Research Paper
Computational Methods for Differential Equations
http://cmde.tabrizu.ac.ir
Vol. 12, No. 2, 2024, pp. 207-225
DOI:10.22034/cmde.2023.56140.2345

Prolific new M-fractional soliton behaviors to the Schrödinger type Ivancevic option pricing model by two efficient techniques

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Abstract

The principal purpose of this research is to study the M-fractional nonlinear quantum-probability grounded Schrödinger kind Ivancevic option pricing model (IOPM). This well-known economic model is an alternative of the standard Black–Scholes pricing model which represents a controlled Brownian motion in an adaptive setting with relation to nonlinear Schrodinger equation. The exact solutions of the underlying equation have been derived through the well-organized extended modified auxiliary equation mapping and generalized exponential rational function methods. Different forms of optical wave structures including dark, bright, and singular solitons are derived. To the best of our knowledge, verified solutions using Maple are new. The results obtained will contribute to the enrichment of the existing literature of the model under consideration. Moreover, some sketches are plotted to show more about the dynamic behavior of this model.

Keywords. The extended modified auxiliary equation mapping method, The generalized exponential rational function method, Exact solutions.
2010 Mathematics Subject Classification. 97M30,34G20,93E35.

1. Introduction

Today, the study of physical, chemical, and biological phenomena is very popular. Many different effective and reliable methods are used to solve these phenomena such as the Lie symmetry analysis method [29, 30, 37–39], the generalized exponential rational function method [15, 22, 28, 40], the extended rational sine–cosine method [27], the improved tan $\left(\frac{\Omega(\Upsilon)}{2}\right)$-expansion method [3], the modified Kudryashov method [18, 46], the Jacobi elliptic function method [6, 47], the extended tanh expansion method [36], the modified Khater method [48, 49], the collocation method [5], the Hirota bilinear approach [21], generalized Kudryashov method [2], the $(\frac{G'}{G'+G+A})$-expansion technique [32], the new auxiliary equation method [35], the $(\frac{G'}{G}, \frac{1}{G})$-expansion method [43]. Another area of interest as well as fluid dynamics, plasmas, geochemistry, optical fiber, quantum field theory, and biophysics, and of interest to many experts and non-experts, is economics and finance. Financial problems have been tried to be explained and researched using scientific norms so that both ordinary users and experts can get the maximum benefit and observe the global market more easily. Modeling the problems that arise in a global financial market creates a dynamic information system. This system creates a resource to be used for deeper research. In this context, in recent years, the detailed examination of mathematical models describing financial and economic problems has become inevitable due to its wide application area. The well-known Black–Scholes (BS) model is an important improvement in finance mathematics. In 1970's, F. Black and M. Scholes presented this model to estimate pricing options [7]. It defines the time-evolution of the market value of a stock option [7, 23, 31]. The price function $S = S(t)$, $0 \le t \le T$, supplies the stochastic differential equation describing geometric Brownian motion

$$dS = S(\mu dt + \sigma dW_t), \quad S \in [0, \infty), \tag{1.1}$$

Received: 10 April 2023 ; Accepted: 23 August 2023.
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where μ is the drift parameter, σ is volatility and W_t is the standard Wiener process. Volatility can be defined as a statistical measure of the dispersion of returns for a given security or market index. Volatility can either be measured by using the standard deviation or variance between returns from that same security or market index. Commonly, the higher the volatility, the riskier the security [1]. In context of financial stochastic processes, the Brownian motion is also described as the Wiener Process that is a continuous stochastic process with normally distributed increments. When the market is modeled with a standard Brownian Motion, the probability distribution function of the future price is a normal distribution [33]. Brownian motion is the random motion of particles suspended in a liquid or gas, or the mathematical model used to describe such random motion, and is also used as a suitable tool for modeling motion in the stock market.

For a function $V = V(S,t)$ defined on the domain $0 \leq S < \infty, 0 \leq t \leq T$ and describing the market value of a stock option with the stock (asset) price S, the BS model is given as:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{1.2}$$

where r is the risk-free interest rate with $V(0,t) = 0$, $V(S,t) \to \infty$ as $S \to \infty$, $V(S,t) = max(S - E)$, the parameter $\sigma \in \mathbb{R}$, $\sigma > 0$ shows the volatility of stock returns E is taken as a constant and the price function is

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}. \tag{1.3}$$

Primarily used for pricing European call and put options [19], the BS model is based on the assumption that μ and σ are constants. This assumption limits this model as it does not take into account long-observed properties of the implied volatility surface, like volatility smiley and skew, which show that implied volatility changes with strike price and expiration [24]. On the other hand, in this model it is possible to more accurately simulate derivatives, assuming that the volatility of the underlying price is a stochastic process rather than a constant. [13]. Classical and quantum mechanics are used to explain the variation of physical systems w.r.t. time (t). It has position and momentum information about all particles to describe the classical mechanical system. However, in quantum mechanics, the data about the system is included in the solution of a Schrödinger equation (SE). This equation produces the wave function $q(x,t)$. The trajectories of pricing dynamics or changes can be observed as trajectories expressed by the mathematical equations of quantum mechanics. This is thanks to the applications of quantum mechanics in financial markets. The well-known linear SE for free particles is expressed for the wave function $q(x,t)$ as:

$$iq_t = -h/2mq_{xx}, \tag{1.4}$$

where m is the mass of the particle and h is the Planck constant. To explain the linear SE with an external potential $V = V(x,t)$, the dynamics of the wave function $q(x,t)$ can be presented as:

$$iq_t = -\frac{h}{2m}q_{xx} + Vq, \tag{1.5}$$

and the nonlinear SE (NLSE) is most commonly expressed as:

$$iq_t = -\frac{h}{2m}q_{xx} + Vq + \beta|q|^2 q, \tag{1.6}$$

where $|q|^2$ gives the probability density for finding the particle at position x. $V = V(x,t)$ is the external potential. In order to explain the behavior of economic systems, it is possible to describe an economic system as a physical system with the SE over the wave function solution (see[25]). Vukovic [51] improved a link between two equations, namely the BS model equation and the SE through quantum physics of the Hamilton operator. It has also been analyzed that the BS equation can be obtained by applying the emerging tools in quantum mechanics from SE [10]. As with the BS option pricing model, the Brownian motion can also be used to derive the IOPM [13]. This model, which explains the controlled the Brownian motion of the financial markets, is a non-linear adaptive wave model and is also a wave alternative to the standard BS option pricing model [8, 23, 52]. IOPM is given with

$$i\frac{\partial w}{\partial t} + \frac{1}{2}\sigma\frac{\partial^2 w}{\partial S^2} + \beta|w|^2 w = 0, \quad i^2 = -1. \tag{1.7}$$

The complex-valued equation was first time developed by V. G. Ivancevic [23], in which the $w(S,t)$ is the option-pricing wave function at time t, $0 \leq t < T$. S, $0 \leq S < \infty$ is used to explain the asset price of a product, σ means the volatility, which demonstrates either a stochastic process itself or just a constant, β is cited as the Landau coefficient representing adaptive market potential, $|w|^2$ denotes the probability density function for the option price and is used to represent the potential field. Edeki et al. obtained analytical solutions of (1.7) using projected differential transform method [11] and He's amplitude frequency formulation [12]. In [9], using the tanh expansion method and the trial function method, dark and rogue wave solutions were found for Eq. (1.7). Chen et al. [8] applied the rational sine-Gordon expansion and modified exponential method to Eq. (1.7) and observed the modulation instability analysis.
In this work, the truncated M-fractional IOPM is discussed. This model is given as follows [34]:

$$iD^{\epsilon,\theta}_{M,t}q + \frac{\delta}{2}D^{2\epsilon,\theta}_{M,2s}q + \Omega q\,|q|^2 = 0, \quad i = \sqrt{-1}, \tag{1.8}$$

where $q = q(s,t)$ shows the option price wave profile, t is the time variable and s is the asset price of the model. δ represents the volatility. Ω is cited to the Landau coefficient which describes adaptive market potential. $|q|^2$ shows the probability density function which denotes the potential field.

In recent years, modeling of real world problems in many scientific and engineering fields using fractional differential equations has become very popular. Besides that, modeling problems in finance and economics using fractional differential equations are also remarkably significant. The advantage of the fractional derivative is that it has nonlocal property. The value of the current state depends on both recent values and historical values of the objective function. This excellent property is suitable for modeling many financial variable series, mainly because of the fact that the financial and economic variable series always exhibit time-dependent memory effect, such as interest rate, stock price, exchange amount of the future, and so on [18–21]. these features make the model we are under considering worth examining [26, 53].

This work is constructed as follows: In section 2, the basic definitions and properties of the fractional derivative operator are given. In section 3, The extended modified auxiliary equation mapping and generalized exponential rational function methods are comprehensively reviewed. In section 4, the governing equation is mentioned. In section 5, proposed efficient techniques are implemented to the underlying equation. In section 6, some graphical representations are given. Section 7 is devoted to discussion part. Finally, we provide the conclusion section in section 8.

2. Truncated M-fractional derivative operator

Definition 2.1. Assume that $f(t) : [0,\infty) \to \mathbb{R}$, then truncated M-fractional derivative of f of order ϵ is given [20, 44, 45]:

$$D^{\epsilon,\theta}_{M,t}f(t) = \lim_{\tau\to 0}\frac{f\left(tE_\theta\left(\tau t^{1-\epsilon}\right)\right) - f\left(t\right)}{\tau},\ 0 < \epsilon < 1, \theta > 0, \tag{2.1}$$

where $E_\theta(.)$ demonstrates truncated Mittag Leffler function of one parameter that is presented as [50]:

$$E_\theta\left(z\right) = \sum_{j=0}^{i}\frac{z^j}{\Gamma\left(\theta j + 1\right)},\ \theta > 0 \text{ and } z \in \mathbb{C}. \tag{2.2}$$

Theorem 2.2. *[45]: If $\epsilon \in (0,1], \theta > 0, m, n \in \mathbb{R}$, and g, h are $\epsilon-$differentiable at $t > 0$, then:*

- ***(i)*** $D^{\epsilon,\theta}_{M,t}\left(mg(t) + nh(t)\right) = mD^{\epsilon,\theta}_{M,t}\left(g(t)\right) + nD^{\epsilon,\theta}_{M,t}\left(h(t)\right).$
- ***(ii)*** $D^{\epsilon,\theta}_{M,t}\left(g(t).h(t)\right) = g(t)D^{\epsilon,\theta}_{M,t}\left(h(t)\right) + h(t)D^{\epsilon,\theta}_{M,t}\left(g(t)\right).$
- ***(iii)*** $D^{\epsilon,\theta}_{M,t}\left(\frac{g(t)}{h(t)}\right) = \frac{h(t)D^{\epsilon,\theta}_{M,t}(g(t)) - g(t)D^{\epsilon,\theta}_{M,t}(h(t))}{(h(t))^2}.$
- ***(iv)*** $D^{\epsilon,\theta}_{M,t}\left(C\right) = 0$, *where C is a constant.*
- ***(v)*** *If $g(t)$ is differentiable, then* $D^{\epsilon,\theta}_{M,t}\left(g(t)\right) = \frac{t^{1-\epsilon}}{\Gamma(\theta+1)}\frac{dg(t)}{dt}.$

3. Methodologies

Here, the main phases of two effective and popular methods to be used to search for solutions of the equation under consideration will be summarized. For this, consider the following partial differential equation (PDE)

$$\Delta(w, w_t, w_x, w_{tt}, w_{tt}, ...) = 0, \tag{3.1}$$

where Δ is a polynomial in dependent function w and its corresponding derivatives. Using the traveling wave transformation $w = w(\xi), \xi = k(x - ct)$, where c is a constant, Eq. (3.1) can be turned into an ordinary differential equation (ODE)

$$\Theta(w, w', w'', ...) = 0, \tag{3.2}$$

where $(.)' = \frac{d}{d\xi}(.)$.

3.1. **The extended modified auxiliary equation mapping method.** Exact solutions of the Eq. (3.2) can be constructed as [4, 41, 42]:

$$u(\xi) = \sum_{j=0}^{m} a_j \mathcal{F}^j(\xi) + \sum_{j=-1}^{-m} b_{-j} \mathcal{F}^j(\xi) + \sum_{j=2}^{m} c_j \mathcal{F}^{j-2}(\xi) \mathcal{F}'(\xi) + \sum_{j=1}^{m} d_j \left(\frac{\mathcal{F}'(\xi)}{\mathcal{F}(\xi)}\right)^j, \tag{3.3}$$

where a_j, b_j, c_j, d_j are unknown constants. $F(\xi)$ holds the following auxiliary ODE:

$$\mathcal{F}'^2 = \left(\frac{dF}{d\xi}\right)^2 = \mu_1 \mathcal{F}^2(\xi) + \mu_2 \mathcal{F}^3(\xi) + \mu_3 \mathcal{F}^4(\xi), \tag{3.4}$$

where μ_1, μ_2, and μ_3 are arbitrary constants. Various solutions according to μ_1, μ_2, and μ_3 are given in the Table 2 in the Appendix . The homogeneous balance between the leading terms gives us the value m. Putting Eq. (3.3) with Eq. (3.4) into ODE and gathering coefficients of $\mathcal{F}^j(\xi)(\mathcal{F}'(\xi))^p$ $(p = 0, 1; j = 0, 1, 2, ..., m)$, by matching them to zero, yields a system of algebraic equations. Solving this system, we obtained a_j, b_j, c_j, and d_j. By inserting all the values of constants into Eq. (3.3), we attain the required solutions of considered equation.

3.2. **The generalized exponential rational function method (GERFM).** Exact solutions of the Eq. (3.2) can be constructed as [14, 16, 17]:

$$u(\xi) = A_0 + \sum_{k=1}^{N} A_k \phi(\xi)^k + \sum_{k=1}^{N} B_k \phi(\xi)^{-k}, \tag{3.5}$$

where

$$\phi(\xi) = \frac{p_1 e^{\theta_1 \xi} + p_2 e^{\theta_2 \xi}}{p_3 e^{\theta_3 \xi} + p_4 e^{\theta_4 \xi}}. \tag{3.6}$$

Here p_j, θ_j $(1 \leq j \leq 4)$, A_0, A_i and B_i $(i = 1, ..., N)$ are constants. The homogeneous balance rule can be used to determine the positive integer N. Inserting Eq. (3.5) into Eq. (3.2) and gathering all terms, Eq. (3.2) gives an algebraic equation $P(\xi, e^{\theta_1\xi}, e^{\theta_2\xi}, e^{\theta_3\xi}, e^{\theta_4\xi}) = 0$. Equalizing coefficients of powers of P to 0, a system in terms of A_0, A_i and B_i and p_j, θ_j is yielded. Solving the set of equations by use of Maple, the values of A_0, A_i and B_i and p_j, θ_j are determined. Plugging obtained values in the Eq. (3.5), the soliton solutions of Eq. (3.1) are obtained.

4. The governing equation

Let us suppose the travelling wave transformation given as follows;

$$q(s,t) = u(\xi) e^{i\left(\mu \frac{\Gamma(\theta+1)}{\epsilon} s^\epsilon + \rho \frac{\Gamma(\theta+1)}{\epsilon} t^\epsilon\right)}, \tag{4.1}$$

$$\xi = \lambda \frac{\Gamma(\theta+1)}{\epsilon} s^\epsilon + \tau \frac{\Gamma(\theta+1)}{\epsilon} t^\epsilon, \tag{4.2}$$

where $u(\xi)$ shows the amplitude of wave function while ρ and τ represent the time velocity. Parameters μ and λ are obtaining from asset price of the product. Inserting Eq.(4.1) with (4.2) into Eq. (1.8), result in the form of real and imaginary parts given as follows:
Real part:

$$2\Omega u^3 + \delta\lambda^2 u'' - \left(\delta\mu^2 + 2\rho\right) u = 0. \tag{4.3}$$

Imaginer part:

$$\left(\tau + \lambda\delta\mu\right) u' = 0. \tag{4.4}$$

From Eq. (4.4), we get the velocity of wave function as $\tau = -\lambda\delta\mu$. Balancing u'' with u^3 in Eq. (4.3) gives $3m = m+2$ and $m = 1$. Now we will gain the exact solutions of Eq. (4.3) by using above mentioned methods.

5. Exact solutions of truncated M-fractional IOPM

In this section, the extended modified auxiliary equation mapping and the generalized exponential rational function approaches, briefly summarized in section 3, will be applied to the model under consideration, i.e Eq. (1.8).

5.1. **Application of extended modified auxiliary equation mapping method.** This subsection is dedicated to solving Eq. (4.3) using the generalized auxiliary equation method formulated in section 3.1. According to (3.3), we search the solution of Eq. (4.3) as

$$u = a_0 + a_1\,\mathcal{F}(\xi) + b_1/\mathcal{F}(\xi) + d_1\,\mathcal{F}'(\xi)/\mathcal{F}(\xi), \tag{5.1}$$

where a_0, a_1, b_1 and d_1 are constants. Here, $\mathcal{F}(\xi)$ is satisfying the following auxiliary ODE with its derivatives:

$$\begin{aligned}
\mathcal{F}'' &= \mu_1\mathcal{F}(\xi) + \frac{3}{2}\mu_2\mathcal{F}^2(\xi) + 2\mu_3\mathcal{F}^3(\xi), \\
\mathcal{F}''' &= \left(\mu_1 + 3\mu_2\mathcal{F}(\xi) + 6\mu_3\mathcal{F}^2(\xi)\right)\mathcal{F}'(\xi).
\end{aligned}$$

Inserting Eq. (5.1) with Eq. (3.4) into Eq. (4.3), and by summing all coefficients of $\mathcal{F}^j(\xi)\left(\mathcal{F}'(\xi)\right)^p$ $(p = 0, 1 \quad \text{and} \quad j = 0, 1, ..., m)$, and equating them to zero, gives an algebraic system. Maple computer package can be used to analyze the obtained system. Hence the solution of this system leads to the following one case:
Set 1:

$$\left\{a_0 = 0, a_1 = \sqrt{\frac{\delta\,\mu^2\mu_3 + 2\,\rho\,\mu_3}{2\Omega\,\mu_1}}, b_1 = 0, d_1 = \sqrt{\frac{\delta\,\mu^2 + 2\,\rho}{2\Omega\,\mu_1}}, \lambda = \sqrt{-\frac{2\,\delta\,\mu^2 + 4\,\rho}{\delta\,\mu_1}}.\right\} \tag{5.2}$$

In Set 1, eight situations occur according to the signs of a_1 , d_1 and λ. For these three coefficients, only the positive case will be considered. Since the remaining cases are similar, we omitted them. Inserting all these values in (5.2) into Eq. (5.1) together with Eq. (4.1) and (4.2), we yield the following solutions using proposed method.

$$\begin{aligned}
q_1(s,t) &= -\sqrt{\frac{\delta\,\mu^2 + 2\,\rho}{\Omega\,\mu_1}}\,\frac{\sqrt{2}}{2\mu_1\left(-{\mu_2}^2 + \mu_1\mu_3\left(\tanh\left(\frac{\sqrt{\mu_1}\xi}{2}\right) + 1\right)^2\right)} \\
&\quad\times\left(-\sqrt{\mu_3}{\mu_1}^2\mu_2\left(sech\left(\frac{\sqrt{\mu_1}\xi}{2}\right)\right)^2 - \mu_1\sqrt{\mu_1}\tanh\left(\frac{\sqrt{\mu_1}\xi}{2}\right){\mu_2}^2\right. \\
&\quad\left.+\left(\sqrt{\mu_1}\right)^5\mu_3\left(\tanh\left(\frac{\sqrt{\mu_1}\xi}{2}\right) + 1\right)^2\right)e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon} + \rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)},
\end{aligned} \tag{5.3}$$

where $\mu_1 > 0$, ξ is given in Eq. (4.2) and $\tau = -\lambda\delta\mu$.

$$\begin{aligned} q_2(s,t) &= -\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega\,\mu_1}}\frac{\sqrt{2}}{2\left(-{\mu_2}^2+\mu_1\,\mu_3\left(1+\coth\left(\frac{\sqrt{\mu_1}\xi}{2}\right)\right)^2\right)\mu_1} \\ &\times\left(\sqrt{\mu_3}{\mu_1}^2\mu_2\left(csch\left(\frac{\sqrt{\mu_1}\xi}{2}\right)\right)^2-{\mu_1}^{3/2}\coth\left(\frac{\sqrt{\mu_1}\xi}{2}\right){\mu_2}^2\right. \\ &\left.+\left(\sqrt{\mu_1}\right)^5\mu_3\left(\coth\frac{1}{2}\xi\sqrt{\mu_1}+1\right)^2\right)e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^\epsilon+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^\epsilon\right)}, \end{aligned} \tag{5.4}$$

where $\mu_1>0$, ξ is given in Eq. (4.2) and $\tau=-\lambda\delta\mu$.

$$\begin{aligned} q_3(s,t) &= -\frac{\sqrt{2}}{2\left(-\sqrt{{\mu_2}^2-4\,\mu_1\,\mu_3}+\mu_2\,sech\left(\sqrt{\mu_1}\xi\right)\right)\mu_1} \\ &\times\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega\,\mu_1}}\left(2\sqrt{\mu_3}\,{\mu_1}^2 sech\left(\sqrt{\mu_1}\xi\right)-{\mu_1}^{3/2}\tanh\left(\sqrt{\mu_1}\xi\right)\sqrt{{\mu_2}^2-4\,\mu_1\,\mu_3}\right) \\ &\times e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^\epsilon+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^\epsilon\right)}, \end{aligned} \tag{5.5}$$

where $\mu_1>0$, $\mu_2^2-4\mu_1\mu_3>0$, ξ is given in Eq. (4.2) and $\tau=-\lambda\delta\mu$.

$$\begin{aligned} q_4(s,t) &= -\frac{\sqrt{2}}{2\left(-\sqrt{{\mu_2}^2-4\,\mu_1\,\mu_3}+\mu_2\,\sec\left(\sqrt{-\mu_1}\xi\right)\right)} \\ &\times\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega\,\mu_1}}\left(2\sqrt{\mu_3}\mu_1\,\sec\left(\sqrt{-\mu_1}\xi\right)+\tan\left(\sqrt{-\mu_1}\xi\right)\sqrt{-\mu_1}\sqrt{{\mu_2}^2-4\,\mu_1\,\mu_3}\right) \\ &\times e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^\epsilon+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^\epsilon\right)}, \end{aligned} \tag{5.6}$$

where $\mu_1<0$, $\mu_2^2-4\mu_1\mu_3>0$, ξ is given in Eq. (4.2) and $\tau=-\lambda\delta\mu$.

$$\begin{aligned} q_5(s,t) &= -\frac{\sqrt{2}}{2\mu_1\left(\mu_2+2\sqrt{\mu_1\,\mu_3}\tanh\left(\frac{\sqrt{\mu_1}\xi}{2}\right)\right)} \\ &\times\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega\,\mu_1}}\left(\sqrt{\mu_3}{\mu_1}^2\left(sech\left(\frac{\sqrt{\mu_1}\xi}{2}\right)\right)^2\right. \\ &\left.+{\mu_1}^{3/2}\left(\tanh\left(\frac{\sqrt{\mu_1}\xi}{2}\right)\mu_2+\sqrt{\mu_1\,\mu_3}\left(\tanh\left(\frac{\sqrt{\mu_1}\xi}{2}\right)\right)^2+\sqrt{\mu_1\,\mu_3}\right)\right) \\ &\times e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^\epsilon+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^\epsilon\right)}, \end{aligned} \tag{5.7}$$

where $\mu_1 > 0$, $\mu_3 > 0$, ξ is given in Eq. (4.2) and $\tau = -\lambda\delta\mu$.

$$\begin{aligned} q_6(s,t) &= -\frac{\sqrt{2}}{2\left(\mu_2 + 2\sqrt{-\mu_1\,\mu_3}\tan\left(\frac{\sqrt{-\mu_1}\xi}{2}\right)\right)} \\ &\quad \times\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega\,\mu_1}}\left(\sqrt{\mu_3}\mu_1\left(\sec\left(\frac{\sqrt{-\mu_1}\xi}{2}\right)\right)^2 - \sqrt{-\mu_1}\tan\left(\frac{\sqrt{-\mu_1}\xi}{2}\right)\mu_2\right. \\ &\quad \left.-\sqrt{-\mu_1}\sqrt{-\mu_1\,\mu_3}\left(\tan\left(\frac{\sqrt{-\mu_1}\xi}{2}\right)\right)^2 + \sqrt{-\mu_1}\sqrt{-\mu_1\,\mu_3}\right) \\ &\quad \times e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon}+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \end{aligned} \tag{5.8}$$

where $\mu_1 < 0$, $\mu_2^2 - 4\mu_1\mu_3 > 0$, ξ is given in Eq. (4.2) and $\tau = -\lambda\delta\mu$.

$$\begin{aligned} q_7(s,t) &= \frac{\sqrt{2}}{2\mu_1\left(\mu_2 + 2\sqrt{\mu_1\,\mu_3}\coth\left(\frac{\sqrt{\mu_1}}{2}\xi\right)\right)}\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega\,\mu_1}} \\ &\quad \times\left(\left(csch\left(\frac{\sqrt{\mu_1}}{2}\xi\right)\right)^2{\mu_1}^2\sqrt{\mu_3} - {\mu_1}^{3/2}\coth\left(\frac{\sqrt{\mu_1}}{2}\xi\right)\mu_2\right. \\ &\quad \left.-{\mu_1}^{3/2}\sqrt{\mu_1\,\mu_3}\left(\coth\left(\frac{\sqrt{\mu_1}}{2}\xi\right)\right)^2 - {\mu_1}^{3/2}\sqrt{\mu_1\,\mu_3}\right) \\ &\quad \times e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon}+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \end{aligned} \tag{5.9}$$

where $\mu_1 > 0$, $\mu_3 > 0$, ξ is given in Eq. (4.2) and $\tau = -\lambda\delta\mu$.

$$\begin{aligned} q_8(s,t) &= \left(-\sqrt{\frac{\delta\,\mu^2\mu_3+2\,\rho\;\;\mu_3}{2\Omega\,\mu_1}}\mu_1\left(1+\tanh\left(\frac{\sqrt{\mu_1}\xi}{2}\right)\right){\mu_2}^{-1}\right. \\ &\quad \left.+\sqrt{\frac{\delta\,\mu^2+2\,\rho}{2\Omega\,\mu_1}}\frac{\sqrt{\mu_1}\left(1-\left(\tanh\left(\frac{\sqrt{\mu_1}\xi}{2}\right)\right)^2\right)}{2\left(1+\tanh\left(\frac{\sqrt{\mu_1}\xi}{2}\right)\right)}\right)e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon}+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \end{aligned} \tag{5.10}$$

where $\mu_1 > 0$, $\mu_2^2 - 4\mu_1\mu_3 = 0$, ξ is given in Eq. (4.2) and $\tau = -\lambda\delta\mu$.

$$\begin{aligned} q_9(s,t) &= \left(-\sqrt{\frac{\delta\,\mu^2\mu_3+2\,\rho\,\mu_3}{2\Omega\,\mu_1}}\mu_1\left(1+\coth\left(\frac{\sqrt{\mu_1}\xi}{2}\right)\right){\mu_2}^{-1}\right. \\ &\quad \left.+\sqrt{\frac{\delta\,\mu^2+2\,\rho}{2\Omega\,\mu_1}}\frac{\sqrt{\mu_1}\left(1-\left(\coth\left(\frac{\sqrt{\mu_1}\xi}{2}\right)\right)^2\right)}{2\left(1+\coth\left(\frac{\sqrt{\mu_1}\xi}{2}\right)\right)}\right) \\ &\quad \times e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon}+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \end{aligned} \tag{5.11}$$

where $\mu_1 > 0$, $\mu_2^2 - 4\mu_1\mu_3 = 0$, ξ is given in Eq. (4.2) and $\tau = -\lambda\delta\mu$.

$$\begin{aligned} q_{10}(s,t) &= -\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega\,\mu_1}}\frac{\left(-4\sqrt{\mu_3}{\mu_1}^2e^{\sqrt{\mu_1}\xi} + {\mu_1}^{3/2}\left(e^{\sqrt{\mu_1}\xi}\right)^2 - {\mu_1}^{3/2}{\mu_2}^2 + 4\,\mu_3\,{\mu_1}^{5/2}\right)}{2\sqrt{2}\mu_1\left(\left(e^{\sqrt{\mu_1}\xi}\right)^2 - 2\,e^{\sqrt{\mu_1}\xi}\mu_2 + {\mu_2}^2 - 4\,\mu_1\,\mu_3\right)} \\ &\quad \times e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon}+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \end{aligned} \tag{5.12}$$

where $\mu_1 > 0$, ξ is given in Eq. (4.2) and $\tau = -\lambda\delta\mu$.

5.2. **Application of GERFM.** The proposed method assumes the solution of Eq. (4.3) in the form

$$u(\xi) = A_0 + A_1\phi(\xi) + \frac{B_1}{\phi(\xi)}. \tag{5.13}$$

Applying to the methodology presented in the Section 3.2, we get the following results:

Group 1: $p = [1-i, -1-i, -1, 1]$ and $\theta = [i, -i, i, -i]$ provides:

$$\phi(\xi) = \frac{-\sin(\xi) + \cos(\xi)}{\sin(\xi)}. \tag{5.14}$$

Set 1.1:

$$\left\{A_0 = \pm\frac{1}{2}\sqrt{-\frac{2\,\delta\,\mu^2 + 4\,\rho}{\Omega}}, A_1 = 0, B_1 = \pm\sqrt{-\frac{2\,\delta\,\mu^2 + 4\,\rho}{\Omega}}, \lambda = \sqrt{\frac{\delta\,\mu^2 + 2\,\rho}{2\delta}}.\right\} \tag{5.15}$$

The solution of Eq. (5.13) using Eq. (5.14) and (5.15) is given by

$$u_{1.1}(\xi) = \pm\frac{1}{2}\sqrt{-\frac{4\,\rho + 2\,\delta\,\mu^2}{\Omega}}\left(\sin(\xi) + \cos(\xi)\right)/\left(-\sin(\xi) + \cos(\xi)\right). \tag{5.16}$$

Then, the solution of Eq. (1.8) is obtained in the following form:

$$q_{1.1}(s,t) = \pm\frac{1}{2}\sqrt{-\frac{4\,\rho + 2\,\delta\,\mu^2}{\Omega}}\,\frac{\sin(\xi) + \cos(\xi)}{-\sin(\xi) + \cos(\xi)}e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon} + \rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \tag{5.17}$$

where ξ is given with (4.2).

Set 1.2:

$$\left\{A_0 = \pm\sqrt{-\frac{\delta\,\mu^2 + 2\,\rho}{2\Omega}}, A_1 = \pm\sqrt{-\frac{\delta\,\mu^2 + 2\,\rho}{2\Omega}}, B_1 = 0, \lambda = \sqrt{\frac{2\,\rho + \delta\,\ \mu^2}{2\delta}}.\right\} \tag{5.18}$$

Inserting these values and (5.14) into Eq. (5.13) we get

$$u_{1.2}(\xi) = \pm\frac{1}{2}\sqrt{-\frac{4\,\rho + 2\,\delta\,\mu^2}{\Omega}}\,\frac{\cos(\xi)}{\sin(\xi)}. \tag{5.19}$$

Consequently, we get

$$q_{1.2}(s,t) = \pm\frac{1}{2}\sqrt{-\frac{4\,\rho + 2\,\delta\,\mu^2}{\Omega}}\,\frac{\cos(\xi)}{\sin(\xi)}e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon} + \rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \tag{5.20}$$

where ξ is given with (4.2).

Group 2: $p = [2-i, 2+i, 1, 1]$ and $\theta = [i, -i, i, -i]$ provides:

$$\phi(\xi) = \frac{2\,\cos(\xi) + \sin(\xi)}{\cos(\xi)}. \tag{5.21}$$

Set 2.1:

$$\left\{A_0 = \pm\frac{2}{5}\sqrt{-\frac{50\,\rho + 25\,\delta\,\mu^2}{2\Omega}}, A_1 = 0, B_1 = \mp\sqrt{-\frac{50\,\rho + 25\,\delta\,\mu^2}{2\Omega}}, \lambda = \sqrt{\frac{\delta\,\mu^2 + 2\,\rho}{2\delta}}.\right\} \tag{5.22}$$

Putting the obtained values (5.22) and (5.21) into Eq. (5.13), we have

$$u_{2.1}(\xi) = \pm\frac{1}{2}\sqrt{-\frac{2\,\delta\,\mu^2 + 4\,\rho}{\Omega}}\,\frac{\cos(\xi) - 2\,\sin(\xi)}{2\,\cos(\xi) + \sin(\xi)}. \tag{5.23}$$

As a consequence, we have

$$q_{2.1}(s,t) = \pm\frac{1}{2}\sqrt{-\frac{2\,\delta\,\mu^2+4\,\rho}{\Omega}}\frac{\cos(\xi)-2\,\sin(\xi)}{2\,\cos(\xi)+\sin(\xi)}e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon}+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \tag{5.24}$$

where ξ is given with (4.2).

Set 2.2:

$$\left\{A_0 = \pm 2\sqrt{-\frac{\delta\,\mu^2+2\,\rho}{2\Omega}}, A_1 = \mp\sqrt{-\frac{\delta\,\mu^2+2\,\rho}{2\Omega}}, B_1 = 0, \lambda = \sqrt{\frac{\delta\,\mu^2+2\,\rho}{2\delta}}.\right\} \tag{5.25}$$

Putting the these values with (5.21) into Eq. (5.13), we have

$$u_{2.2}(\xi) = \pm\sqrt{-\frac{2\,\delta\,\mu^2+4\,\rho}{\Omega}}\frac{\tan(\xi)}{2}. \tag{5.26}$$

Hencefore, following solution of Eq. (1.8) is yielded as

$$q_{2.2}(s,t) = \pm\frac{1}{2}\sqrt{-\frac{2\,\delta\,\mu^2+4\,\rho}{\Omega}}\sin(\xi)\left(\cos(\xi)\right)^{-1}e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon}+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \tag{5.27}$$

where ξ is given with (4.2).

Group 3: $p = [i, -i, 1, 1]$ and $\theta = [i, -i, i, -i]$ provides:

$$\phi(\xi) = -\sin(\xi)\,/\cos(\xi)\,. \tag{5.28}$$

Set 3.1:

$$\left\{A_0 = 0, A_1 = \pm\sqrt{-\frac{\delta\,\mu^2+2\,\rho}{8\Omega}}, B_1 = \mp\sqrt{-\frac{\delta\,\mu^2+2\,\rho}{8\Omega}}, \lambda = \sqrt{\frac{2\,\rho+\delta\,\mu^2}{8\delta}}.\right\} \tag{5.29}$$

The solution of Eq. (5.13) with using (5.29) and (5.28) is given by

$$u_{3.1}(\xi) = \pm\frac{1}{4}\sqrt{-\frac{2\,\delta\,\mu^2+4\,\rho}{\Omega}}\frac{2\,\left(\cos(\xi)\right)^2-1}{\cos(\xi)\sin(\xi)}. \tag{5.30}$$

Resultantly, we get:

$$q_{3.1}(s,t) = \pm\frac{1}{4}\sqrt{-\frac{2\,\delta\,\mu^2+4\,\rho}{\Omega}}\frac{2\,\left(\cos(\xi)\right)^2-1}{\cos(\xi)\sin(\xi)}e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon}+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \tag{5.31}$$

where ξ is given with (4.2).

Group 4: $\theta = [0, 0, 0, 1]$ and $p = [-1, 0, 1, 1]$ provides:

$$\phi(\xi) = -\left(1+\cosh(\xi)+\sinh(\xi)\right)^{-1}. \tag{5.32}$$

Set 4.1:

$$\left\{A_0 = -\frac{\sqrt{2}}{2}\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega}}, A_1 = -\sqrt{\frac{2\delta\,\mu^2+4\,\rho}{\Omega}}, B_1 = 0, \lambda = \sqrt{-\frac{2\,\delta\,\mu^2+4\,\rho}{\delta}}.\right\} \tag{5.33}$$

Substituting the above values into Eq. (5.13), we have

$$u_{4.1}(\xi) = \pm\frac{\sqrt{2}}{2}\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega}}\frac{\sinh(\xi)}{1+\cosh(\xi)}. \tag{5.34}$$

As a result, we achieve

$$q_{4.1}(s,t) = \pm\frac{\sqrt{2}}{2}\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega}}\sinh(\xi)\left(1+\cosh(\xi)\right)^{-1}e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon}+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \tag{5.35}$$

where ξ is given with (4.2).

Group 5: $\theta = [0, 1, 0, 1]$ and $p = [-3, -2, 1, 1]$ gives:

$$\phi(\xi) = (-3 - 2\cosh(\xi) - 2\sinh(\xi)) / (1 + \cosh(\xi) + \sinh(\xi)). \tag{5.36}$$

Set 5.1:

$$\left\{ A_0 = \pm\frac{5}{12}\sqrt{\frac{72\,\delta\,\mu^2 + 144\,\rho}{\Omega}}, A_1 = 0, B_1 = \pm\sqrt{\frac{72\,\delta\ \mu^2 + 144\,\rho}{\Omega}}, \lambda = \sqrt{-\frac{4\,\rho + 2\,\delta\,\mu^2}{\delta}}. \right\} \tag{5.37}$$

Inserting the above values into Eq. (5.13), we have

$$u_{5.1}(\xi) = \mp\frac{\sqrt{2}}{2}\sqrt{\frac{\delta\,\mu^2 + 2\,\rho}{\Omega}}\,(12\sinh(\xi) - 5)\,(13 + 12\cosh(\xi))^{-1}. \tag{5.38}$$

As a consequence, we have:

$$q_{5.1}(s,t) = \mp\frac{\sqrt{2}}{2}\sqrt{\frac{\delta\,\mu^2 + 2\,\rho}{\Omega}}\,(12\sinh(\xi) - 5)\,(13 + 12\cosh(\xi))^{-1}\,e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon} + \rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \tag{5.39}$$

where ξ is given with (4.2).

Group 6: $\theta = [1, -1, 1, -1]$ and $p = [1, 1, 1, -1]$ provides:

$$\phi(\xi) = \frac{\cosh(\xi)}{\sinh(\xi)}. \tag{5.40}$$

Set 6.1:

$$\left\{ A_0 = 0, A_1 = \pm\sqrt{\frac{2\,\rho + \delta\,\mu^2}{8\Omega}}, B_1 = \pm\sqrt{\frac{2\,\rho + \delta\,\mu^2}{8\Omega}}, \lambda = \sqrt{-\frac{\delta\,\mu^2 + 2\,\rho}{8\delta}}. \right\} \tag{5.41}$$

Inserting these values and (5.40) into Eq. (5.13), one get

$$u_{6.1}(\xi) = \pm\frac{\sqrt{2}}{4}\sqrt{\frac{\delta\,\mu^2 + 2\,\rho}{\Omega}}\left(2\,(\cosh(\xi))^2 - 1\right)(\sinh(\xi))^{-1}\,(\cosh(\xi))^{-1}. \tag{5.42}$$

Consequently, we get

$$q_{6.1}(s,t) = \pm\frac{\sqrt{2}}{4}\sqrt{\frac{\delta\,\mu^2 + 2\,\rho}{\Omega}}\left(2\,(\cosh(\xi))^2 - 1\right)(\sinh(\xi))^{-1}\,(\cosh(\xi))^{-1}\,e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon} + \rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \tag{5.43}$$

where ξ is given with (4.2).

Set 6.2:

$$\left\{ A_0 = 0, A_1 = \pm\sqrt{-\frac{\delta\,\mu^2 + 2\,\rho}{4\Omega}}, B_1 = \mp\sqrt{-\frac{\delta\,\mu^2 + 2\,\rho}{4\Omega}}, \lambda = -\sqrt{\frac{\delta\,\mu^2 + 2\,\rho}{4\delta}}. \right\} \tag{5.44}$$

Inserting the these values into Eq. (5.13), we get

$$u_{6.2}(\xi) = \pm\frac{1}{2}\sqrt{-\frac{\delta\,\mu^2 + 2\,\rho}{\Omega}}\,(\sinh(\xi))^{-1}\,(\cosh(\xi))^{-1}. \tag{5.45}$$

Resultantly, we obtain the following solution of Eq. (1.8):

$$q_{6.2}(s,t) = \pm\frac{1}{2}\sqrt{-\frac{\delta\,\mu^2 + 2\,\rho}{\Omega}}\,(\sinh(\xi))^{-1}\,(\cosh(\xi))^{-1}\,e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon} + \rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \tag{5.46}$$

where ξ is given with (4.2).

Set 6.3:

$$\left\{A_0 = 0, A_1 = \pm\sqrt{\frac{2\,\rho+\delta\,\mu^2}{2\Omega}}, B_1 = 0, \lambda = \sqrt{-\frac{\delta\,\mu^2+2\,\rho}{2\delta}}.\right\} \tag{5.47}$$

Using these values and (5.40), Eq. (5.13) can be written as

$$u_{6.3}(\xi) = \pm\frac{\sqrt{2}}{2}\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega}}\cosh(\xi)\,(\sinh(\xi))^{-1}. \tag{5.48}$$

Then, the solution of Eq. (1.8) is given by:

$$q_{6.3}(s,t) = \pm\frac{\sqrt{2}}{2}\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega}}\cosh(\xi)\,(\sinh(\xi))^{-1}\,e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon}+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \tag{5.49}$$

where ξ is given with (4.2).

Set 6.4:

$$\left\{A_0 = 0, A_1 = 0, B_1 = -\sqrt{\frac{2\,\rho+\delta\,\mu^2}{2\Omega}}, \lambda = \sqrt{-\frac{\delta\,\mu^2+2\,\rho}{2\delta}}.\right\} \tag{5.50}$$

The solution of Eq. (5.13) corresponding to these values and (5.40) is given by

$$u_{6.4}(\xi) = \pm\frac{\sqrt{2}}{2}\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega}}\sinh(\xi)\,(\cosh(\xi))^{-1}. \tag{5.51}$$

Therefore, we get the following solution of Eq. (1.8):

$$q_{6.4}(s,t) = \pm\frac{\sqrt{2}}{2}\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega}}\sinh(\xi)\,(\cosh(\xi))^{-1}\,e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon}+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \tag{5.52}$$

where ξ is given with (4.2).

Group 7: $\theta = [0, 1, 0, 1]$ and $p = [-2, -1, 1, 1]$ provides:

$$\phi(\xi) = \frac{-2-\cosh(\xi)-\sinh(\xi)}{1+\cosh(\xi)+\sinh(\xi)}. \tag{5.53}$$

Set 7.1:

$$\left\{A_0 = \pm\frac{3}{4}\sqrt{\frac{16\,\rho+8\,\delta\,\mu^2}{\Omega}}, A_1 = 0, B_1 = \pm\sqrt{\frac{16\,\rho+8\,\delta\,\mu^2}{\Omega}}, \lambda = \sqrt{-\frac{4\,\rho+2\,\delta\,\mu^2}{\delta}}.\right\} \tag{5.54}$$

Putting the these values into Eq. (5.13), we have

$$u_{7.1}(\xi) = \mp\frac{\sqrt{2}}{2}\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega}}\,(4\,\sinh(\xi)-3)\,(5+4\,\cosh(\xi))^{-1}. \tag{5.55}$$

Henceforth, we obtain the solution of Eq. (1.8):

$$q_{7.1}(s,t) = \mp\frac{\sqrt{2}}{2}\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega}}\,(4\,\sinh(\xi)-3)\,(5+4\,\cosh(\xi))^{-1}\,e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon}+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \tag{5.56}$$

where ξ is given with (4.2).

Group 8: $\theta = [1, -1, 1, -1]$, $p = [-3, -1, 1, 1]$ provides:

$$\phi(\xi) = \frac{-4\,\cosh(\xi)-2\,\sinh(\xi)}{2\cosh(\xi)}. \tag{5.57}$$

Set 8.1:

$$\left\{A_0 = -\frac{2}{3}\sqrt{\frac{18\,\rho + 9\,\delta\,\mu^2}{2\Omega}}, A_1 = 0, B_1 = -\sqrt{\frac{18\,\rho + 9\,\delta\,\mu^2}{2\Omega}}, \lambda = \sqrt{-\frac{\delta\,\mu^2 + 2\,\rho}{2\delta}}.\right\} \tag{5.58}$$

Putting the these values into Eq. (5.13), we have

$$u_{8.1}(\xi) = \pm\frac{\sqrt{2}}{2}\sqrt{\frac{\delta\,\mu^2 + 2\,\rho}{\Omega}}\left(\cosh(\xi) + 2\,\sinh(\xi)\right)\left(2\,\cosh(\xi) + \sinh(\xi)\right)^{-1}. \tag{5.59}$$

As a result, we get

$$q_{8.1}(s,t) = \pm\frac{\sqrt{2}}{2}\sqrt{\frac{\delta\,\mu^2 + 2\,\rho}{\Omega}}\left(\cosh(\xi) + 2\,\sinh(\xi)\right)\left(2\,\cosh(\xi) + \sinh(\xi)\right)^{-1} e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon} + \rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \tag{5.60}$$

where ξ is given with (4.2).

Group 9: $\theta = [-1, 1, -1, 1]$ and $p = [1, 2, 1, 1]$ provides:

$$\phi(\xi) = \frac{3\,\cosh(\xi) + \sinh(\xi)}{2\cosh(\xi)}. \tag{5.61}$$

Set 9.1:

$$\left\{A_0 = \pm\frac{3}{4}\sqrt{\frac{16\,\rho + 8\,\delta\,\mu^2}{\Omega}}, A_1 = 0, B_1 = \mp\sqrt{\frac{16\,\rho + 8\,\delta\;\;\mu^2}{\Omega}}, \lambda = \sqrt{-\frac{\delta\,\mu^2 + 2\,\rho}{2\delta}}.\right\} \tag{5.62}$$

Substituting the above values into Eq. (5.13), we get

$$u_{9.1}(\xi) = \pm\frac{\sqrt{2}}{2}\sqrt{\frac{\delta\,\mu^2 + 2\,\rho}{\Omega}}\left(\cosh(\xi) + 3\,\sinh(\xi)\right)\left(3\,\cosh(\xi) + \sinh(\xi)\right)^{-1}. \tag{5.63}$$

In conclusion, we yield the solution of Eq. (1.8):

$$q_{9.1}(s,t) = \pm\frac{\sqrt{2}}{2}\sqrt{\frac{\delta\,\mu^2 + 2\,\rho}{\Omega}}\left(\cosh(\xi) + 3\,\sinh(\xi)\right)\left(3\,\cosh(\xi) + \sinh(\xi)\right)^{-1} e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon} + \rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}, \tag{5.64}$$

where ξ is given with (4.2).

6. Graphical Demonstrations

The methods applied in this study are capable of producing several types of solutions in different ways, unlike some classical methods that can only produce a small number of solutions. The dynamics of the selected solutions to exemplify the results obtained in this work are analyzed by representing several three- and two-dimensional plots. One of the two-dimensional subgraphs describes the wave propagation pattern of the wave with different values of t. The other one is plotted to show the effect of changing of fractional order (ϵ) on the wave. For this purpose, the solutions q_1, q_3,$q_{4.1}$,$q_{6.2}$ and $q_{7.1}$ are used for particular classes of the parameters.

In Figure 1, the 3D plot visualizes the graphical representation of q_1 as dark soliton with $\mu = 0.1$, $\mu_1 = 1$, $\mu_2 = 1, \mu_3 = 3, \rho = 1, \Omega = 1, \delta = 0.2, \epsilon = 0.2, \theta = 0.5$. The first 2D plot is given for $t = 0, 1, 2$ and the second one is plotted for $\epsilon = 0.2, 0.5, 0.9$, and $t = 1$.

Figure 2 is 3D graph of q_3 as dark soliton with associated values of parameters $\mu = 0.1, \mu_1 = 1, \mu_2 = 4, \mu_3 = 1, \rho = 2, \Omega = 1, \delta = 0.1, \epsilon = 0.3, \theta = 0.5$. The first 2D plot is given for $t = 0, 1, 3$ and the second one is plotted for $\epsilon = 0.1, 0.3, 0.9$ and $t = 1$.

In Figure 3, the 3D plot visualizes the graphical representation of $q_{4.1}$ as singular solution with $\mu = 1, \rho = 0.1, \Omega = 2, \delta = 0.5, \epsilon = 0.1, \theta = 0.5$. The first 2D plot is given for $t = 0, 1, 2$ and the second one is plotted for $\epsilon = 0.1, 0.2, 0.3$ and $t = 1$.

Figure 4 represents bright solitary wave solution $q_{6.2}$ for $\mu = 1, \rho = 0.1, \Omega = 0.1, \delta = 0.2, \epsilon = 0.2, \theta = 0.5$. The first 2D

plot is given for $t = 0, 1, 2$ and the second one is plotted for $\epsilon = 0.2, 0.3, 0.5$ and $t = 1$.
In Figure 5, the 3D plot visualizes the graphical representation of $q_{7.1}$ as singular solution with $\mu = 2, \rho = 0.2, \Omega = 0.1, \delta = 0.1, \epsilon = 0.3, \theta = 0.5$. The first 2D plot is given for $t = 0, 1, 2$ and the second one is plotted for $\epsilon = 0.1, 0.2, 0.3$ and $t = 1$.

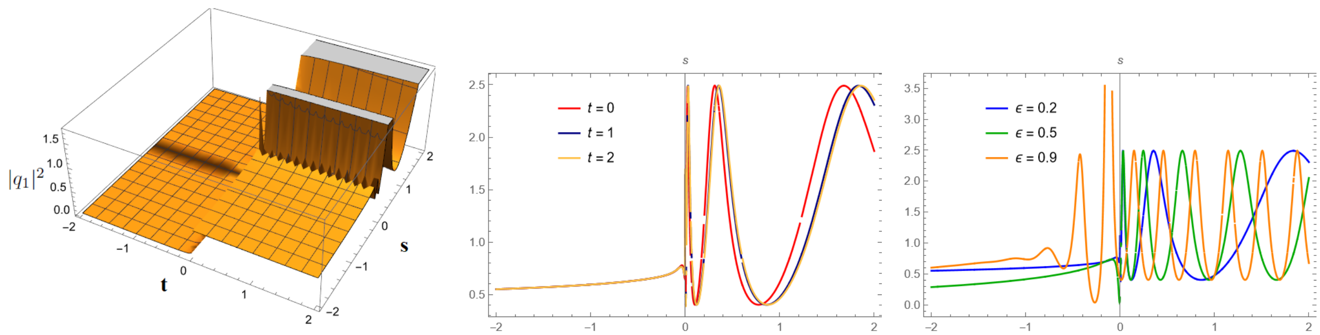

FIGURE 1. 3D plot of $|q_1|^2$ given in (5.3) between $-2 \le s, t \le 2$, 2D plot of $|q_1|^2$ for different values of time $t = 0, 1, 2$ and 2D plot of $|q_1|^2$ for different values of fractional parameter $\epsilon = 0.2, 0.5, 0.9$ and $t = 1$, respectively.

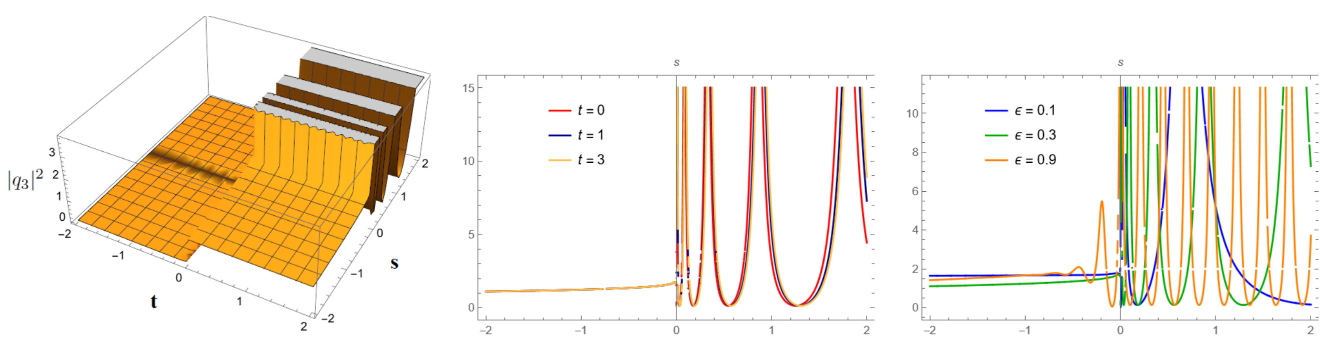

FIGURE 2. 3D plot of $|q_3|^2$ given in (5.5), 2D plot of $|q_3|^2$ for varying time values $t = 0, 1, 3$ and 2D plot of $|q_3|^2$ for different values of fractional parameter $\epsilon = 0.1, 0.3, 0.9$ and $t = 1$, respectively.

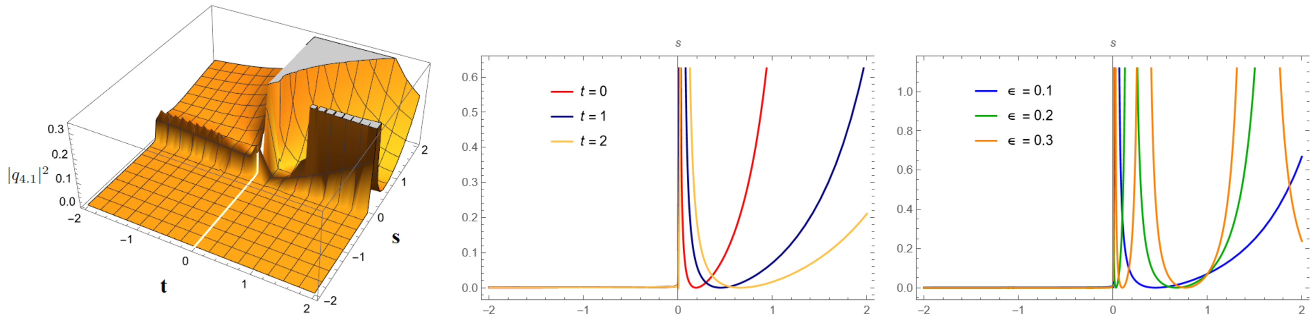

FIGURE 3. 3D plot of $|q_{4.1}|^2$ given in (5.35) between $-2 \leq s, t \leq 2$, 2D plot of $|q_{4.1}|^2$ for varying time values $t = 0, 1, 2$ and 2D plot of $|q_{4.1}|^2$ for different values of fractional parameter $\epsilon = 0.1, 0.2, 0.3$ and $t = 1$, respectively.

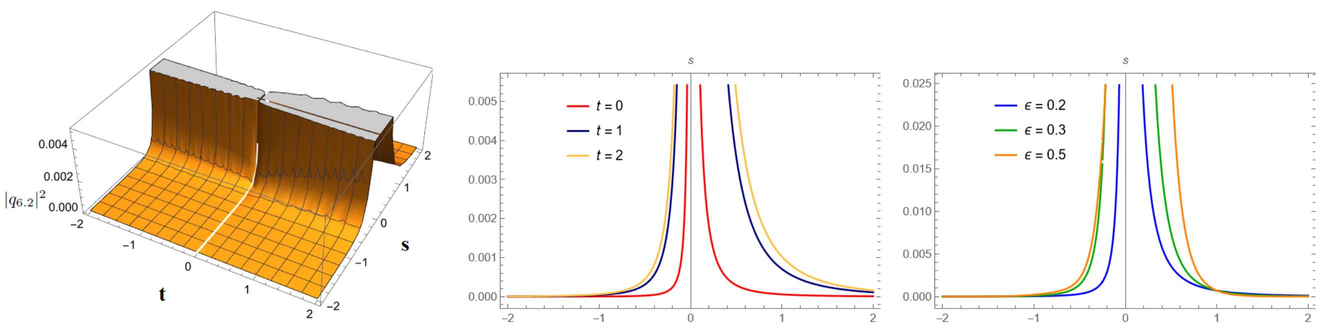

FIGURE 4. 3D plot of $|q_{6.2}|^2$ given in (5.46) between $-2 \leq s, t \leq 2$, 2D plot of $|q_{6.2}|^2$ for varying time values $t = 0, 1, 2$ and 2D plot of $|q_{6.2}|^2$ for different values of fractional parameter $\epsilon = 0.2, 0.3, 0.5$ and $t = 1$, respectively.

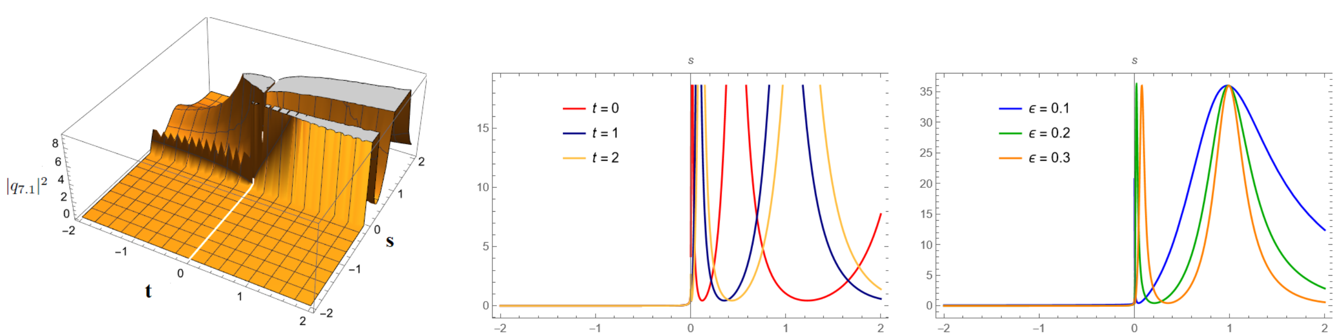

FIGURE 5. 3D plot of $|q_{7.1}|^2$ given in (5.56) between $-2 \leq s, t \leq 2$, 2D plot of $|q_{7.1}|^2$ for varying time values $t = 0, 1, 2$ and 2D plot of $|q_{7.1}|^2$ for different values of fractional parameter $\epsilon = 0.1, 0.2, 0.3$ and $t = 1$, respectively.

7. Discussions

Raheel et al. [34] explored analytical solutions of the truncated M-fractional IOPM (1.8) based on exp_a function, extended sinh- Gordon equation expansion, and extended $(\frac{G'}{G})$-expansion methods and obtained trigonometric, hyperbolic, and exponential type solutions. The comparison is ascertained as follows:

Table 1. Comparison of solutions.

Raheel et al.[34]	Our solution
For $\alpha_0 = 0, \alpha_1 = -\frac{i\sqrt{\delta}\lambda}{\sqrt{\Omega}}, \beta_1 = 0,$ $\rho = -\frac{\delta\left(2\lambda^2+\mu^2\right)}{2}$ in Set 1.	For $A_0 = 0, A_1 = \pm\sqrt{\frac{2\,\rho+\delta\,\mu^2}{2\Omega}}, B_1 = 0,$ $\lambda = \sqrt{-\frac{\delta\,\mu^2+2\,\rho}{2\delta}}$ in Set 6.3.
$q_1\left(s,t\right) = \mp\frac{i\sqrt{\delta}\lambda}{\sqrt{\Omega}}\coth\left(\zeta\right)$ $\times\exp\left(i(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{\delta\left(2\lambda^2+\mu^2\right)}{2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})\right),$	$q_{6.3}(s,t) = \pm\frac{\sqrt{2}}{2}\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega}}\coth\left(\xi\right)e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon}+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)},$
$q_2\left(s,t\right) = \mp\frac{i\sqrt{\delta}\lambda}{\sqrt{\Omega}}\tanh\left(\zeta\right)$ $\times\exp\left(i(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{\delta\left(2\lambda^2+\mu^2\right)}{2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}\right).$	$q_{6.4}(s,t) = \pm\frac{\sqrt{2}}{2}\sqrt{\frac{\delta\,\mu^2+2\,\rho}{\Omega}}\tanh\left(\xi\right)e^{i\left(\mu\frac{\Gamma(\theta+1)}{\epsilon}s^{\epsilon}+\rho\frac{\Gamma(\theta+1)}{\epsilon}t^{\epsilon}\right)}.$

Moreover, the solutions obtained for Set 2 in [34] are in the same form as those in the Table 1. Then, our results (5.49) and (5.52) has similarity with results [41, 42] and [44, 45] mentioned in [34] which is approximately the same. Our remain results are new which is not reported using another different method. We believe that the results obtained by both methods in this study will bring innovation to the literature.

8. Conclusions

Equations containing fractional derivatives may model the problem in more detail than integer-order equations. This provides the opportunity to interpret the results in a wider range. One of the various fractional derivative operators that have attracted attention recently is the M-fractional derivative operator. In this research, we discussed the M-fractional Ivancevic option pricing model, which is an important equation for financial circles. We highlight the importance of this model and its solutions for understanding financial derivatives markets from a mathematical and physical perspective. The exact solutions of underlying model have been explored using two different methods. One of these efficient methods is the extended modified auxiliary equation mapping method and the other one is the generalized exponential rational function method. These methods reveal the bright dark, singular solitary wave solutions. In addition, graphical representations showing the applicability of the proposed method are presented. The results obtained in this study can be used to explain some of the deeper features of the economic model under consideration. Their option price wave fluctuations are given with the real physical meanings of IOPM model and stable option price pulses. All the extracted solutions of the IOPM economy model may have numerous applications in many branches of nonlinear sciences, including finance, economy, the option price and so on. Application of the analytical techniques presented in this paper to study fractional PDEs to different economic models will be the focus of our future research.

Declarations

Ethical Approval Not applicable.

Competing interests The authors declare no conflict of interests.

Authors' contributions YSO: writing, review and editing. EY: Supervision, methodology, writing, review and editing.

Funding The authors have not received any funding for this research.

Availability of data and materials Not applicable.

9. Appendices

Table 2. Solution of the auxiliary equation Eq. 3.4, $\Lambda = \mu_2^2 - 4\mu_1\mu_3$.

	$w(\xi)$		$w(\xi)$
1.	$-\frac{\mu_1\mu_2 sech\left(\frac{\sqrt{\mu_1}\,\xi}{2}\right)^2}{\mu_2^2-\mu_1\mu_3\left(1+\tanh\left(\frac{\sqrt{\mu_1}\,\xi}{2}\right)\right)^2}, \mu_1 > 0$	9.	$\frac{\mu_1 csch\left(\frac{\sqrt{\mu_1}\,\xi}{2}\right)^2}{\mu_2+2\sqrt{\mu_1\mu_3}\,\coth\left(\frac{\sqrt{\mu_1}\,\xi}{2}\right)}, \mu_1 > 0, \mu_3 > 0$
2.	$\frac{\mu_1\mu_2 csch\left(\frac{\sqrt{\mu_1}\,\xi}{2}\right)^2}{\mu_2^2-\mu_1\mu_3\left(1+\coth\left(\frac{\sqrt{\mu_1}\,\xi}{2}\right)\right)^2}, \mu_1 > 0$	10.	$-\frac{\mu_1\mu_3\left(\csc^2\left(\frac{\sqrt{-\mu_1}\,\xi}{2}\right)\right)}{\mu_2+2\sqrt{-\mu_1\mu_3}\,\cot\left(\frac{\sqrt{-\mu_1}\,\xi}{2}\right)}, \mu_1 < 0, \mu_3 > 0$
3.	$\frac{2\mu_1\, sech\left(\sqrt{\mu_1}\,\xi\right)}{\sqrt{-4\mu_1\mu_3+\mu_2^2}-\mu_2\, sech\left(\sqrt{\mu_1}\,\xi\right)}, \mu_1 > 0, \Lambda > 0$	11.	$-\frac{\mu_1\left(1+\tanh\left(\frac{\sqrt{\mu_1}\,\xi}{2}\right)\right)}{\mu_2}, \mu_1 > 0, \Lambda = 0$
4.	$\frac{2\mu_1\sec\left(\sqrt{-\mu_1}\,\xi\right)}{\sqrt{-4\mu_1\mu_3+\mu_2^2}-\mu_2\sec\left(\sqrt{-\mu_1}\,\xi\right)}, \mu_1 < 0, \Lambda > 0$	12.	$-\frac{\mu_1\left(1+\coth\left(\frac{\sqrt{\mu_1}\,\xi}{2}\right)\right)}{\mu_2}, \mu_1 > 0, \Lambda = 0$
5.	$\frac{2\mu_1\, csch\left(\sqrt{\mu_1}\,\xi\right)}{\sqrt{-4\mu_1\mu_3+\mu_2^2}-\mu_2\, csch\left(\sqrt{\mu_1}\,\xi\right)}, \mu_1 > 0, \Lambda > 0$	13.	$\frac{4\mu_1\, e^{\sqrt{\mu_1}\,\xi}}{\left(e^{\sqrt{\mu_1}\,\xi}-\mu_2\right)^2-4\mu_1\mu_3}, \mu_1 > 0$
6.	$\frac{2\mu_1\csc\left(\sqrt{-\mu_1}\,\xi\right)}{\sqrt{4\mu_1\mu_3-\mu_2^2}-\mu_2\csc\left(\sqrt{-\mu_1}\,\xi\right)}, \mu_1 < 0, \Lambda > 0$	14.	$\frac{4\mu_1\, e^{\sqrt{\mu_1}\,\xi}}{1-\mu_1\mu_3\, e^{\sqrt{\mu_1}\,\xi}}, \mu_1 > 0, \mu_2 = 0$
7.	$-\frac{\mu_1 sech\left(\frac{\sqrt{\mu_1}\,\xi}{2}\right)^2}{\mu_2+2\sqrt{\mu_1\mu_3}\,\tanh\left(\frac{\sqrt{\mu_1}\,\xi}{2}\right)}, \mu_1 > 0, \mu_3 > 0$	15.	$\frac{\mu_1\mu_2}{\mu_2^2\xi^2-\mu_1\mu_3}, \mu_1 = 0$
8.	$-\frac{\mu_1\left(\sec^2\left(\frac{\sqrt{-\mu_1}\,\xi}{2}\right)\right)}{\mu_2+2\sqrt{-\mu_1\mu_3}\,\tan\left(\frac{\sqrt{-\mu_1}\,\xi}{2}\right)}, \mu_1 < 0, \Lambda > 0$	16.	$\frac{1}{\sqrt{\mu_3}\,\xi}, \mu_1 = 0, \mu_2 = 0$

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