMIs.

First, we provide the following admissibility result for the SMJS in (5) with partially unknown transition probabilities in the case of $K_i = 0$ and $\omega(k) = 0$.

Theorem 1: Consider the nominal form of SMJS (5) with partially unknown transition probabilities, and suppose d and p are given positive integers. In the case of $K_i = 0$ and $\omega(k) = 0$, the corresponding system is stochastically admissible if there exist matrices $P_i > 0$, $Q > 0$, $Z > 0$, N_{2i}, N_{3i}, N_{4i}, M_{1i}, M_{2i}, and M_{3i} such that

$$\begin{aligned} \Omega = {} & W_p^T\left(\bar{P}_k^{(i)} + d^2 Z\right)W_p + W_{p3}^T\left(\bar{P}_k^{(i)} - P_i\right)W_{p3} \\ & + W_Q^T\bar{Q}W_Q - W_z^T E^T ZEW_z \\ & + Her\left(W_{p1}^T E^T \bar{P}_k^{(i)} W_p + \hat{N}_i^T W_{p2} + \hat{M}_i R^T W_p\right) \\ & < 0, \qquad\qquad \forall j \in \mathrm{I}_{\mathrm{UK}}^{(i)} \end{aligned} \tag{6}$$

where $R \in R^{n\times(n-r)}$ is chosen as a full column rank matrix satisfying $R^T E = 0$, and

$$W_p = \begin{bmatrix} 0_{n,(p+1)n} & I_n \end{bmatrix}, W_{p1} = \begin{bmatrix} I_n & 0_{n,(p+1)n} \end{bmatrix}$$

$$W_{p2} = \begin{bmatrix} A_i - E & 0_{n,(p-1)n} & \bar{A}_{di} & -I_n \end{bmatrix},$$

$$W_{p3} = \begin{bmatrix} E & 0_{n,(p+1)n} \end{bmatrix}, W_z = \begin{bmatrix} I_n & -I_n & 0_{n,pn} \end{bmatrix},$$

$$W_Q = \begin{bmatrix} I_{pn} & 0_{pn,n} & 0_{pn,n} \\ 0_{pn,n} & I_{pn} & 0_{pn,n} \end{bmatrix},$$

$$\hat{M}_i = \begin{bmatrix} M_{1i}^T & M_{2i}^T & 0_{n-r,(p+1)n} & M_{3i}^T \end{bmatrix}^T,$$

$$\hat{N}_i = \begin{bmatrix} N_{2i} & N_{4i} & 0_{n,(p-1)n} & N_{3i} \end{bmatrix},$$

$$\bar{Q} = \begin{bmatrix} Q & 0_{pn,pn} \\ * & -Q \end{bmatrix}, Q = \begin{bmatrix} Q_{11} & \cdots & Q_{1p} \\ * & \ddots & \vdots \\ * & * & Q_{pp} \end{bmatrix},$$

$$P_k^{(i)} = \sum\nolimits_{j \in \mathrm{I}_{\mathrm{K}}^{(i)}} \lambda_{ij} P_j, \bar{P}_k^{(i)} = P_k^{(i)} + \left(1 - \lambda_{\mathrm{K}}^{(i)}\right)P_j.$$

Proof: First, it should be noted that

$$0 \le \hat{\lambda}_{ij} \Big/ 1 - \lambda_{\mathrm{K}}^{(i)} \le 1 \quad \forall j \in \mathrm{I}_{\mathrm{UK}}^{(i)}$$

$$\sum\nolimits_{j \in \mathrm{I}_{\mathrm{UK}}^{(i)}} \hat{\lambda}_{ij} \Big/ 1 - \lambda_{\mathrm{K}}^{(i)} = 1$$

Analogous to the proof of Theorem 3 in [31], it can be concluded that $\Omega < 0$ is equivalent to

$$\begin{aligned} & W_p^T\left(P_k^{(i)} + d^2 Z\right)W_p + W_{p3}^T\left(P_k^{(i)} - P_i\right)W_{p3} \\ & + W_Q^T\bar{Q}W_Q - W_z^T E^T ZEW_z \\ & + Her\left(W_{p1}^T E^T \bar{P}_k^{(i)} W_p + \hat{N}_i^T W_{p2} + \hat{M}_i R^T W_p\right) \\ & + \sum\nolimits_{j \in \mathrm{I}_{\mathrm{UK}}^{(i)}}\left(\hat{\lambda}_{ij}\left(W_p^T P_j W_p + W_{p3}^T P_j W_{p3}\right) + Her\left(W_{p1}^T E^T P_j W_p\right)\right) \\ & < 0. \end{aligned}$$

Since $\bar{P}_k^{(i)} = P_k^{(i)} + \sum\nolimits_{j \in \mathrm{I}_{\mathrm{UK}}^{(i)}} \hat{\lambda}_{ij} P_j$, the above inequality can be rewritten as

$$\begin{aligned} \Omega' = {} & \Big(W_p^T\left(\bar{P}_i + d^2 Z\right)W_p + W_{p3}^T\left(\bar{P}_i - P_i\right)W_{p3} \\ & + W_Q^T\bar{Q}W_Q - W_z^T E^T ZEW_z \\ & + Her\left(W_{p1}^T E^T \bar{P}_k^{(i)} W_p + \hat{N}_i^T W_{p2} + \hat{M}_i R^T W_p\right)\Big) < 0. \end{aligned} \tag{7}$$

First, we prove that the corresponding system is regular and causal. Let

$$\tilde{E} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}_{2n\times 2n}, \tilde{A}_i = \begin{bmatrix} E & I \\ A_i - E & -I \end{bmatrix}_{2n\times 2n},$$

$$\tilde{\bar{P}}_i = \begin{bmatrix} \bar{P}_i & 0 \\ 0 & 0 \end{bmatrix}_{2n\times 2n}, \tilde{P}_i = \begin{bmatrix} P_i & 0 \\ 0 & 0 \end{bmatrix}_{2n\times 2n},$$

$$\tilde{Q} = \begin{bmatrix} Q_{11} & 0 \\ 0 & \bar{P}_i + d^2 Z \end{bmatrix}_{2n\times 2n}, \tilde{R} = \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix}_{2n\times(2n-r)},$$

$$\tilde{H}_i = \begin{bmatrix} M_{1i} & N_{2i}^T \\ M_{3i} & N_{3i}^T \end{bmatrix}_{2n\times(2n-r)}, \tilde{Z} = \begin{bmatrix} Z & 0 \\ 0 & 0 \end{bmatrix}_{2n\times 2n}.$$

Since $\mathrm{rank}\left(\tilde{E}\right) = r \le n$, it always possible to find two nonsingular matrices U and V of appropriate dimensions such that

$$U\tilde{E}V = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Let

$$U\tilde{A}_iV=\left[\begin{array}{c|c}A_{11i} & A_{12i}\\ \hline A_{21i} & A_{22i}\end{array}\right]$$

$$U^{-T}\tilde{R}=\begin{bmatrix}0\\ \hline I\end{bmatrix}S$$

$$V^T\tilde{H}_i=\begin{bmatrix}H_{11i_{r\times(2n-r)}}\\ \hline H_{21i_{(2n-r)\times(2n-r)}}\end{bmatrix}$$

in which $S\in R^{(2n-r)\times(2n-r)}$ is an invertible matrix. Left- and right-multiplying of (7) by

$$T=\begin{bmatrix}I_n & 0 & 0_{n,(p-1)n} & 0\\ 0 & 0 & 0_{n,(p-1)n} & I_n\\ 0_{(p-1)n,n} & 0_{(p-1)n,n} & I_{(p-1)n} & 0_{(p-1)n,n}\\ 0 & I_n & 0 & 0\end{bmatrix}$$

and its transpose, leads to

$$T\Omega'T^T=\begin{bmatrix}\hat{\Omega}_{11i} & \hat{\Omega}_{12i} & \bullet & \bullet\\ \hat{\Omega}_{12i}^T & \hat{\Omega}_{22i} & \bullet & \bullet\\ \bullet & \bullet & \bullet & \bullet\\ \bullet & \bullet & \bullet & \bullet\end{bmatrix}<0, \quad (8)$$

where

$$\hat{\Omega}_{11i}=E^T(\bar{P}_i-P_i)E+Q_{11}-E^TZE+Her(N_{2i}^T(\bar{A}_i-E))$$

$$\hat{\Omega}_{12i}=E^T\bar{P}_i-N_{2i}^T+(\bar{A}_i-E)^TN_{3i}+M_{1i}R^T$$

$$\hat{\Omega}_{22i}=\bar{P}_i+d^2Z-Her(N_{3i}^T)+Her(M_{3i}R^T)$$

It can be concluded from (8) that

$$\tilde{A}_i^T\tilde{\bar{P}}_i\tilde{A}_i-\tilde{E}^T\tilde{P}_i\tilde{E}+\tilde{H}_i\tilde{R}^T\tilde{A}_i+\tilde{A}_i^T\tilde{R}\tilde{H}_i^T-\tilde{E}^T\tilde{Z}\tilde{E}+\tilde{Q}_i<0,$$

which implies that

$$-\tilde{E}^T\tilde{P}_i\tilde{A}_i+\tilde{H}_i\tilde{R}^T\tilde{A}_i+\tilde{A}_i^T\tilde{R}\tilde{H}_i^T-\tilde{E}^T\tilde{Z}\tilde{E}<0, \quad (9)$$

Left- and right-multiplying of (9) by V^T and its transpose, leads to

$$\begin{bmatrix}\bullet & \bullet\\ \bullet & Her\left(H_{21i}S^TA_{22i}\right)\end{bmatrix}<0.$$

The (2,2) entry of the above inequality reads $Her\left(H_{21i}S^TA_{22i}\right)<0$, and therefore the matrix A_{22i} is nonsingular. Thus,

$$\det\left(z\tilde{E}-\tilde{A}_i\right)=\det\left(U^{-1}\right)\det\left(zI_r-A_{11i}+A_{12i}A_{22i}^{-1}A_{21i}\right)$$
$$\det\left(-A_{22i}\right)\det\left(V^{-1}\right)$$

can not identical to zero and $\deg\left(\det\left(z\tilde{E}-\tilde{A}_i\right)\right)=r$. Therefore, according to Definition 1, the system with the pair $\left(\tilde{E},\tilde{A}_i\right)$ meets the regularity and causality conditions. Note that

a) $\det\left(z\tilde{E}-\tilde{A}_i\right)=\det\left(zE-A_i\right)$

b) $\deg\left(\det\left(z\tilde{E}-\tilde{A}_i\right)\right)=\deg\left(\det\left(zE-A_i\right)\right)$,

and therefore for the case of $K_i=0$ and $\omega(k)=0$, the nominal form of system (5) is regular and causal.

Next, we prove the stability. Choose the following Lyapunov–Krasovskii functional:

$$V(x(k),k)=\sum_{i=1}^{3}V_i(x(k),k) \quad (10)$$

where

$$V_1(x(k),k)=x^T(k)E^TP_iEx(k),$$

$$V_2(x(k),k)=\sum_{l=k-d}^{k-1}\Upsilon^T(l)Q\Upsilon(l),$$

$$V_3(x(k),k)=d\sum_{l=-d}^{-1}\sum_{v=k+l}^{k-1}y^T(v)Zy(v),$$

in which

$$\Upsilon(q)=\begin{bmatrix}x(q)\\ x(q-d)\\ x(q-2d)\\ \vdots\\ x(q-dp+d)\end{bmatrix},\ y(v)=E\left(x\left(v+1\right)-x\left(v\right)\right).$$

Defining $\Delta V\left(x\left(k\right),k\right)=V\left(x(k+1),k+1\right)-V(x(k),k)$, we have

$$\begin{aligned}&\Delta V_1(k)=x^T(k+1)E^T\bar{P}_iEx(k+1)-x^T(k)E^TP_iEx(k)\\ &=y^T(k)\bar{P}_iy(k)+2x^T(k)E^T\bar{P}_iy(k)\\ &+x^T(k)E^T\bar{P}_iEx(k)-x^T(k)E^TP_iEx(k)\\ &+2\left[x^T(k)N_{2i}^T+y^T(k)N_{3i}^T+x^T(k-d)N_{4i}^T\right]\\ &\times\left[(A_i-E)x(k)-y(k)+\bar{A}_{di}x(k-d)\right]\\ &+2\left[x^T(k)M_{1i}R^T+x^T(k-d)M_{2i}R^T+y^T(k)M_{3i}R^T\right]\times y(k)\end{aligned} \quad (11)$$

$$\begin{aligned}&\Delta V_2(x(k),k)=\sum_{l=k-d+1}^{k}\Upsilon^T(l)Q\Upsilon(l)-\sum_{l=k-d}^{k-1}\Upsilon^T(l)Q\Upsilon(l)\\ &=\Upsilon^T(k)Q\Upsilon(k)-\Upsilon^T(k-d)Q\Upsilon(k-d)\end{aligned} \quad (12)$$

$$\begin{aligned}&\Delta V_3(x(k),k)=d\left(\sum_{l=-d}^{-1}\sum_{v=k+l+1}^{k}y^T(v)Zy(v)\right.\\ &\left.-\sum_{l=-d}^{-1}\sum_{v=k+l}^{k-1}y^T(v)Zy(v)\right)\\ &=d\left(\sum_{l=-d}^{-1}\left(y^T(k)Zy(k)-y^T(k+l)Zy(k+l)\right)\right)\\ &=d^2y^T(k)Zy(k)-d\sum_{l=k-d}^{k-1}y^T(l)Zy(l)\end{aligned} \quad (13)$$

By Lemma 2, we have

$$\begin{aligned}&\Delta V_3(k)\le d^2y^T(k)Zy(k)\\ &\quad-\left[x(k)-x(k-d)\right]^TE^TZE\left[x(k)-x(k-d)\right]\end{aligned} \quad (14)$$

Defining $\eta(k)=\left[\Upsilon^T(k)\quad x^T(k-pd)\quad y^T(k)\right]^T$, it can be concluded from (11)-(14) that

$$\begin{aligned}&\Delta V(k)\le\eta^T(k)W_p^T\left(\bar{P}_i+d^2Z\right)W_p\eta(k)\\ &+\eta^T(k)W_{p3}^T\left(\bar{P}_i-P_i\right)W_{p3}\eta(k)\\ &+\eta^T(k)W_Q^T\bar{Q}W_Q\eta(k)-\eta^T(k)W_z^TE^TZEW_z\eta(k)\\ &+\eta^T(k)\mathrm{Her}\left(W_{p1}^TE^T\bar{P}_iW_p+\hat{N}_i^TW_{p2}+\hat{M}_iR^TW_p\right)\eta(k)\end{aligned}$$

Thus (7) implies that $\Delta V(k)<0$, and therefore the corresponding system is stochastically stable. This completes the proof. ■

The bounded real problem for the nominal form of system (5) with $K_i=0$ is given in the following theorem.

Theorem 2: Given positive scalars d, p and γ, the nominal form of SMJS (5) with $K_i=0$ meets the H_∞ performance requirements defined in Definition 1, if there exist matrices $P_i>0$, $Q>0$, $Z>0$, N_{2i}, N_{3i}, N_{4i}, M_{1i}, M_{2i}, and M_{3i} such that

$$\bar{\Omega}=\bar{W}_p^T\left(\bar{P}_k^{(i)}+d^2Z\right)\bar{W}_p+\bar{W}_{p3}^T\left(\bar{P}_k^{(i)}-P_i\right)\bar{W}_{p3}$$

$$+\bar{W}_Q^T\bar{Q}\bar{W}_Q+\bar{W}_C^T\bar{W}_C-\gamma^2\bar{W}_W^T\bar{W}_W-\bar{W}_z^TE^TZE\bar{W}_z$$
$$+Her\left(\bar{W}_{p1}^TE^T\bar{P}_k^{(i)}\bar{W}_p+\bar{N}_i^TW_{p2}+\bar{M}_iR^TW_p\right)<0,$$
$$\forall j\in\mathcal{J}_{\mathcal{UK}}^{(i)} \tag{15}$$

where

$$W_p=\begin{bmatrix}0_{n,(p+1)n} & I_n & 0_{n,p_\omega}\end{bmatrix},$$
$$\bar{W}_{p1}=\begin{bmatrix}I_n & 0_{n,(p+1)n} & 0_{n,p_\omega}\end{bmatrix},$$
$$\bar{W}_{p2}=\begin{bmatrix}A_i-E & 0_{n,(p-1)n} & \bar{A}_{di} & -I_n & \bar{B}_{\omega i}\end{bmatrix},$$
$$\bar{W}_{p3}=\begin{bmatrix}E & 0_{n,(p+1)n+p_\omega}\end{bmatrix},\bar{Q}=\begin{bmatrix}Q & 0_{pn,pn}\\ * & -Q\end{bmatrix},$$
$$\bar{W}_Q=\begin{bmatrix}I_{pn} & 0_{pn,n} & 0_{pn,n+p_\omega}\\ 0_{pn,n} & I_{pn} & 0_{pn,n+p_\omega}\end{bmatrix},$$
$$\bar{M}_i=\begin{bmatrix}M_{1i}^T & M_{2i}^T & 0_{n-r,(p-1)n} & M_{3i}^T & 0_{n-r,p_\omega}\end{bmatrix}^T,$$
$$\bar{N}_i=\begin{bmatrix}N_{2i} & N_{4i} & 0_{n-r,(p-1)n} & N_{3i} & 0_{n,p_\omega}\end{bmatrix},$$
$$\bar{W}_C=\begin{bmatrix}C_i & 0_{S_z,(p+1)n+p_\omega}\end{bmatrix},$$
$$\bar{W}_z=\begin{bmatrix}I_n & -I_0 & 0_{n,pn+p_\omega}\end{bmatrix},$$
$$\bar{W}_W=\begin{bmatrix}0_{p_\omega,(p+2)n} & I_{p_\omega}\end{bmatrix}.$$

Proof: Note that (15) implies (6), and therefore the nominal form of SMJS (5) ($K_i=0$) is admissible. We now aim to investigate the H_∞ performance of the system. Toward this end, we introduce the following performance index:

$$J_{z\omega}=\sum_{k=0}^{N}\left[z^T(k)z(k)-\gamma^2\omega^T(k)\omega(k)\right]$$

Under zero initial condition, it can be verified that for all $\omega(k)\in L_2[0,\infty)$

$$J_{z\omega}=\sum_{k=0}^{N}\left[z^T(k)z(k)-\gamma^2\omega^T(k)\omega(k)-V(k)+V(k+1)\right]$$
$$-V(U+1)\le\sum_{k=0}^{N}\left[z^T(k)z(k)-\gamma^2\omega^T(k)\omega(k)+\Delta V(k)\right]$$
$$=\sum_{k=0}^{N}\bar{\xi}^T(k)\bar{\Omega}\bar{\xi}(k)$$

where

$$\bar{\xi}(k)=\begin{bmatrix}\Upsilon^T(k) & x^T(k-pd) & y^T(k) & \omega^T(k)\end{bmatrix}^T.$$

Considering (15), $J_{z\omega}<0$ for $\omega(k)\in L_2[0,\infty)$, and therefore the corresponding system meets the H_∞ performance requirements. The proof is completed. ■

Next, the analysis of robust H_∞ performance of the sliding mode dynamics (5) is investigated using Theorem 2, and a design condition is established using LMIs for obtaining the tuning parameter K_i in (3).

Theorem 3: Let λ_1, λ_2, λ_3, p, $d>0$, and $\gamma>0$ be given scalars. The sliding mode dynamics (5) meets the robust H_∞ performance requirements, if there exist matrices $P_i>0$, $Q>0$, $Z>0$, invertible matrix X_i, matrices M_{1i}, M_{2i}, M_{3i}, Y_i and a scalar $\varepsilon>0$, such that

$$\begin{bmatrix}\tilde{\Omega}+\varepsilon\tilde{\Pi}_i^T\tilde{\Pi}_i & \tilde{\Phi}_i^T\\ * & -\varepsilon I\end{bmatrix}<0,\quad \forall j\mathcal{I}_{\mathrm{UK}}^{(i)}, \tag{16}$$

where

$$\tilde{\Omega}=\tilde{W}_p^T\left(\bar{P}_k^{(i)}+d^2Z\right)\tilde{W}_p+\tilde{W}_{p3}^T\left(\bar{P}_k^{(i)}-P_i\right)\tilde{W}_{p3}$$
$$+\tilde{W}_Q^T\bar{Q}\tilde{W}_Q+\tilde{W}_C^T\tilde{W}_C-\gamma^2\tilde{W}_W^T\tilde{W}_W-\tilde{W}_z^TEZE^T\tilde{W}_z$$
$$+Her\left(\tilde{W}_{p1}^TE\bar{P}_k^{(i)}\tilde{W}_p+\tilde{M}_iR^T\tilde{W}_p+\Lambda_i^T\tilde{W}_p\right)<0,$$

in which

$$\tilde{W}_p=\begin{bmatrix}0_{n,(p+1)n} & I_n & 0_{n,s_z}\end{bmatrix},$$
$$\tilde{W}_{p1}=\begin{bmatrix}I_n & 0_{n,(p+1)n+s_z}\end{bmatrix},$$
$$\tilde{W}_{p3}=\begin{bmatrix}E^T & 0_{n,(p+1)n+s_z}\end{bmatrix},\bar{Q}=\begin{bmatrix}Q & 0_{pn,pn}\\ * & -Q\end{bmatrix},$$
$$\tilde{W}_Q=\begin{bmatrix}I_{pn} & 0_{pn,n} & 0_{pn,n+s_z}\\ 0_{pn,n} & I_{pn} & 0_{pn,n+s_z}\end{bmatrix},$$
$$\tilde{M}_i=\begin{bmatrix}M_{1i}^T & M_{2i}^T & 0_{n-r,(p-1)n} & M_{3i}^T & 0_{n-r,s_z}\end{bmatrix}^T,$$
$$\tilde{W}_z=\begin{bmatrix}I_n & -I_n & 0_{n,pn+s_z}\end{bmatrix},\tilde{W}_W=\begin{bmatrix}0_{s_z,(p+2)n} & I_{s_z}\end{bmatrix}$$
$$\tilde{W}_C=\begin{bmatrix}\bar{B}_{\omega i}^T & 0_{p_\omega,(p+1)n+s_z}\end{bmatrix},$$
$$\tilde{W}_E=\begin{bmatrix}\lambda_1I_n & \lambda_3I_n & 0_{n,(p-1)n} & \lambda_2I_n & 0_{n,s_z}\end{bmatrix},$$
$$\tilde{\Pi}_i=\begin{bmatrix}E^T\tilde{J}_i^T & E^T\tilde{J}_i^T & 0_{n,pn+s_z}\end{bmatrix}^T,$$
$$\tilde{\Phi}_i=\begin{bmatrix}\lambda_1F_iX_i & \lambda_3F_iX_i & 0_{n,(p-1)n} & \lambda_2F_iX_i & 0_{n,s_z}\end{bmatrix},$$
$$\Lambda_i=\left[X_i^T\left(A_i-E\right)^T+Y_i^TB_i^T \quad 0_{n,(p-1)n}\right.$$
$$\left.X_i^T\bar{A}_{di}^T \quad -X_i^T \quad X_i^TC_i^T\right].$$

Furthermore, if the LMI condition (16) has a feasible solution, then the tuning matrix of the sliding surface (3) can be given by $K_i=Y_iX_i^{-1}$.

Proof: It can be easily verified that the sliding mode dynamics (5) is equivalent to the following system:

$$E^T\varphi(k+1)=\left(\bar{A}_i+\Delta\bar{A}_i\right)^T\varphi(k)+\left(\bar{A}_{di}+\Delta\bar{A}_{di}\right)^T\varphi(k-d)$$
$$+C_i^T\delta(k)$$
$$\mu(k)=\bar{B}_{\omega i}^T\varphi(k). \tag{17}$$

Substituting E ,A_i, $\bar{A}_{di}$, $\bar{B}_{\omega i}$, C_i and p_ω with E^T, $\left(\bar{A}_i+\Delta\bar{A}_i\right)^T$, $\left(\bar{A}_{di}+\Delta\bar{A}_{di}\right)^T$, C_i^T, $\bar{B}_{\omega i}^T$ and s_z in (15), leads to

$$\tilde{W}_p^T\left(\bar{P}_k^{(i)}+d^2Z\right)\tilde{W}_p+\tilde{W}_{p3}^T\left(\bar{P}_k^{(i)}-P_i\right)\tilde{W}_{p3}$$
$$+\tilde{W}_Q^T\bar{Q}\tilde{W}_Q+\tilde{W}_C^T\tilde{W}_C-\gamma^2\tilde{W}_W^T\tilde{W}_W-\tilde{W}_z^TEZE^T\tilde{W}_z$$
$$+Her\left(\tilde{W}_{p1}^TE\bar{P}_k^{(i)}\tilde{W}_p+\tilde{M}_iR^T\tilde{W}_p\right)$$
$$+\mathrm{Her}\left(\begin{bmatrix}A_i+B_iK_i-E\\ \bar{A}_{di}\\ -I_n\\ C_i\end{bmatrix}\tilde{N}_i+\begin{bmatrix}\tilde{J}_iE_i\Delta(k)F_i\\ \tilde{J}_iE_i\Delta(k)F_{di}\\ -I_n\\ C_i\end{bmatrix}\tilde{N}_i\right)<0, \tag{18}$$

where $\tilde{N}_i=\begin{bmatrix}N_{2i} & N_{4i} & 0_{n,(p-1)n} & N_{3i} & 0_{n,s_z}\end{bmatrix}$. By considering $N_{2i}=\lambda_1X_i$, $N_{3i}=\lambda_2X_i$, $N_{4i}=\lambda_3X_i$, $Y_i=K_iX_i$, and then applying Lemma 1 to the above inequality together with Schur complement completes the proof. ■

3.3. Sliding mode controller design

After designing an appropriate sliding surface using the results established in Theorem 3, the next step is to synthesize an SMC law such that 1) the system trajectories are globally driven onto the predefined quasi-sliding surface, and 2) resulting in a non-increasing zigzag motion on the sliding surface. Before proceeding further, denote $\Delta_a(k)=G_i\Delta A_ix(k)$ and $\Delta_d(k)=$

$G_i \Delta A_{di} x(k-d)$. It is assumed that there exist $\underline{\delta}_a^j$, $\bar{\delta}_a^j$, $\underline{\delta}_d^j$ and $\bar{\delta}_d^j$ $\bar{\delta}_d^j$ ($j = 1, 2, \ldots, m_u$) satisfying

$\underline{\delta}_a^j \le \delta_a^j(k) \le \bar{\delta}_a^j, \quad \underline{\delta}_d^j \le \delta_d^j(k) \le \bar{\delta}_d^j.$

In the following theorem, a design technique for the robust SMC law is presented.

Theorem 4: Consider the SMJS (1) with the quasi-sliding surface (3), where K_i is the solution of (16). The system trajectories are globally pushed to the sliding surface $s(k) = 0$ by the following SMC law:

$$\begin{aligned} u(k) &= K_i x(k) - G_i A_{di} x(k-d) + (I - \mathrm{V}) s(k) \\ &-\mathrm{U}\operatorname{sgn}(s(k)) - \left(\hat{\Delta}_a + \tilde{\Delta}_a \operatorname{sgn}(s(k))\right) \\ &-\left(\hat{\Delta}_d + \tilde{\Delta}_d \operatorname{sgn}(s(k))\right) - \rho(k)\operatorname{sgn}(s(k)) \end{aligned} \tag{19}$$

where $\rho(k) = \lambda \|x(k)\| + \eta \varpi$, μ_j and v_j ($j = 1, 2, \ldots, m_u$) are properly chosen constants, and

$$\mathrm{U} = diag\left\{\mu_j, \mu_2, \ldots, \mu_{m_u}\right\} \in R^{m_u \times m_u},$$

$$\mathrm{V} = diag\left\{v_j, v_2, \ldots, v_{m_u}\right\} \in R^{m_u \times m_u},$$

$$\hat{\Delta}_a = \begin{bmatrix} \hat{\delta}_a^1 & \hat{\delta}_a^2 & \cdots & \hat{\delta}_a^{m_u} \end{bmatrix}^T, \hat{\delta}_a^j = \frac{\bar{\delta}_a^j + \underline{\delta}_a^j}{2},$$

$$\tilde{\Delta}_a = diag\left\{\tilde{\delta}_a^1, \tilde{\delta}_a^2, \ldots, \tilde{\delta}_a^{m_u}\right\}, \tilde{\delta}_a^j = \frac{\bar{\delta}_a^j - \underline{\delta}_a^j}{2},$$

$$\hat{\Delta}_d = \begin{bmatrix} \hat{\delta}_d^1 & \hat{\delta}_d^2 & \cdots & \hat{\delta}_d^{m_u} \end{bmatrix}^T, \hat{\delta}_d^j = \frac{\bar{\delta}_d^j + \underline{\delta}_d^j}{2},$$

$$\tilde{\Delta}_d = diag\left\{\tilde{\delta}_d^1, \tilde{\delta}_d^2, \ldots, \tilde{\delta}_d^{m_u}\right\}, \tilde{\delta}_d^j = \frac{\bar{\delta}_d^j - \underline{\delta}_d^j}{2},$$

$$\eta \ge \max_{i \in \mathrm{I}} \left\{ \sqrt{\lambda_{\max}\left(G_i B_{\omega i} B_{\omega i}^T G_i^T\right)} \right\}.$$

Proof: Combining Eqs. (1), (3), and (19), the incremental change of $s(k)$ is given by

$$\begin{aligned} \Delta s(k) &= s(k+1) - s(k) \\ &= -\mathrm{V}s(k) - \mathrm{U}\operatorname{sgn}(s(k)) + G_i \Delta A_i(k) x(k) \\ &-\left(\hat{\Delta}_a + \tilde{\Delta}_a \operatorname{sgn}(s(k))\right) + G_i \Delta A_{di}(k) x(k-d) \\ &-\left(\hat{\Delta}_d + \tilde{\Delta}_d \operatorname{sgn}(s(k))\right) + \left(f(x(k),k) - \lambda \|x(k)\| \operatorname{sgn}(s(k))\right) \\ &+\left(G_i B_{\omega i} \omega(k) - \eta \varpi \operatorname{sgn}(s(k))\right). \end{aligned} \tag{20}$$

Analogous to the proof of Theorem 3 in [32], it can be easily verified that

$$\begin{cases} \Delta s_j(k) \le -\mu_j - v_j s_j(k), \text{ if } s_j(k) > 0 \\ \Delta s_j(k) \ge \mu_j - v_j s_j(k), \text{ if } s_j(k) < 0 \end{cases} \tag{21}$$

$$j = 1, 2, \ldots, m_u$$

Therefore, the signs of $s_j(k)$ and $\Delta s_j(k)$ are opposite of each other at each sampling instant k, and thus the sliding variable will converge to zero together with non-increasing zigzag motion along the surface. The proof is completed.

■

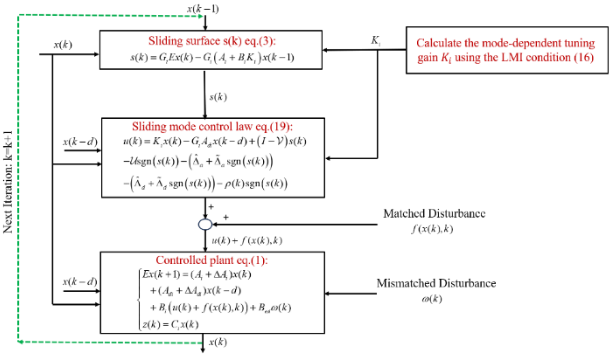

Fig.1. Schematic diagram of the proposed method

The detailed block diagram of the proposed control method is depicted in Fig 1.

Remark 1: To deal with the potential of undesirable chattering phenomenon caused by implementing the function $\operatorname{sgn}(s(k))$ in (19), a practical solution is to replace this function with either a high smooth function $\tanh(s(k))$ or a high slope function $\operatorname{sat}(s(k))$.

4. Numerical Examples

In this section, two numerical examples are presented to illustrate the effectiveness of the proposed control scheme.

Example 1: This example highlights the benefits of the LMI condition derived in equation (16) for the development of static output feedback controller for a DC servo system operating in a network communication channel. Considering the the network-induced delays from the sensor to-controller communication link, the objective is to ensure that the resulting closed-loop system becomes stochastically stable with γ -disurbance attenuation. The plant dynamics is represented by the following equation

$$\begin{cases} x_p(k+1) = A_p x_p(k) + B_p u_p(k) + B_{\omega p}\omega(k), \\ z(k) = C_{1p} x_p(k) + D_{\omega p}\omega(k), \\ y(k) = C_{2p} x_p(k), \end{cases} \tag{22}$$

where

$$A_p = \begin{bmatrix} 1.12 & 0.213 & -0.335 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B_p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, B_{\omega p} = \begin{bmatrix} 0.1 \\ 0.1 \\ 0 \end{bmatrix},$$

$$C_{1p} = \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix}, C_{2p} = \begin{bmatrix} 0.0541 & 0.115 & 0.0001 \end{bmatrix},$$

$$D_{\omega p} = \begin{bmatrix} 0.1 \end{bmatrix}, \omega(k) = \frac{1}{(k+1)^2}$$

Let the variable $\{\tau_k, k \ge 0\}$ signifies the time-varying delay caused by the communication network when transmitting data from the sensor to the controller. Due to the presence of correlation between the delays in consecutive time steps within real communication networked, the behaviour of the delay can be appropriately represented using the Markov chain. Here, τ_k is assumed to have values in the set $\tau_k = \{1,2,3\}$, with transition probability matrix specifies as follows:

$$\Lambda = \begin{bmatrix} 0.2 & \hat{\lambda}_{12} & \hat{\lambda}_{13} \\ 0.18 & \hat{\lambda}_{22} & \hat{\lambda}_{23} \\ 0.4 & 0.5 & 0.1 \end{bmatrix}$$

The output feedback controller that has been implemented takes the form of

$$u_p(k) = K(\tau_k)C_{2p}x_p(k-\tau_k) \quad (23)$$

Replacing (11) into the controlled plant (1), and subsequently utilizing the augmentation technique presented in [14], the closed-loop system dynamics is represented in the following manner:

$$\begin{cases} X_p(k+1) = \left(\tilde{A} + \tilde{B}K(\tau_k)\tilde{R}(\tau_k)\right)X_p(k) + \tilde{B}_\omega \omega(k) \\ z(k) = \tilde{C}_1 X_p(k) + \tilde{D}_\omega \omega(k) \end{cases} \quad (24)$$

Considering $U_p(k) = K(\tau_k)\tilde{R}(\tau_k)X_p(k)$, and defining the augmented state vector $X(t) = \left[X_p(k)^T \quad U_p(k)^T\right]^T$, resulting in the formulation of closed-loop system as a singular MJS (25):

$$\begin{cases} EX(k+1) = A(\tau_k)X(k) + B_\omega \omega(k) \\ z(k) = CX(k) + D_\omega \omega(k) \end{cases} \quad (25)$$

where

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, A(\tau_k) = \begin{bmatrix} \tilde{A} & \tilde{B} \\ K(\tau_k)\tilde{R}(\tau_k) & -I \end{bmatrix},$$

$$B_\omega = \begin{bmatrix} \tilde{B}_\omega \\ 0 \end{bmatrix}, C = \begin{bmatrix} \tilde{C}_1 & 0 \end{bmatrix}, D_\omega = \tilde{D}_\omega.$$

The tuning scalars λ_1, λ_2, λ_3 in Theorem 3 are selected as $\lambda_1 = -0.3$, $\lambda_2 = -0.8$, and $\lambda_3 = -0.1$, respectively. By solving the LMI condition (16) with $\gamma_{\min} = 0.64$, we can find the possible solutions for the tuning matrices K_i ($i = 1,2,3$) as

$$K(0) = 0.9210,\ K(1) = 0.7176,\ K(2) = 1.0345.$$

For an initial condition of $\varphi(t) = [-0.2 \quad 0.1 \quad 0.3]^T, k \in [-2, 0]$, the trajectories of the closed-loop system described in equation (22) are depicted in Fig. 2. It is evident the system trajectories converge toward a region close to the origin, even in the presence of networked-induced delay and external disturbance. As a result, the static output feedback controller derived from the LMI condition of (16) possesses the benefit of meeting the H_∞ performance requirements for a practical plant represented in the form of MJS.

Example 2: This example deals with the design of the sliding-mode controller for the delayed singular MJS (1) with two operating modes as follows:

$$A_1 = \begin{bmatrix} -1.5 & 1 \\ -1.03 & -5.2 \end{bmatrix}, A_{d1} = \begin{bmatrix} 0.02 & 0 \\ -0.01 & 0.5 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -2.5 & -1 \\ 0 & -5 \end{bmatrix}, B_{\omega 1} = \begin{bmatrix} 0.02 & 0 \\ -0.04 & 0.1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -0.12 & -0.02 \\ 0.1 & 0.05 \end{bmatrix}, E_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0.2 & 0.1 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 0.01 & 0.1 \\ 0.01 & 0.02 \end{bmatrix}, F_{d1} = \begin{bmatrix} 0.02 & 0.2 \\ 0.01 & 0.01 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.5 & 6 \\ 0 & -1 \end{bmatrix}, A_{d1} = \begin{bmatrix} 0 & 0.12 \\ 0 & 0.2 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.5 \end{bmatrix}, B_{\omega 1} = \begin{bmatrix} 0.1 & 0.02 \\ 0.01 & 0.3 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} -0.02 & -0.02 \\ 0.1 & 0.15 \end{bmatrix}, E_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0 & 0.2 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 0.02 & 0 \\ 0.1 & 0.02 \end{bmatrix}, F_{d2} = \begin{bmatrix} 0.24 & 0 \\ 0.1 & 0.01 \end{bmatrix},$$

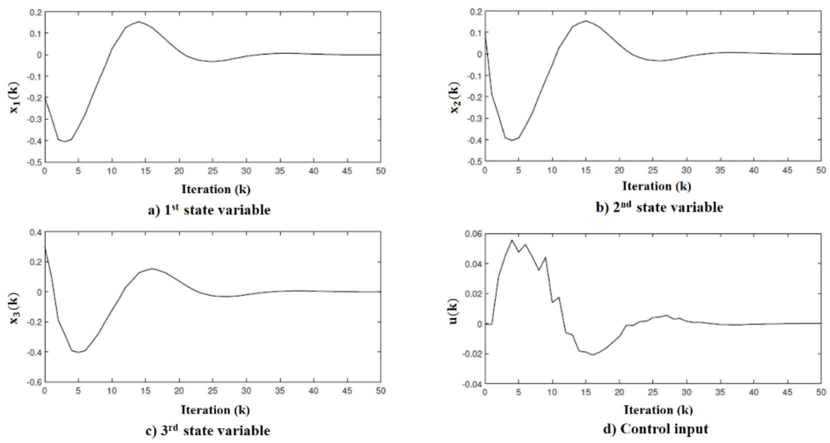

Fig.2. Evolution of the state trajectories and control input

$$f\left(x(k),k\right) = \begin{bmatrix} e^{-0.1k}\sin\left(0.1k\right)x_1(k)x_2(k) \\ 0.7e^{-k}\cos\left(0.1k\right)x_2^2(k) \end{bmatrix},$$

$$\omega(k) = \begin{cases} \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, & 5 < k < 45 \\ 0, & \text{ow.} \end{cases}$$

The singular matrix and the transition probability matrix are, respectively, given as follows:

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0.7 & 0.3 \\ \hat{\lambda}_{21} & \hat{\lambda}_{22} \end{bmatrix}$$

Selecting the tuning parameters as $\lambda_1 = -0.6110$, $\lambda_2 = -0.4935$, and $\lambda_3 = -0.1679$, Table I summarizes the minimum values of the H_∞ performance $\gamma_{\min}$ for various d via Theorem 3 in this paper and the method in [33].

Table 1. Comparisons of $\gamma_{\min}$ given $d > 0$

Methods	$d = 15$	$d = 30$	$d = 45$	$d = 55$
Theorem 4 in [33]	**0.514**	**0.571**	**0.713**	**Infeasible**
Theorem 3 (*p=1*)	**0.485**	**0.534**	**0.668**	**Infeasible**
Theorem 3 (*p=3*)	**0.469**	**0.519**	**0.644**	**0.901**
Theorem 3 (*p=5*)	**0.461**	**0.492**	**0.631**	**0.892**

It is clear that the results of Theorem 3 are less conservative than those in [33]. Furthermore, it can be seen that the conservatism decreases as the partitioning size m increases. In particular, when $d = 55$, the stabilization condition in [33] and Theorem 3 with $m = 1$ are failed to give a feasible solution. However, the LMI condition (16) in Theorem 3 gives a feasible solution as p chooses greater than 2.

To evaluate the effectiveness of the proposed technique, the simulation results are compared to the results obtained from the control approach of [33]. We select $d = 30$ and $p = 3$, then the feasible solution for the tuning matrix K_i is obtained as

$$K_1 = \begin{bmatrix} -0.6883 & 1.3164 \\ -0.2429 & -1.9880 \end{bmatrix}, \quad K_2 = \begin{bmatrix} -3.8384 & 44.8247 \\ 4.8886 & 69.5005 \end{bmatrix}$$

In our simulation, the design parameters of the controller (19) are selected as $\lambda = 0.8$, $\varpi = 0.05$, $\eta = 0.05$, $\mu_1 = 0.04$, $\mu_2 = 0.05$, $v_1 = 0.8$, and $v_2 = 0.9$. For an initial condition of $\Psi(k) = [0.2 \quad 0.7]\ k \in [-30, 0]$, the simulation results under the SMC law (19) and the approach of reference [33] are depicted in Figures 3-6. The simulation results indicate that our proposed method yields faster transient responses.

Based on the simulation results from Example 1 and Example 2, it can be deduced that the proposed SMC method effectively fulfills the H_∞ performance requirements.

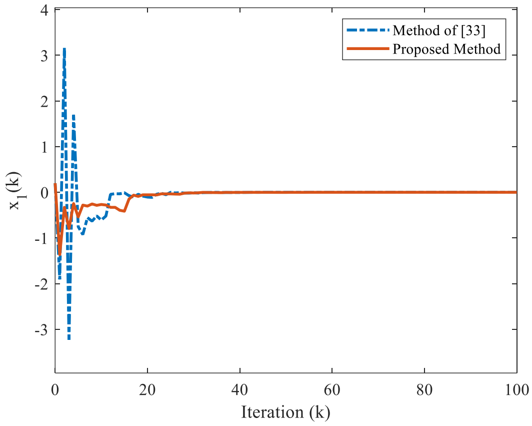

Fig.3. 1st state variable

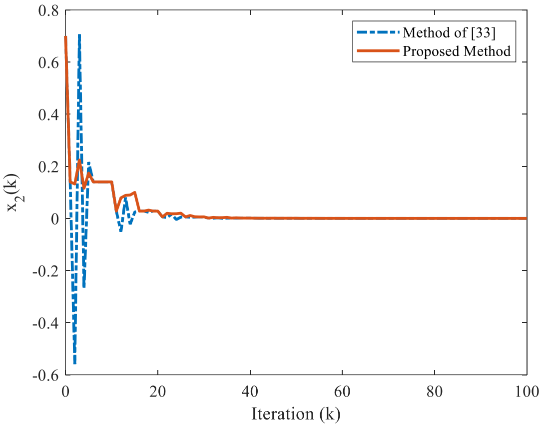

Fig.4. 2nd state variable

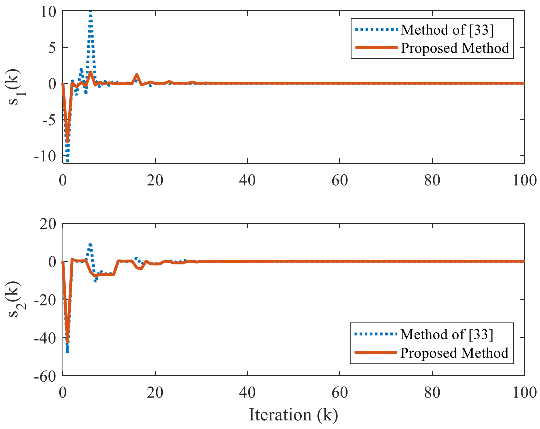

Fig.5. Sliding surface

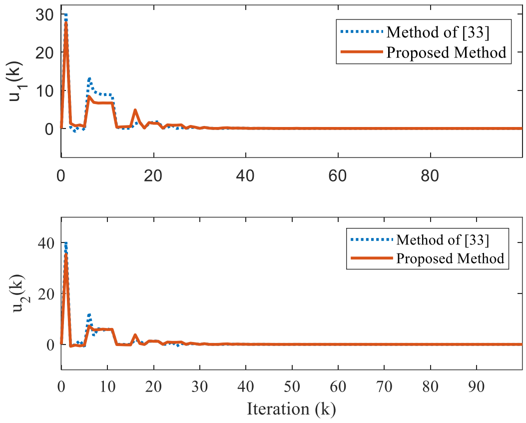

Fig.6. Control input

5. Conclusion

This paper has solved the problem of SMC for delayed singular MJSs subject to parameter uncertainties and both of matched/mismatched disturbances. The Lyapunov functional approach has been used together with the delay partitioning technique in driving an LMI-based solvability condition for the existence of the desired sliding surface, which ensures the sliding mode dynamics to be robust stochastically admissible with a prescribed H_∞ performance. The derived LMI condition has the merit of introducing some slack matrices to reduce the conservatism in the solutions. Then, a sliding mode controller has been synthesized to ensure that the state trajectories of the system can be driven onto the pre-designed surface, while achieving a non-increasing zigzag movement along the surface. Finally, two numerical examples have been presented which clearly demonstrate the advantages and merits of the developed control scheme. Due to the limitations of MJSs in practice, extending the proposed design method of this paper to the case of semi-Markovian singular systems might be an interesting topic for the future research.

6. References

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