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Synchronization a chaotic system with Quadratic terms using the contraction Method

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Abstract

In this article, Synchronization and control methods are discussed as essential topics in science. The contraction method is an exciting method that has been studied for the synchronization of chaotic systems with known and unknown parameters. The controller and the dynamic parameter estimation are obtained using the contraction theory to prove the stability of the synchronization error and the low parameter estimation. The control scheme does not employ the Lyapunov method. For demonstrate the ability of the proposed method, we performed a numerical simulation and compared the result with the previous literature.

Keywords. Contraction theory, Chaos, Synchronization.
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1. Introduction

Over the last decade, there has been a significant shift towards the examination and control of complex systems within the fields of engineering and science. Researchers from various disciplines, including mechanics, physics, and mathematics, have recognized the practical implications of dynamical systems and have dedicated their attention to studying them. These systems exhibit unpredictable and intricate behaviors that are evident in diverse fields such as laser theory [3], plasma dynamics [18], weather prediction [14], and fluid mixing[34]. One key characteristic of these systems is their sensitivity to initial conditions, contributing to their chaotic nature. This chaos can manifest in instability, lack of stability, and the emergence of limit cycles. The applications and examples of these systems are widespread and can be found in biomedical sciences [2, 23], image processing techniques [6], information theory [8], secure communication practices [19, 20, 27], the chaotic system of HIV disease fractional order [17], and electrical power systems [4].

Some researchers have conducted extensive studies on different systems, including the Van der Pol oscillator [30] and the Moore-Greitzer engine model [7]. These investigations have explored the effectiveness of various control techniques, such as adaptive control, sliding mode, and fuzzy or hybrid controllers. However, a recent groundbreaking approach called the contraction method has emerged for system control [9]. This innovative method, introduced by Slotine distinguishes itself from previous stability analysis methods by eliminating the need for a Lyapunov function [9, 27]. In 2017, a research paper introduced a novel distributed model predictive control algorithm for continuous-time Nonlinear systems. The algorithm utilized contraction theory to estimate the prediction error, resulting in the establishment of feasibility and stability conditions [13]. Additionally, a 2020 study developed a controller for synchronization between two identical modified chaotic oscillators, employing contraction theory. The performance of this controller was compared to well-known controllers for chaotic systems, revealing that the contraction theory-based controller exhibited superior performance in terms of synchronization time and negligible steady-state error [28]. Recently, in 2022, an article was published focusing on the dynamic stability of a group of fractional-order Nonlinear systems. The article explored the application of fixed point contraction theory to analyze the system's Stability [12]. Interconnected systems are a fundamental aspect of systems theory, and a notable advantage of contraction systems is their ability to

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handle multiple internal connections, resulting in a contraction system. In line with this, a research paper titled "A sufficient condition for k-contraction of the series connection of two systems" was published in 2022, shedding light on the development of a sufficient condition for k-contraction in the series connection of two systems [21].
Synchronizing dynamic systems is a complex control problem that aims to establish conditions for accommodating the dynamic behavior of two systems with different initial conditions over time. The sensitivity of chaotic systems to initial conditions poses a challenge, making it seemingly impossible to synchronize them. However, groundbreaking research conducted by Pecora and Carroll in 1990 demonstrated the possibility of synchronizing chaotic systems [22]. Since then, numerous researchers have explored the synchronization of chaotic systems and their diverse applications [19, 20]. various approaches such as adaptive control, sliding mode, tracking control, and non-linear control and feedback, to achieve synchronization and control, methods have been employed. These methods often utilize the Lyapunov stability theorem to design controllers. However, it is worth noting that these controllers tend to be complex and Nonlinear [11, 19, 20, 29, 31, 32].
Motivated by the above facts and considering the controller is hard to find, the controller is usually Nonlinear and complex. It seems that the contraction method for synchronizing dynamic, chaotic systems is possible. There is an important difference between the contraction theory and the Lyapunov-based control method because there is no requirement to know the specific movement of states. The inherent advantages of synchronization of the chaotic systems using the contraction theory can be referred to as simple controller design. also for designing the controller, there is no problem with the selection of a Lyapunov function, and it can be used to discuss the stability analysis of complex systems [28].
For more discussion, the following sections of the paper have been organized: In section 2, we introduce the fundamental principles that form the basis of system behavior and highlight the application of the contraction method. Section 3 delves into the synchronization of the system, considering parameters are known and unknown. A comprehensive discussion on this topic is provided. In section 4, we validate the accuracy and effectiveness of the proposed method through numerical simulations. Additionally, we compare its advantages with previous approaches. Lastly, in section 5, we present our conclusions based on the findings and analysis presented in the preceding sections.

2. Preliminary

In this section, we describe two components of system behavior and contraction theory. The system that will be studied in this paper is a system with quadratic terms [1]. For more details, we study its behavior of it.

2.1. **Behavior of system with quadratic terms.** The system is characterized by quadratic terms that were introduced in [1, 16] exhibits chaotic behavior and can be described as follows:

$$\begin{cases} \dot{x} = -ax + by - zy, \\ \dot{y} = x + xz, \\ \dot{z} = -cz + y^2, \end{cases} \tag{2.1}$$

where a, b ,and c are parameters, and x, y ,and z are the state variables of the system. According to the initial calculations, the equilibrium points of the system are as follows:

$$\{x = \frac{\sqrt{-c}\,(b+1)}{a}, y = \sqrt{-c}, z = -1\}, \{x = -\frac{\sqrt{-c}\,(b+1)}{a}, y = -\sqrt{-c}, z = -1\},$$
$$\{x = 0, y = 0, z = 0\}, \{x = 0, y = \sqrt{bc}, z = b\}, \{x = 0, y = -\sqrt{bc}, z = b\}.$$

To illustrate the chaotic behavior of the system (2.1), we calculate and plot the Lyapunov exponent, bifurcation diagram attractor, and sensitivity diagram concerning the initial condition for some parameters. The Lyapunov exponent is a mathematical concept that measures the rate of divergence or convergence of nearby trajectories in a complex dynamical system. It quantifies the sensitivity of the system to initial conditions, indicating whether it is chaotic or stable. A positive Lyapunov exponent indicates exponential divergence of trajectories, implying chaotic behavior, while a negative exponent implies convergence and stability. It provides valuable insight into the long-term predictability and behavior of complex systems, such as weather patterns, population dynamics, and fluid dynamics

[5, 15]. The bifurcation diagram provides a visual representation of how a system behaves as a parameter is varied, illustrating the different stable states or attractors that emerge from the system dynamics. It shows the points of divergence, where the system transitions from one attractor to multiple attractors or from stability to chaos. On the other hand, a sensitivity diagram examines the response of a system to slight changes or perturbations in its initial conditions or parameters, highlighting the sensitivity or robustness of the system. It reveals regions of parameter values where slight variations can lead to significantly different outcomes, helping understand the stability and predictability of the system[15]. The experimental findings depicted in Figures 1 - 6 provide evidence that the system (2.1) exhibits chaotic behavior for $a = 4$, $b = 28$, and $0 \leq c \leq 10$. The results show that when c tends to 2, the chaos decreases. In other words, the sensitivity of the system decreases.

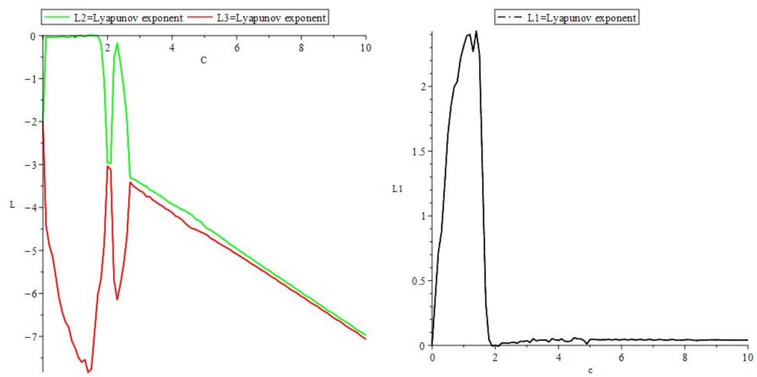


FIGURE 1. Lyaponuv Exponent of system (2.1) for $a = 4$, $b = 28$, and $0 \leq c \leq 10$.

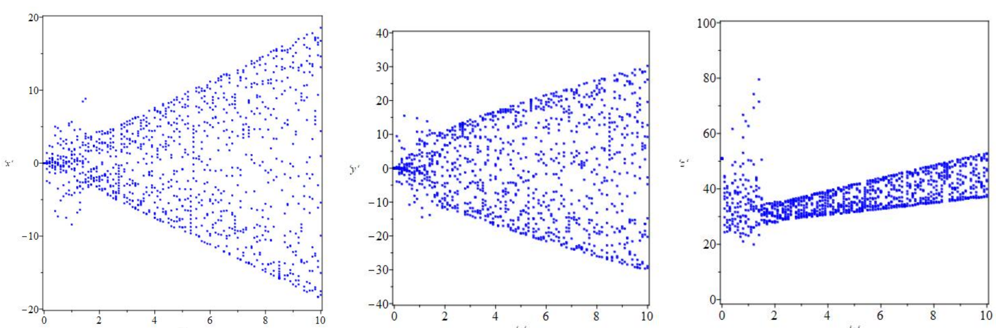
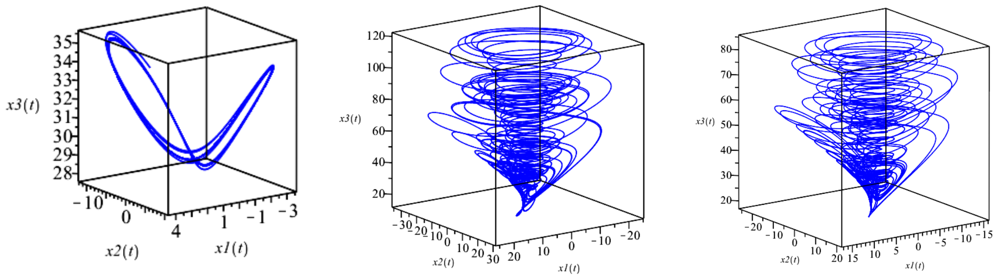


FIGURE 2. Bifurcation diagram of system (2.1) for $a = 4$, $b = 28$, and $0 \leq c \leq 10$.



$c = 2$ $c = 1.5$ $c = 1$

FIGURE 3. Attractors of system (2.1) for $a = 4$, $b = 28$.

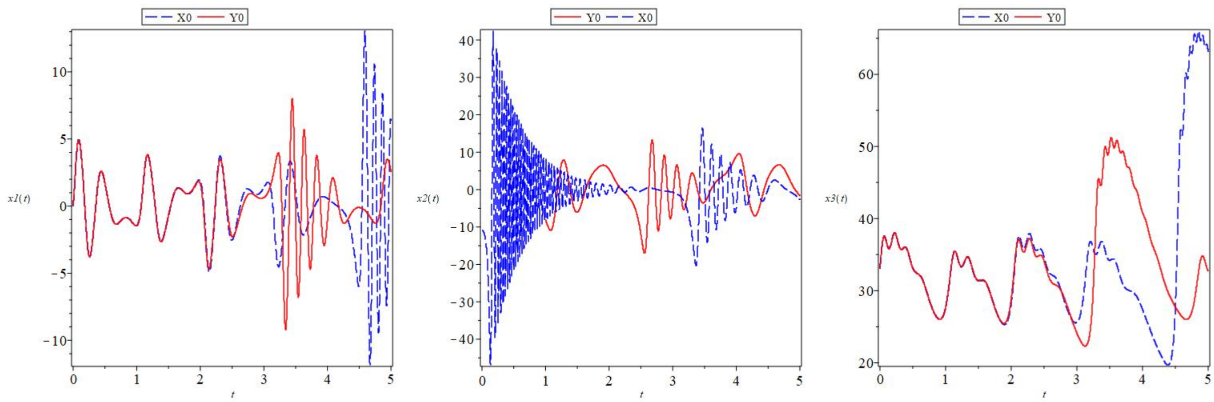


FIGURE 4. Sensitivity for $c = 1$ for initial conditions $(0, -12, 33)$ and $(0.01, -12, 33.01)$.

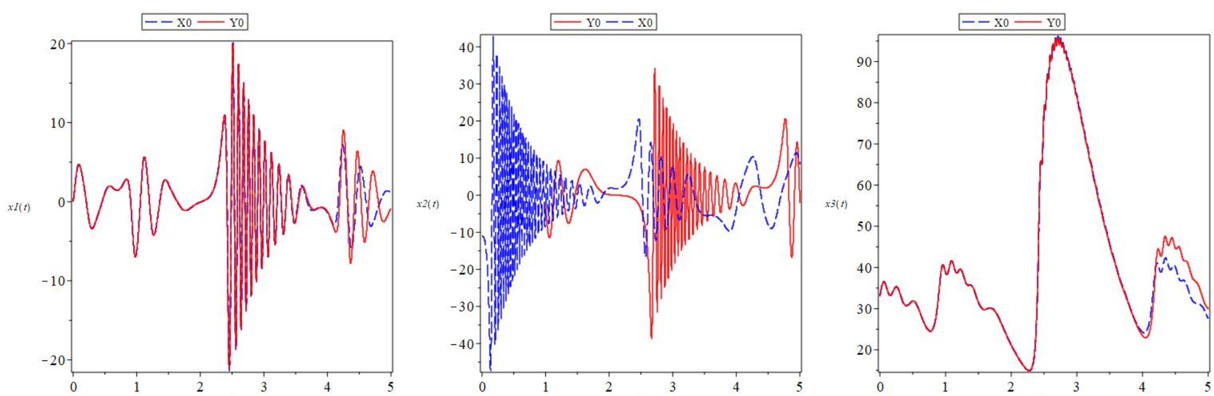


FIGURE 5. Sensitivity for $c = 1.5$ for initial conditions $(0, -12, 33)$ and $(0.01, -12, 33.01)$.

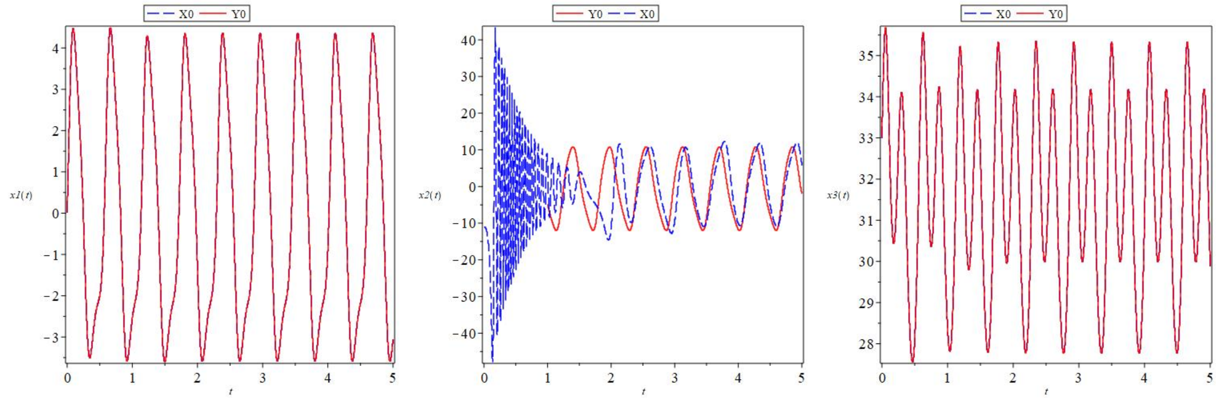


FIGURE 6. Sensitivity for $c = 2$ for initial conditions $(0, -12, 33)$ and $(0.01, -12, 33.01)$.

2.2. **Contraction theory.** Contraction theory concerns the subject of the convergence between two trajectories of an arbitrary system. If initial conditions or temporary disturbances are ignored fast and exponentially, called a contracting Nonlinear dynamic system, i.e., the trajectories of the perturbed system converge to their nominal behavior with an exponential convergence rate. Now, we briefly summarize the primary definitions and main theorems of contraction theory. Consider a Nonlinear system as follows:

$$\dot{x} = f(x, t), \tag{2.2}$$

where $x \in R^{m\times 1}$ is the state vector of the dynamical system (2.2) and f is a vector function whit $m \times 1$ dimensional. Function $f(x,t)$ is considered to be a continuous differentiable function. Let δx be the virtual displacement in the state x, infinitesimal displacements at a fixed time. To introduce the concept of virtual dynamics, we define the variation of the system in (2.2) as follows:

$$\delta\dot{x} = \frac{\partial f(x,t)}{\partial x}\delta x. \tag{2.3}$$

By considering this equation, we have:

$$\frac{d}{dt}\left(\delta x^T \delta x\right) = 2\delta x^T \delta\dot{x} = 2\delta x^T \frac{\partial f}{\partial x}\delta x \leq 2\lambda_m(x,\, t)\,\delta x^T \delta x, \tag{2.4}$$

here, the Jacobian matrix is denoted as $J = \frac{\partial f}{\partial x}$ and $\lambda_m(x,\, t)$ is the largest eigenvalue of the symmetric part of the Jacobian matrix (J). If the eigenvalue $\lambda_m(x,t)$ is strictly and uniformly negative, then any infinitesimal distance $\|\delta x\|$ between neighboring paths converges exponentially to zero. Where $\delta x^T \delta x$ represents the distance between the neighboring trajectories. The path integration in (2.4) shows that all solution paths of the system in (2.2) converge exponentially to a single path that is independent of initial conditions.

Definition 2.1. Given the system equations $\dot{x} = f(x,t)$, a region (open, connected space) of state space is called a contracting region if the Jacobian matrix $J = \frac{\partial f}{\partial x}$ is uniformly negative definite in that region [33].

The Jacobian matrix $J = \frac{\partial f(x,\, t)}{\partial x}$ if uniformly negative definite, if there exists a scalar $\alpha > 0$, $\forall x$, $\forall t \geq 0$ $s.t$ $\frac{\partial f}{\partial x} \leq -\alpha I < 0$. Also based on matrix inequalities, the symmetric part of the square matrix $\frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{\partial f^T}{\partial x}\right) \leq -\alpha I$ are negative definite. Considering the above definition, the basic lemmas related to exponential convergence of the trajectories can be stated as follows [24, 28, 33]:

Lemma 2.2. *Let the system equations $\dot{x} = f(x,t)$, any trajectory which starts in a ball of constant radius centered about a given trajectory and contained at all times in a contraction region, remain in that ball and converge exponentially to the given trajectory. Further, global exponential convergence to this given trajectory is guaranteed if the whole state space region is contracting [9].*

To investigate the above concepts in a more general way, we consider a coordinate transformation as follows:

$$\delta z = \theta \delta x, \tag{2.5}$$

where $\theta(x,t)$ is a uniformly invertible matrix. Related definitions and theorems in the generalized domain can be found in [24, 28, 33].
In some cases, where the representation provided in equation (2.2) is applicable, the Jacobian matrix $\frac{\partial f}{\partial x}$ could be negative semi-definite. These systems are referred to as semi-contracting systems, an extension of Definition (2.1). By applying contraction theory results, asymptotic stability can be guaranteed for such systems. The lemma below is formulated to analyze the asymptotic stability of semi-contracting systems [10].

Lemma 2.3. *For the system $\dot{x} = f(x,t)$, let the stable reference system is given by $\dot{y} = f(y,t)$, and the error is given as $e = y - x$. For the error system $\dot{e} = f_1(e,x,t)$, if the Jacobian matrix $J = \frac{\partial f_1(e,x,t)}{\partial x}$ is uniformly negative semi-definite, i.e., in terms of virtual displacement in differential framework, if*

$$\begin{bmatrix} \partial \dot{e_1} \\ \dots \\ \partial \dot{e_2} \end{bmatrix} = \begin{bmatrix} J_{11} & \vdots & W(x,t) \\ \dots & \dots & \dots \\ -W^T(x,t) & \vdots & 0 \end{bmatrix} \begin{bmatrix} \partial e_1 \\ \dots \\ \partial e_2 \end{bmatrix}, \tag{2.6}$$

where submatrix J_{11} is uniformly negative definite, then the system will be semi-contracting. For such cases, the states of the actual system state and reference system corresponding to the error vector e_1 converge to each other asymptotically with e_2 in remaining bounded [11].

3. Synchronization

In this section, we study the synchronization of the system (2.1). The synchronization of this system is discussed in two cases, with known and unknown parameters.
In the synchronization process, two systems are considered slave and master systems. Let the master and slave be defined as follows, respectively:

$$\dot{x} = f(x), \tag{3.1}$$

$$\dot{y} = f(y) + u, \tag{3.2}$$

where u is a control function. In synchronization, the goal is to find controller u, such that for any initial condition, we have:

$$\lim_{t \to \infty} \|x - y\| = 0.$$

Now, for synchronization, we define the master system as follows:

$$\begin{cases} \dot{x_1} = -ax_1 + bx_2 - x_2 x_3, \\ \dot{x_2} = x_1 + x_1 x_3, \\ \dot{x_3} = -cx_3 + x_2^2, \end{cases} \tag{3.3}$$

then, we define the slave system in two cases, in the following subsections.

3.1. **Synchronization with the known parameter.** Let, the slave system identical of (3.3) as follows:

$$\begin{cases} \dot{y_1} = -ay_1 + by_2 - y_2 y_3 + u_1, \\ \dot{y_2} = y_1 + y_1 y_3 + u_2, \\ \dot{y_3} = -cy_3 + y_2^2 + u_3. \end{cases} \tag{3.4}$$

Now, we define the error system between (3.3) and (3.4) , so we have:

$$\begin{cases} e_1 = y_1 - x_1, \\ e_2 = y_2 - x_2, \\ e_3 = y_3 - x_3, \end{cases} \implies \begin{cases} \dot{e_1} = -ae_1 + be_2 + (x_2x_3 - y_2y_3) + u_1, \\ \dot{e_2} = e_1 + (y_1y_3 - x_1x_3) + u_2, \\ \dot{e_3} = -ce_3 + (y_2^2 - x_2^2) + u_3. \end{cases} \tag{3.5}$$

For every choice of initial conditions in (3.3) and (3.4) the control functions u_i have been designed, such that in the system (3.5) we have:

$$\lim_{t\to\infty} e_i(t) = 0.$$

For designing the controllers based on contraction theory, we let the controller as follows:

$$\begin{cases} u_1 = -c_{11}e_1 = c_{11}(x_1 - y_1), \\ u_2 = -c_{22}e_2 = c_{22}(x_2 - y_2), \\ u_3 = -c_{33}e_3 = c_{33}(x_3 - y_3), \end{cases} \tag{3.6}$$

3.2. **Stability using virtual system concept.** Consider the dynamical system $\dot{x} = h\,(x(t), p)$ for where p is the parameter vector and the condition $h\,(x(t), p) \leq \beta I, \beta > 0$ are satisfied. To analyze the exponential stability of the system, the virtual system can be defined as $\dot{w} = h\,(x(t), p)\,w$ The first virtual system change is as $\delta\dot{w} = \frac{\partial h(w(t))}{\partial w}\partial w$. Suppose the symmetric part of $\frac{\partial h(w(t))}{\partial w}$ is uniformly negative definite, then we can say that the virtual system is contracting [9, 26]. Because the real system is a specific solution of the virtual system, the states of the real system converge to each other exponentially.

Based on [27], for the design of the controller, consider the virtual system of systems (3.3) and (3.4), as follows:

$$\begin{cases} \dot{z_1} = -az_1 + bz_2 - z_2z_3 + c_{11}(x_1 - z_1), \\ \dot{z_2} = z_1 + z_1z_3 + c_{22}(x_2 - z_2), \\ \dot{z_3} = -cz_3 + z_2^2 + c_{33}(x_3 - z_3). \end{cases} \tag{3.7}$$

Now, we calculate the Jacobian matrix of the system (3.7). So, we have:

$$J = \begin{pmatrix} -a - c_{11} & b - z_3 & -z_2 \\ 1 + z_3 & -c_{22} & z_1 \\ 0 & 2z_2 & -c - c_{33} \end{pmatrix}. \tag{3.8}$$

For satisfying the contraction theory, the symmetric part of Jacobian matrix $J_s = \frac{1}{2}(J + J^T)$, should be negative definite($-J_s$ should be positive definite). The J_s and $-J_s$ are as follows:

$$J_s = \frac{1}{2}\left(J + J^T\right) = \begin{pmatrix} -a - c_{11} & \frac{b+1}{2} & \frac{-z_2}{2} \\ \frac{b+1}{2} & -c_{22} & \frac{z_1+2z_2}{2} \\ \frac{-z_2}{2} & \frac{z_1+2z_2}{2} & -c - c_{33} \end{pmatrix}, \tag{3.9}$$

and

$$-J_s = \begin{pmatrix} a + c_{11} & \frac{-b-1}{2} & \frac{z_2}{2} \\ \frac{-b-1}{2} & c_{22} & \frac{-z_1-2z_2}{2} \\ \frac{z_2}{2} & \frac{-z_1-2z_2}{2} & c + c_{33} \end{pmatrix}. \tag{3.10}$$

So the controllers should be such that:

- $a + c_{11} > 0,$
- $(a + c_{11})c_{22} - \frac{(-b-1)^2}{4} > 0,$
- $\frac{x\,y}{4} - \frac{c_{33}}{4} - \frac{b\,c}{2} - \frac{b\,c_{33}}{2} - \frac{c}{4} - \frac{b^2\,c}{4} - \frac{b^2\,c_{33}}{4} - \frac{a\,x^2}{4} - a\,y^2 + \frac{b\,y^2}{2} - \frac{c_{11}\,x^2}{4} - c_{11}\,y^2 - \frac{c_{22}\,y^2}{4} + \frac{y^2}{2} + a\,c\,c_{22} + a\,c_{22}\,c_{33} + c\,c_{11}\,c_{22} + c_{11}\,c_{22}\,c_{33} - a\,x\,y + \frac{b\,x\,y}{4} - c_{11}\,x\,y > 0.$

The above condition, illustrates that the contraction theory occurs in synchronization of (3.3) and (3.4), with known parameters. Now, we investigate it for unknown parameters.

3.3. **Synchronization with the unknown parameter.** Now consider system (3.3) as the master system and the slave system with unknown parameters as follows:

$$\begin{cases} \dot{y_1} = -\hat{a}y_1 + \hat{b}y_2 - y_2y_3 + k_{11}(x_1 - y_1), \\ \dot{y_2} = y_1 + y_1y_3 + k_{22}(x_2 - y_2), \\ \dot{y_3} = -\hat{c}y_3 + y_2^2 + k_{33}(x_3 - y_3), \end{cases} \tag{3.11}$$

where $k_{ii}(x_i - y_i)$ for $i = 1, 2, 3$ are controllers, and $\hat{a}$, $\hat{b}$ and $\hat{c}$ are unknown parameters. Such that

$$\begin{cases} \hat{a} = \tilde{a} + a, \\ \hat{b} = \tilde{b} + b, \\ \hat{c} = \tilde{c} + c, \end{cases} \tag{3.12}$$

where $\tilde{a}$, $\tilde{b}$, and $\tilde{c}$ are approximation errors of parameters respectively. For the estimation of parameters, we use the following rules based on [25, 27]:

$$\begin{cases} \dot{\hat{a}} = \dot{\tilde{a}}, \\ \dot{\hat{b}} = \dot{\tilde{b}}, \\ \dot{\hat{c}} = \dot{\tilde{c}}, \end{cases} \implies \begin{cases} \dot{\hat{a}} = -y_1(x_1 - y_1), \\ \dot{\hat{b}} = y_2(x_1 - y_1), \\ \dot{\hat{c}} = -y_3(x_3 - y_3). \end{cases} \tag{3.13}$$

Now to calculate the contraction theory condition, we write the virtual system based on (3.3), (3.11), and (3.13) as follows:

$$\begin{cases} \dot{z_1} = -z_1z_4 + z_2z_5 - z_2z_3 + k_{11}(x_1 - z_1), \\ \dot{z_2} = z_1 + z_1z_3 + k_{22}(x_2 - z_2), \\ \dot{z_3} = -z_3z_6 + z_2^2 + k_{33}(x_3 - z_3), \\ \dot{z_4} = \dot{\hat{a}} = -z_1(x_1 - z_1), \\ \dot{z_5} = \dot{\hat{b}} = z_2(x_1 - z_1), \\ \dot{z_6} = \dot{\hat{c}} = -z_3(x_3 - z_3). \end{cases} \tag{3.14}$$

Now, we calculate the Jacobian matrix of the system (3.14). So, we have:

$$J = \begin{pmatrix} -z_4 - k_{11} & z_5 - z_3 & -z_2 & -z_1 & z_2 & 0 \\ 1 + z_3 & -k_{22} & z_1 & 0 & 0 & 0 \\ 0 & 2z_2 & -z_6 - k_{33} & 0 & 0 & -z_3 \\ -x_1 + 2z_1 & 0 & 0 & 0 & 0 & 0 \\ -z_2 & x_1 - z_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2z_3 - x_3 & 0 & 0 & 0 \end{pmatrix}. \tag{3.15}$$

For satisfying the contraction theory, the symmetric part of the Jacobian matrix $J_s = \frac{1}{2}(J + J^T)$, should be negative definite($-J_s$ should be positive definite). The J_s and $-J_s$ are as follows:

$$J_s = \frac{1}{2}\left(J + J^T\right) = \begin{pmatrix} -z_4 - k_{11} & \frac{z_5+1}{2} & \frac{-z_2}{2} & \frac{z_1-x_1}{2} & 0 & 0 \\ \frac{z_5+1}{2} & -k_{22} & \frac{z_1+2z_2}{2} & 0 & \frac{x_1-z_1}{2} & 0 \\ \frac{-z_2}{2} & \frac{z_1+2z_2}{2} & -z_6 - k_{33} & 0 & 0 & \frac{z_3-x_3}{2} \\ \frac{z_1-x_1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{x_1-z_1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{z_3-x_3}{2} & 0 & 0 & 0 \end{pmatrix}, \tag{3.16}$$

and

$$-J_s = \begin{pmatrix} z_4 + k_{11} & \frac{-z_5-1}{2} & \frac{z_2}{2} & \frac{x_1-z_1}{2} & 0 & 0 \\ \frac{-z_5-1}{2} & k_{22} & \frac{-z_1-2z_2}{2} & 0 & \frac{z_1-x_1}{2} & 0 \\ \frac{z_2}{2} & \frac{-z_1-2z_2}{2} & z_6 + k_{33} & 0 & 0 & \frac{x_3-z_3}{2} \\ \frac{x_1-z_1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{z_1-x_1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{x_3-z_3}{2} & 0 & 0 & 0 \end{pmatrix}. \tag{3.17}$$

So the controllers should be such that:

- $z_4 + k_{11} > 0$,
- $-\frac{z_5^2}{4} - \frac{z_5}{2} + k_{11}k_{22} + k_{22}z_4 - \frac{1}{4} > 0$,
- $\frac{z_1 z_2}{4} - \frac{z_6}{4} - \frac{k_{33} z_5}{2} - \frac{k_{33}}{4} - \frac{z_5 z_6}{2} - \frac{k_{11} z_1^2}{4} - k_{11} z_2^2 - \frac{k_{22} z_2^2}{4} - \frac{k_{33} z_5^2}{4} - \frac{z_1^2 z_4}{4} - z_2^2 z_4 + \frac{z_2^2 z_5}{2} - \frac{z_5^2 z_6}{4} + \frac{z_2^2}{2} + k_{11} k_{22} k_{33} + k_{11} k_{22} z_6 + k_{22} k_{33} z_4 - k_{11} z_1 z_2 + k_{22} z_4 z_6 - z_1 z_2 z_4 + \frac{z_1 z_2 z_5}{4} > 0$,
- $\frac{(x_1 - z_1)^2 \left(z_1^2 + 4 z_1 z_2 + 4 z_2^2 - 4 k_{22} k_{33} - 4 k_{22} z_6\right)}{16} > 0$,
- $\frac{(k_{33} + z_6)(x_1 z_1)^4}{16} > 0$,
- $-\frac{(x_1 - z_1)^4 (x_3 - z_3)^2}{64} > 0$.

In the comparison of [1], the designed controllers are very simple. The obtained controllers, are based on contraction theory. i.e., known and unknown parameters, and also the estimation rule of unknown parameters is simple.

4. Numerical simulation

To analyze the synchronization based on contraction, numerical results were obtained using the controller designed in the previous section. The 'ode45' MATLAB function was utilized for this purpose. Based on the mentioned behavior of the system (2.1), we know this system is chaotic for $a = 4$, $b = 28$, and $c = 2$. According to contraction conditions, the suitable controllers gain for synchronization cases known, and unknown parameters are given in Table 1. The results are given in figures 7-13. The results show that the error of synchronization and parameters estimation rule tends to zero in a short time(quickly).

Table 1. Controller gain.

Known Parameters	Unknown Parameters
$c_{11} = -3$	$k_{11} = 0$
$c_{22} = 211.25$	$k_{22} = 812.25$
$c_{33} = 0/78525$	$k_{33} = 0$

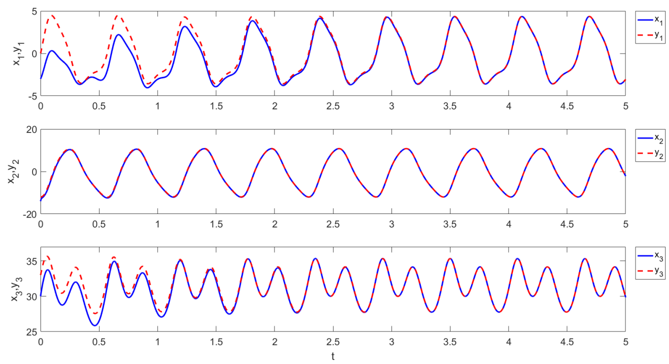


FIGURE 7. Synchronization of a chaotic system with known parameter.

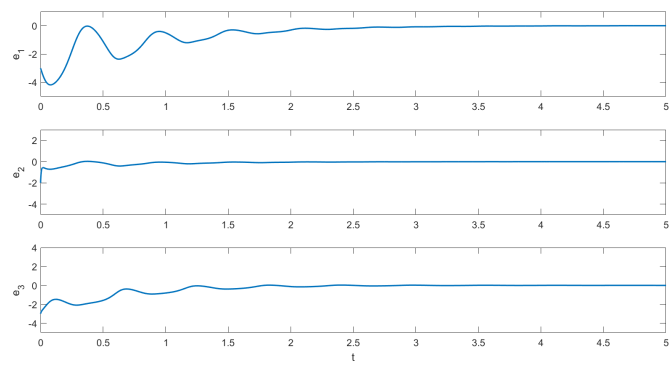


FIGURE 8. Synchronization error of chaotic system with known parameter.

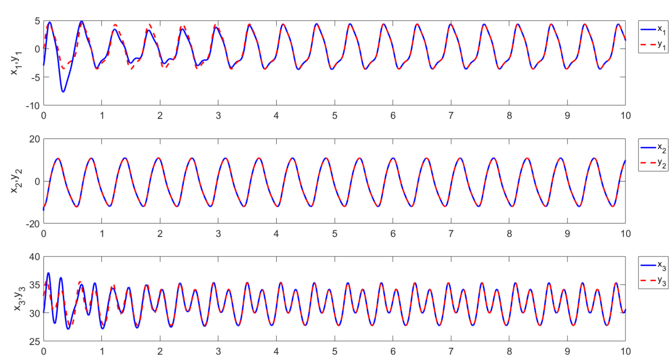


FIGURE 9. Synchronization of a chaotic system with unknown parameter.

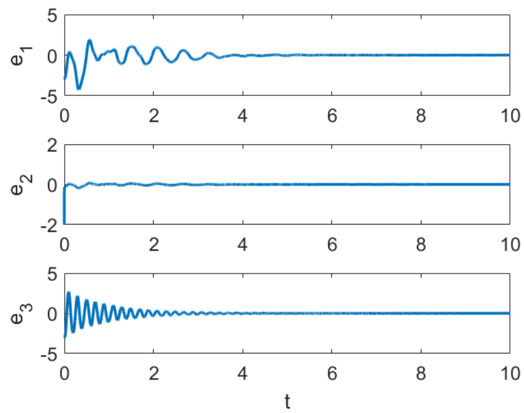


FIGURE 10. Synchronization error of chaotic system with unknown parameter.

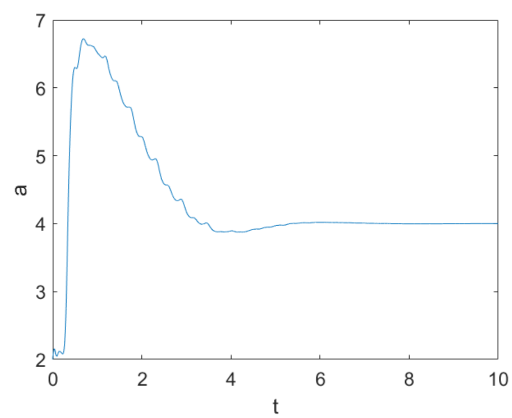


FIGURE 11. Estimation plot with parameter a.

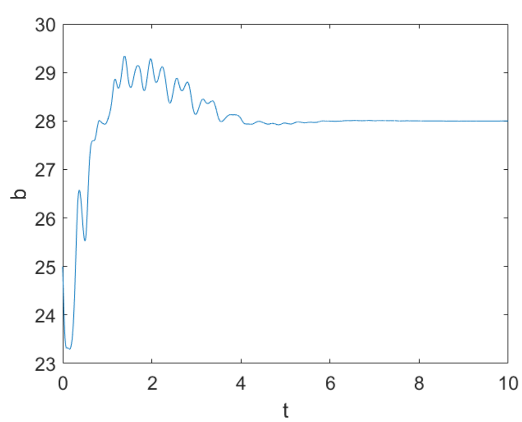


FIGURE 12. Estimation plot with parameter b.

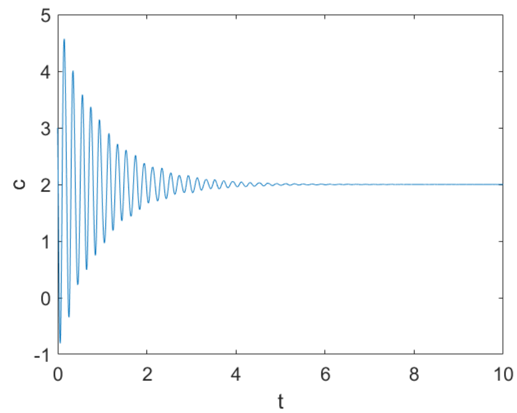


FIGURE 13. Estimation plot with parameter c.

5. Conclusion

In this article, our focus is on examining the dynamic characteristics of a chaotic quadratic system. We delve into various aspects of the system that contribute to its chaotic behavior, such as Lyapunov exponents, bifurcation, attractors, and sensitivity to initial conditions. Furthermore, we explore the application of contraction theory in achieving synchronization of chaotic quadratic systems, both with known and unknown parameters. Our findings demonstrate that the controllers developed in this study exhibit linearity and simplicity, setting them apart from previous research. Importantly, we establish the system's stability without relying on the traditional Lyapunov method. Through numerical simulations, we provide evidence showcasing the effectiveness of our proposed approach.

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