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The Legendre Wavelet Method for Solving Singular Integro-differential Equations

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Abstract In this paper, we present Legendre wavelet method to obtain numerical solution of a singular integro-differential equation. The singularity is assumed to be of the Cauchy type. The numerical results obtained by the present method compare favorably with those obtained by various Galerkin methods earlier in the literature.

Keywords. Legendre wavelet, Singular integro-differential equation, Operational matrix.

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1. Introduction

The singular integro-differential equation

$$2\frac{d\phi}{dx} + \lambda \int_{-1}^{1} \frac{\phi(t)}{t-x} dt = f(x), \quad -1 < x < 1, \quad \lambda > 0 \tag{1.1}$$

with specified end conditions, $\phi(\pm 1) = 0$, and a special forcing function $f(x) = -\frac{x}{2}$, was solved earlier by frankel [3], Chakrabarti and Hamsapriye [2], and recently by Mandal and Bera [7].

Applications in many important fields, like fracture mechanics [5], elastic contact problems [1], the theory of porous filtering [4] and combined infrared radiation and molecular conduction [3], contain integral and singular integro-differential equation with singular kernel. The solution of some of these problems may be obtained analytically by the method introduced in [9].

The forcing function $f(x) = -\frac{x}{2}$, is importance case, because it arises in the study of problems concerning conduction and radiation. Also singular integro-differential equations arise in connection with solving some special type of mixed boundary value problems involving the two dimensional Laplace's equation in the quarter plane. Here we applied Legendre wavelets for solving such singular integro-differential equation. The numerical results show that the method has good accuaracy. We suppose that ϕ is $L^2[-1,1]$ function and also Holder continuous.

2. Legendre wavelet

In recent years, wavelets have found their place in many applications such as signal processing, image processing, and solution of many equations. The main characteristic of wavelets is its ability to convert the given differential and integral equations to a system of linear or nonlinear differential equations, which are then solved by existing numerical methods.

Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter 'a' and the translation parameter 'b' vary continuously, we have the following family of continuous wavelets as:

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}}\psi(\frac{t-b}{a}), \quad a, b \in \mathbb{R}, \quad a \neq 0.$$

If we restrict parameters a and b to discrete values as:

$$a = a_0^{-k}, \quad b = nb_0a_0^{-k}, \quad a_0 > 1, \quad b_0 > 0,$$

where n, k are positive integers then we have the following family of discrete wavelets:

$$\psi_{k,n}(t) = |a|^{\frac{-1}{2}}\psi(a_0^k t - nb_0),$$

where $\psi_{k,n}(t)$ form a basis for $L^2(\mathbb{R})$.

Legendre wavelets $\psi_{nm}(t) = \psi(k, \hat{n}, m, t)$ have four arguments:

$$\hat{n} = 2n-1, \quad n = 1, 2, 3, \ldots, 2^{k-1}, \quad k \in \mathbb{Z}^+,$$

where m is the order of Legendre polynomials and t is the normalized time. They are defined on the interval [0,1) as

$$\psi_{nm}(t) = \begin{cases} \sqrt{m+\frac{1}{2}}\, 2^{\frac{k}{2}}\, p_m\left(2^k t - \hat{n}\right) & \frac{\hat{n}-1}{2^k} \leq t < \frac{\hat{n}+1}{2^k}, \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

where $m = 0, 1, 2, \ldots, M-1, n = 1, 2, 3, \ldots, 2^{k-1}$. The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, the dilation parameter is $a = 2^{-k}$ and translation parameter is $b = \hat{n}2^{-k}$.

Here $p_m(t)$ are well-known Legendre polynomials of order m with the aid of the following recurrence formulas:

$$\begin{aligned} &p_0(t) = 1, \\ &p_1(t) = t, \\ &p_{m+1}(t) = t\left(\frac{2m+1}{m+1}\right)p_m(t) - \left(\frac{m}{m+1}\right)p_{m-1}(t) \quad m = 1, 2, 3, \ldots. \end{aligned}$$

3. Function approximation

A function $f(t)$ defined over [0,1) may be expanded using Legendre wavelet as

$$f(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t), \tag{3.1}$$

where $c_{nm} = \langle f(t), \psi_{nm}(t)\rangle$, and $\langle ., .\rangle$ denotes the inner product. If the infinite series (3.1) is truncated, then it can be written as

$$f(t) \cong \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\Psi(t), \tag{3.2}$$

where C and $\Psi(t)$ are $2^{k-1}M \times 1$ matrices given by

$$C = [c_{10}, c_{11}, \dots, c_{1M-1}, c_{20}, \dots, c_{2M-1}, \dots, c_{2^{k-1}0}, \dots, c_{2^{k-1}M-1}]^T, \tag{3.3}$$

$$\Psi = [\psi_{10}, \psi_{11}, \dots, \psi_{1M-1}, \psi_{20}, \dots, \psi_{2M-1}, \dots, \psi_{2^{k-1}0}, \dots, \psi_{2^{k-1}M-1}]^T. \tag{3.4}$$

Also a two variables function $h(t, s) \in L^2([0, 1) \times [0, 1))$ can be written as:

$$h(t, s) \simeq \Psi^T(t)H\Psi(s),$$

where H is $2^{k-1}M \times 2^{k-1}M$ matrix with

$$H_{ij} = (\psi_i(t), (h(t, s), \psi_j(s))).$$

3.1. **The operational matrices of derivative.** The differentiation of vectors Ψ in (3.4) can be expressed as

$$\Psi'(x) = D\Psi(x), \tag{3.5}$$

where D is operational matrix of derivative for Legendre wavelets. If we assume that $k = 1$, vectors Ψ and differential of vectors Ψ can be written as follows

$$\Psi(t) = [\psi_{10}(t), \psi_{11}(t), \dots, \psi_{1M-1}(t)], \tag{3.6}$$

$$\Psi'(t) = [\psi'_{10}(t), \psi'_{11}(t), \dots, \psi'_{1M-1}(t)]. \tag{3.7}$$

The matrix D can be obtained by the following process:

$$d(i, j) = \int_0^1 \psi'_{1i}(t)\psi_{1j}(t)dt, \quad j = 1, \dots, i - 1 \tag{3.8}$$

$$d(i, j) = 0, \quad j \geq i. \tag{3.9}$$

4. Applying the method

The Legendre wavelet method terminology is used in [11] in 2001 by Razzaghi et al. Also, many authors, e.g. [10,17–19] have handled Legendre wavelets for the solution of varieties of differential and integral equations. Venkatesh et al. in [6, 12–16] applied Legendre wavelets for the solution of initial value problems of Bratu-type, Higher order Volterra integro-differential equations, Cauchy problems of first order partial differential, advection problems and they gave the theoretical analysis of Legendre wavelets method for the solution of second kind Fredholm integral equations. Also, in [6] Legendre wavelet method used for numerical solutions of partial differential equation.

In the present article, we are concerned with the application of Legendre wavelets to find the approximate solution of Eq. (1.1). The Legendre wavelet method (LWM) consists of reducing the given integral equations to a system of simultaneous nonlinear equations. The properties of Legendre wavelets are utilized to evaluate the unknown coefficients and find an approximate solution to the equation.

The unknown function $\phi(x)$ of (1.1) with $\phi(\pm 1) = 0$, can be represented in the form

$$\phi(x) = \sqrt{1-x^2}g(x), \quad -1 \leq x \leq 1,$$

where $g(x)$ is a well behaved function of x in the interval $-1 \leq x \leq 1$. To find an approximate solution of (1.1), $g(x)$ is approximated using Legendre wavelets in the interval $[-1, 1]$ as

$$g(x) = \sum_{j=0}^{2^{k-1}M-1} c_j \psi_j(x), \tag{4.1}$$

with considering $k = 1$ we can write

$$g(x) = C^T \Psi(x), \tag{4.2}$$

where

$$C^T = [c_0, c_1, \ldots, c_{M-1}], \tag{4.3}$$

$$\Psi^T = [\psi_0, \psi_1, \ldots, \psi_{M-1}], \tag{4.4}$$

and with the above notification we have following expression for the $f(x)$ as

$$f(x) = F^T \Psi(x), \tag{4.5}$$

where

$$F^T = [f_0, f_1, \ldots, f_{M-1}]. \tag{4.6}$$

Substituting (4.1) in (1.1), we get

$$2\frac{d}{dx}\left((1-x^2)^{\frac{1}{2}}g(x)\right) + \int_{-1}^{1} \frac{(1-t^2)^{\frac{1}{2}}g(t)}{t-x}dt = f(x). \tag{4.7}$$

Define vector V as follows

$$V = \left[\int_{-1}^{1} \sqrt{1-t^2}\frac{\psi_0(t)}{t-x}dt, \ldots, \int_{-1}^{1} \sqrt{1-t^2}\frac{\psi_{M-1}(t)}{t-x}dt\right]^T. \tag{4.8}$$

Now Eq. (4.7) can be written in matrix form

$$2\sqrt{1-x^2}C^T D\Psi(x) - C^T \frac{x}{\sqrt{1-x^2}}\Psi(x) + \lambda C^T V = F^T \Psi(x), \tag{4.9}$$

where D is the operational matrix of derivative. We collocate Eq.(4.9) in M collocation points, thus we have M equations and M unknown coefficients c_j:

$$2\sqrt{1-x_i^2}C^T D\psi(x_i) - C^T \frac{x_i}{1-x_i^2}\psi(x_i) + \lambda C^T V(x_i) + \frac{x_i}{2} = 0, \tag{4.10}$$

for $i = 0, \dots, M-1$. We solve this M nonlinear equations by the Newton iterative method. Then the unknown coefficients c_j can be found, so the problem is solved numerically.

5. Numerical results

In order to illustrate the performance of our method in solving integro-differential equations, we consider the following examples. The solution of the examples are obtained for different values of N.

Example 1. Here we solve Eq. (4.9) for $\lambda = 1$ and $f(x) = -\frac{x}{2}$. Using these coefficients the value of $f(x)$ at $x = (0.2)k, \quad k = 0, 1, \cdots, 5$, are presented in Table 1. For a comparison between the present method and that of the method used in [3], values of $\phi(x)$ at these points obtained by Frankel [3] are also given. It is obvious that the result compares favorably with the results of Frankel and also those obtained by Chakrabarti and Hamsapriye [2] and Mandal and Bera [7]. It was further observed that by increasing M the accuracy of the result increases.

Table 1. Numerical results for example 1

	0	0.2	0.4	0.6	0.8	1
n=6	0.06982	0.0669	0.06007	0.04718	0.02876	0
n=8	0.06976	0.06745	0.06002	0.04718	0.02795	0
n=10	0.06966	0.06703	0.06053	0.04717	0.02850	0
frankel	0.06950	0.06712	0.05984	0.04718	0.02891	0

Example 2. Consider the following integro-differnential equation [8]:

$$\phi^{'}(s) + \frac{1}{\pi^5}\int_{-1}^{1}\frac{\phi(t)}{t-s}dt = \frac{2(1+\pi^5)s}{\pi 5} + \frac{(s^2-1)}{\pi^5}\ln\left(\frac{1-s}{1+s}\right), \tag{5.1}$$

with the exact solution $\phi(s) = s^2 - 1$. Table 2 gives the numerical results for this example.

Table 2. Numerical results for example 2

	0	0.2	0.4	0.6	0.8	1
n=6	-1	-0.9400	-0.8423	-0.6411	-0.3719	0
n=8	-0.9938	-0.9590	-0.8365	-0.6396	-0.3589	0
n=10	-1.018	-0.9613	-0.8407	-0.6407	-0.3607	0
exact valuee	-1	-0.96	-0.84	-0.64	-0.36	0

Example 3. Consider the following integro-differential equation:

$$\frac{d\phi}{ds} + \int_{-1}^{1}\frac{\phi(t)}{t-s}dt = (s^2+2s-1)e^s + e^{1-s} - e^{-1-s} \tag{5.2}$$

TABLE 3. Numerical results for example 3

	0	0.2	0.4	0.6	0.8	1
n=6	-1.321	-1.163	-1.067	-0.9082	-0.6369	0
n=8	-1.4	-1.224	-1.128	-0.9777	-0.7566	0
n=10	-1.17	-1.1875	-1.213	-1.096	0.8192	0
exact valuee	-1	-1.1725	-1.2531	-1.1662	-0.8012	0

with the exact solution $\phi(s) = (s^2 - 1)e^s$. Table 3 gives the numerical results of the method.

These examples show the efficiency and accuracy of the method. Table 2 and Table 3 show the exact value of $f(x)$ at $s = (0.2)k, k = 0, 1, \cdots, 5$, and the values of ϕ at these points obtained by presented method.

REFERENCES

[1] V. Alexandrov and E. Kovalenko, *Problems with mixed boundry conditions in continuum mechanics*, Nauka, Moscow, 1986.

[2] A. Chakrabarti and Hamsapriye, *Numerical solution of a singular integro-differential equation*, ZAMM Z. Angew. Math. Mech., *79* (1999), 233–241.

[3] J. I. Frankel, *A Galerkin solution to a regularized Cauchy singular integro-differential equation*, Quart. Appl. Math., *53* (1995), 245–258.

[4] M. Hori and S. Nemat-Nasser, *Asymptotic solution of a class of strongly singular integral equations*, SIAM J. Appl. Math., *50* (1990), 716–725. DOI 10.1137/0150042.

[5] A. C. Kaya and F. Erdogan, *On the solution of integral equations with strongly singular kernels*, Quart. Appl. Math., *45* (1987), 105–122.

[6] N. Liu and E.-B. Lin, *Legendre wavelet method for numerical solutions of partial differential equations*, Numer. Methods Partial Differential Equations, *26* (2010), 81–94. DOI 10.1002/num.20417.

[7] B. N. Mandal and G. H. Bera, *Approximate solution of a class of singular integral equations of second kind*, J. Comput. Appl. Math., *206* (2007), 189–195. DOI 10.1016/j.cam.2006.06.011.

[8] A. Mennouni, *The iterated projection method for integro-differential equations with Cauchy kernel*, J. Appl. Math. Inform., *31* (2013), 661–667. DOI 10.14317/jami.2013.661.

[9] N. I. Muskhelishvili, *Singular integral equations. 1953*, Noordhoff, Groningen.

[10] M. Razzaghi and S. Yousefi, *Legendre wavelets method for the solution of nonlinear problems in the calculus of variations*, Math. Comput. Modelling, *34* (2001), 45–54. DOI 10.1016/S0895-7177(01)00048-6.

[11] M. Razzaghi and S. Yousefi, *The Legendre wavelets operational matrix of integration*, Internat. J. Systems Sci., *32* (2001), 495–502. DOI 10.1080/002077201300080910.

[12] S. Venkatesh, S. Ayyaswamy, and S. Raja Balachandar, *Legendre approximation solution for a class of higher-order volterra integro-differential equations*, Ain Shams Engineering Journal, *3* (2012), 417–422.

[13] S. Venkatesh, S. Ayyaswamy, and S. Raja Balachandar, *Legendre wavelets based approximation method for solving advection problems*, Ain Shams Engineering Journal, *4* (2013), 925–932.

[14] S. G. Venkatesh, S. K. Ayyaswamy, and S. Raja Balachandar, *Convergence analysis of Legendre wavelets method for solving Fredholm integral equations*, Appl. Math. Sci. (Ruse), *6* (2012), 2289–2296.

[15] S. G. Venkatesh, S. K. Ayyaswamy, and S. Raja Balachandar, *The Legendre wavelet method for solving initial value problems of Bratu-type*, Comput. Math. Appl., *63* (2012), 1287–1295. DOI 10.1016/j.camwa.2011.12.069.

[16] S. G. Venkatesh, S. K. Ayyaswamy, S. Raja Balachandar, and K. Kannan, *Legendre wavelets based approximation method for Cauchy problems*, Appl. Math. Sci. (Ruse), *6* (2012), 6281–6286.

[17] S. Yousefi and M. Razzaghi, *Legendre wavelets method for the nonlinear Volterra-Fredholm integral equations*, Math. Comput. Simulation, *70* (2005), 1–8. DOI 10.1016/j.matcom.2005.02.035.

[18] S. A. Yousefi, *Legendre wavelets method for solving differential equations of Lane-Emden type*, Appl. Math. Comput., *181* (2006), 1417–1422. DOI 10.1016/j.amc.2006.02.031.

[19] S. A. Yousefi, *Legendre scaling function for solving generalized Emden-Fowler equations*, Int. J. Inf. Syst. Sci., *3* (2007), 243–250.