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New generalized special functions with two generalized M-series at their kernels and solution of fractional PDEs via double Laplace transform

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Abstract

In this paper, we introduce three types of generalized special functions: beta, Gauss hypergeometric, and confluent hypergeometric, all involving two generalized M-series at their kernels. We then give several properties of these functions, such as integral representations, functional relations, summation relations, derivative formulas, transformation formulas, and double Laplace transforms. Furthermore, we obtain solutions of fractional partial differential equations involving these new generalized special functions and then we present graphs of the approximate behavior of the solutions. Also, we introduce a new generalized beta distribution and incomplete beta function. Finally, we establish relationships between the new generalized special functions and other generalized special functions found in the literature.

Keywords. Double Laplace transform, Fractional partial differential equations, Beta function, Gauss hypergeometric function, Confluent hypergeometric function.

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1. Introduction

Special functions have an important role in many scientific fields such as physics, chemistry, mathematics, and engineering, see [4, 9, 18, 27] for examples. Below are some examples of these special functions. The beta function [2] for $\Re(\sigma) > 0$ and $\Re(\tau) > 0$ is defined by

$$B(\sigma, \tau) = \int_0^1 \omega^{\sigma-1}(1-\omega)^{\tau-1} d\omega. \tag{1.1}$$

The Gauss and confluent hypergeometric functions [16] respectively are given by

$$_2F_1(\vartheta_1, \vartheta_2; \vartheta_3; z) = \sum_{k=0}^{\infty} \frac{(\vartheta_1)_k (\vartheta_2)_k}{(\vartheta_3)_k} \frac{z^k}{k!}, \quad (|z| < 1) \tag{1.2}$$

and

$$\Phi(\vartheta_2; \vartheta_3; z) = \sum_{k=0}^{\infty} \frac{(\vartheta_2)_k}{(\vartheta_3)_k} \frac{z^k}{k!}, \tag{1.3}$$

where $(\cdot)_k$ is known as the Pochhammer symbol [2] and defined as follows:

$$(\vartheta)_k = \vartheta(\vartheta+1)\ldots(\vartheta+k-1) \quad \text{and} \quad (\vartheta)_0 \equiv 1.$$

Note that the following formula for $\Re(\vartheta_3) > \Re(\vartheta_2) > 0$ holds true:

$$B(\vartheta_2 + k, \vartheta_3 - \vartheta_2) = \frac{(\vartheta_2)_k}{(\vartheta_3)_k} B(\vartheta_2, \vartheta_3 - \vartheta_2). \tag{1.4}$$

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Scientists have defined various generalizations of the beta function using special functions with appropriate kernels such as exponential, confluent, Mittag-Leffler, Wright and Fox-Wright functions in the integral representation of Eq. (1.1), see [1, 5–8, 10, 12, 17, 20–22, 25] for examples. Some of these generalized beta functions are given below.
Şahin et al. [29] defined the generalized beta function as follows:

$$B_{p,q}^{(m,n)}(\sigma,\tau)=\int_0^1 \omega^{\sigma-1}(1-\omega)^{\tau-1}\ \exp\left(-\frac{p}{\omega^m}\right)\exp\left(-\frac{q}{(1-\omega)^n}\right)d\omega,$$

where $\Re(p)>0$, $\Re(q)>0$, $\Re(m)>0$, $\Re(n)>0$. Mubeen et al. [19] defined the generalized beta function as follows:

$$B_{p,q}^{\xi,\eta}(\sigma,\tau)=\int_0^1 \omega^{\sigma-1}(1-\omega)^{\tau-1}\ {}_1F_1\left(\xi;\eta;-\frac{p}{\omega}\right){}_1F_1\left(\xi;\eta;-\frac{q}{(1-\omega)}\right)d\omega,$$

where $p,q\in\mathbb{R}_0^+$, $\xi\in\mathbb{C}$, $\eta\in\mathbb{C}\setminus\mathbb{Z}_0^-$, $\Re(\sigma)>0$, $\Re(\tau)>0$.

Rahman et al. [23] defined the generalized beta function as follows:

$$B_{p,q}^{\alpha}(\sigma,\tau)=\int_0^1 \omega^{\sigma-1}(1-\omega)^{\tau-1}\ E_\alpha\left(-\frac{p}{\omega}\right)E_\alpha\left(-\frac{q}{(1-\omega)}\right)d\omega,$$

where $p,q\geq 0$, $\Re(\sigma)>0$, $\Re(\tau)>0$, $\Re(\alpha)>0$. If we replace the beta function on the left side of Eq. (1.4) with generalized beta functions that have the appropriate kernel and then use them in Eqs. (1.2) and (1.3), we obtain various generalized Gauss and confluent hypergeometric functions. For more information, see [11, 13, 15, 24, 28]. Motivated by these studies, this paper describes new generalized beta, Gauss hypergeometric, and confluent hypergeometric functions. We accomplish this by using two generalized M-series [26], which are defined by

$${}_u^{\alpha}M_v^{\beta}(z)={}_u^{\alpha}M_v^{\beta}(\xi_1,\ldots,\xi_u;\eta_1,\ldots,\eta_v;z)=\sum_{k=0}^{\infty}\frac{(\xi_1)_k\ldots(\xi_u)_k}{(\eta_1)_k\ldots(\eta_v)_k}\frac{z^k}{\Gamma(\alpha k+\beta)},$$

where $\xi_1,\ldots,\xi_u,\eta_1,\ldots,\eta_v\neq 0,-1,-2,\ldots$ and $\Re(\alpha)>0$.

We aimed to use the generalized M-series as it has a more general form than most appropriate kernel functions mentioned above. Since the generalized M-series contains more parameters, the application areas of special functions are thought to expand with the results obtained here.

2. Preliminaries

In this section, we give below the partial fractional Caputo derivative, double Laplace and inverse Laplace transforms that we will need throughout this paper.

The partial fractional Caputo derivative [3] is given by

$${}_q^cD_{0^+}^{\varepsilon_2}\ {}_p^cD_{0^+}^{\varepsilon_1}f(p,q)=\frac{1}{\Gamma(m-\varepsilon_1)}\frac{1}{\Gamma(n-\varepsilon_2)}\int_0^q\int_0^p(p-x)^{m-\varepsilon_1-1}(q-y)^{n-\varepsilon_2-1}\frac{\partial^{m+n}f(x,y)}{\partial y^n\partial x^m}dxdy,$$

where $m-1<\Re(\varepsilon_1)\leq m$, $n-1<\Re(\varepsilon_2)\leq n$, $m,n\in\mathbb{N}$.

The double Laplace and inverse Laplace transforms [3, 14] respectively are defined by

$$\mathfrak{L}_q\mathfrak{L}_p\left[f(p,q)\right](s_1,s_2)=\int_0^\infty\int_0^\infty \exp(-s_1p)\exp(-s_2q)f(p,q)dpdq, \tag{2.1}$$

and

$$\mathfrak{L}_q^{-1}\mathfrak{L}_p^{-1}\Big[\mathfrak{L}_q\mathfrak{L}_p\left[f(p,q)\right](s_1,s_2)\Big](p,q) = f(p,q)$$
$$= \frac{1}{(2\pi i)^2}\int_{c-i\infty}^{c+i\infty}\int_{d-i\infty}^{d+i\infty}\exp(s_1p)\exp(s_2q)\mathfrak{L}_q\mathfrak{L}_p\left[f(p,q)\right](s_1,s_2)ds_1ds_2, \tag{2.2}$$

where $\Re(s_1) \geqq c$, $\Re(s_2) \geqq d$.

Theorem 2.1 ([3]). *Application of the double Laplace transform to partial fractional Caputo derivative gives*

$$\begin{aligned}&\mathfrak{L}_q\mathfrak{L}_p\left[{}^c_qD^{\varepsilon_2}_{0+}\ {}^c_pD^{\varepsilon_1}_{0+}f(p,q)\right](s_1,s_2)\\ &= s_1^{\varepsilon_1}s_2^{\varepsilon_2}\Bigg(\mathfrak{L}_q\mathfrak{L}_p\left[f(p,q)\right](s_1,s_2) - \sum_{i=0}^{m-1}s_1^{-1-i}\mathfrak{L}_q\left[\frac{\partial^i f(0,q)}{\partial p^i}\right](s_2)\\ &\quad - \sum_{j=0}^{n-1}s_2^{-1-j}\mathfrak{L}_p\left[\frac{\partial^j f(p,0)}{\partial q^j}\right](s_1) + \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}s_1^{-1-i}s_2^{-1-j}\ \frac{\partial^{i+j}f(0,0)}{\partial q^j\partial p^i}\Bigg),\end{aligned} \tag{2.3}$$

where $\Re(\varepsilon_1), \Re(\varepsilon_2) > 0$, $m-1 < \Re(\varepsilon_1) \leq m$, $n-1 < \Re(\varepsilon_2) \leq n$, $m, n \in \mathbb{N}$.

3. New Generalized Special Functions and Their Fundamental Properties

Throughout the study, we assume that $\Re(p) > 0$, $\Re(q) > 0$, $\Re(\sigma) > 0$, $\Re(\tau) > 0$, $\Re(\alpha) > 0$, $\Re(m) > 0$, $\Re(n) > 0$, $\Re(\vartheta_3) > \Re(\vartheta_2) > 0$, $\Re(s_1) > 0$, and $\Re(s_2) > 0$. For the sake of shortness, we did not wrote these conditions for the rest of the article, unless otherwise stated.

In this section, we define new generalized special functions and present their fundamental properties.

Definition 3.1. The new generalized beta function is defined by

$$\begin{aligned}{}^M\widehat{B}^{(\alpha,\beta;m,n)}_{p,q}(\sigma,\tau) &= {}^MB^{(\alpha,\beta;m,n)}_{p,q}\left[\begin{matrix}\xi_1,\ldots,\xi_u;\eta_1,\ldots,\eta_v\\ \kappa_1,\ldots,\kappa_u;\mu_1,\ldots,\mu_v\end{matrix}\Big|\sigma,\tau\right]\\ &:= \int_0^1\omega^{\sigma-1}(1-\omega)^{\tau-1}\ {}^\alpha_uM^\beta_v\left(\xi_1,\ldots,\xi_u;\eta_1,\ldots,\eta_v;-\frac{p}{\omega^m}\right)\\ &\quad\times {}^\alpha_uM^\beta_v\left(\kappa_1,\ldots,\kappa_u;\mu_1,\ldots,\mu_v;-\frac{q}{(1-\omega)^n}\right)d\omega.\end{aligned} \tag{3.1}$$

Definition 3.2. The new generalized Gauss hypergeometric function is defined by

$$\begin{aligned}{}^M\widehat{F}^{(\alpha,\beta;m,n)}_{p,q}(\vartheta_1,\vartheta_2;\vartheta_3;z) &= {}^MF^{(\alpha,\beta;m,n)}_{p,q}\left[\begin{matrix}\xi_1,\ldots,\xi_u;\eta_1,\ldots,\eta_v\\ \kappa_1,\ldots,\kappa_u;\mu_1,\ldots,\mu_v\end{matrix}\Big|\vartheta_1,\vartheta_2;\vartheta_3;z\right]\\ &:= \sum_{k=0}^{\infty}(\vartheta_1)_k\frac{{}^MB^{(\alpha,\beta;m,n)}_{p,q}\left[\begin{matrix}\xi_1,\ldots,\xi_u;\eta_1,\ldots,\eta_v\\ \kappa_1,\ldots,\kappa_u;\mu_1,\ldots,\mu_v\end{matrix}\Big|\vartheta_2+k,\vartheta_3-\vartheta_2\right]}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\frac{z^k}{k!}.\end{aligned} \tag{3.2}$$

Definition 3.3. The new generalized confluent hypergeometric function is defined by

$$\begin{aligned}{}^M\widehat{\Phi}^{(\alpha,\beta;m,n)}_{p,q}(\vartheta_2;\vartheta_3;z) &= {}^M\Phi^{(\alpha,\beta;m,n)}_{p,q}\left[\begin{matrix}\xi_1,\ldots,\xi_u;\eta_1,\ldots,\eta_v\\ \kappa_1,\ldots,\kappa_u;\mu_1,\ldots,\mu_v\end{matrix}\Big|\vartheta_2;\vartheta_3;z\right]\\ &:= \sum_{k=0}^{\infty}\frac{{}^MB^{(\alpha,\beta;m,n)}_{p,q}\left[\begin{matrix}\xi_1,\ldots,\xi_u;\eta_1,\ldots,\eta_v\\ \kappa_1,\ldots,\kappa_u;\mu_1,\ldots,\mu_v\end{matrix}\Big|\vartheta_2+k,\vartheta_3-\vartheta_2\right]}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\frac{z^k}{k!}.\end{aligned} \tag{3.3}$$

Theorem 3.4. *We have the following integral representations:*

$$^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\sigma,\tau)=2\int_0^{\frac{\pi}{2}}\sin^{2\sigma-1}(\theta)\cos^{2\tau-1}(\theta)\times {}_{u}^{\alpha}M_v^{\beta}\left(-\frac{p}{\sin^{2m}(\theta)}\right){}_{u}^{\alpha}M_v^{\beta}\left(-\frac{q}{\cos^{2n}(\theta)}\right)d\theta,$$

$$^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\sigma,\tau)=\int_0^{\infty}\frac{t^{\sigma-1}}{(1+t)^{\sigma+\tau}}\times {}_{u}^{\alpha}M_v^{\beta}\left(-\frac{p(1+t)^m}{t^m}\right){}_{u}^{\alpha}M_v^{\beta}\Big(-q(1+t)^n\Big)dt,$$

and

$$^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\sigma,\tau)=(b-a)^{1-\sigma-\tau}\int_a^b(t-a)^{\sigma-1}(b-t)^{\tau-1}\times {}_{u}^{\alpha}M_v^{\beta}\left(-\frac{p(b-a)^m}{(t-a)^m}\right){}_{u}^{\alpha}M_v^{\beta}\left(-\frac{q(b-a)^n}{(b-t)^n}\right)dt.$$

Proof. Taking $\omega=\sin^2(\theta)$, $\omega=\frac{t}{1+t}$ and $\omega=\frac{t-a}{b-a}$ in Eq. (3.1) respectively completes the proof. □

Theorem 3.5. *We have the following functional relation:*

$$^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\sigma,\tau+1)+{}^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\sigma+1,\tau)={}^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\sigma,\tau).$$

Proof. Using Eq. (3.1), we have

$$\begin{aligned}
^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\sigma,\tau+1)+{}^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\sigma+1,\tau)&=\int_0^1\omega^{\sigma-1}(1-\omega)^{\tau}\ {}_{u}^{\alpha}M_v^{\beta}\left(-\frac{p}{\omega^m}\right){}_{u}^{\alpha}M_v^{\beta}\left(-\frac{q}{(1-\omega)^n}\right)d\omega\\
&+\int_0^1\omega^{\sigma}(1-\omega)^{\tau-1}\ {}_{u}^{\alpha}M_v^{\beta}\left(-\frac{p}{\omega^m}\right){}_{u}^{\alpha}M_v^{\beta}\left(-\frac{q}{(1-\omega)^n}\right)d\omega\\
&=\int_0^1\left(\omega^{\sigma-1}(1-\omega)^{\tau}+\omega^{\sigma}(1-\omega)^{\tau-1}\right)\ {}_{u}^{\alpha}M_v^{\beta}\left(-\frac{p}{\omega^m}\right){}_{u}^{\alpha}M_v^{\beta}\left(-\frac{q}{(1-\omega)^n}\right)d\omega\\
&=\int_0^1\omega^{\sigma-1}(1-\omega)^{\tau-1}\ {}_{u}^{\alpha}M_v^{\beta}\left(-\frac{p}{\omega^m}\right){}_{u}^{\alpha}M_v^{\beta}\left(-\frac{q}{(1-\omega)^n}\right)d\omega\\
&={}^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\sigma,\tau).
\end{aligned}$$

□

Theorem 3.6. *We have the following summation relation:*

$$^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\sigma,1-\tau)=\sum_{k=0}^{\infty}\frac{(\tau)_k}{k!}\ {}^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\sigma+k,1),\quad(\Re(1-\tau)>0).$$

Proof. From Eq. (3.1), we have

$$^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\sigma,1-\tau)=\int_0^1\omega^{\sigma-1}(1-\omega)^{-\tau}\ {}_{u}^{\alpha}M_v^{\beta}\left(-\frac{p}{\omega^m}\right){}_{u}^{\alpha}M_v^{\beta}\left(-\frac{q}{(1-\omega)^n}\right)d\omega. \tag{3.4}$$

The binomial series [2] is defined by

$$(1-\omega)^{-\tau}=\sum_{k=0}^{\infty}(\tau)_k\frac{\omega^k}{k!},\quad\big(|\omega|<1\big). \tag{3.5}$$

Using Eq. (3.5) in Eq. (3.4), the proof is complete. □

Theorem 3.7. *We have the following integral representations:*

$$\begin{aligned}
^{M}\widehat{F}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_1,\vartheta_2;\vartheta_3;z)=\frac{1}{B(\vartheta_2,\vartheta_3-\vartheta_2)}&\int_0^1\omega^{\vartheta_2-1}(1-\omega)^{\vartheta_3-\vartheta_2-1}(1-z\omega)^{-\vartheta_1}\\
&\times{}_{u}^{\alpha}M_v^{\beta}\left(-\frac{p}{\omega^m}\right){}_{u}^{\alpha}M_v^{\beta}\left(-\frac{q}{(1-\omega)^n}\right)d\omega,
\end{aligned} \tag{3.6}$$

$$ {}^{M}\widehat{F}_{p,q}^{(\alpha,\beta;m,n)}\left(\vartheta_1,\vartheta_2;\vartheta_3;z\right)=\frac{2}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_0^{\frac{\pi}{2}}\sin^{2\vartheta_2-1}(\theta)\cos^{2\vartheta_3-2\vartheta_2-1}(\theta)\left(1-z\sin^2(\theta)\right)^{-\vartheta_1} \times {}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{p}{\sin^{2m}(\theta)}\right){}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{q}{\cos^{2n}(\theta)}\right)d\theta, \tag{3.7} $$

$$ {}^{M}\widehat{F}_{p,q}^{(\alpha,\beta;m,n)}\left(\vartheta_1,\vartheta_2;\vartheta_3;z\right)=\frac{1}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_0^{\infty}t^{\vartheta_2-1}(1+t)^{\vartheta_1-\vartheta_3}\left(1+t(1-z)\right)^{-\vartheta_1} \times {}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{p(1+t)^m}{t^m}\right){}_{u}^{\alpha}M_{v}^{\beta}\Big(-q(1+t)^n\Big)dt, \tag{3.8} $$

and

$$ {}^{M}\widehat{F}_{p,q}^{(\alpha,\beta;m,n)}\left(\vartheta_1,\vartheta_2;\vartheta_3;z\right)=\frac{(b-a)^{1-\vartheta_3}}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_a^{b}(t-a)^{\vartheta_2-1}(b-t)^{\vartheta_3-\vartheta_2-1}\left(1-\frac{z(t-a)}{b-a}\right)^{-\vartheta_1} \times {}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{p(b-a)^m}{(t-a)^m}\right){}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{q(b-a)^n}{(b-t)^n}\right)dt. \tag{3.9} $$

Proof. Rewriting Eq. (3.2), we have

$$ {}^{M}\widehat{F}_{p,q}^{(\alpha,\beta;m,n)}\left(\vartheta_1,\vartheta_2;\vartheta_3;z\right)=\sum_{k=0}^{\infty}(\vartheta_1)_k\frac{{}^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_2+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\frac{z^k}{k!}. \tag{3.10} $$

Using Eqs. (3.1) and (3.5) in Eq. (3.10), we have

$$ {}^{M}\widehat{F}_{p,q}^{(\alpha,\beta;m,n)}\left(\vartheta_1,\vartheta_2;\vartheta_3;z\right)=\frac{1}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_0^{1}\omega^{\vartheta_2-1}(1-\omega)^{\vartheta_3-\vartheta_2-1}(1-z\omega)^{-\vartheta_1} \times {}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{p}{\omega^m}\right){}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{q}{(1-\omega)^n}\right)d\omega, \tag{3.11} $$

which is Eq. (3.6). Also, taking $\omega=\sin^2(\theta)$, $\omega=\frac{t}{1+t}$ and $\omega=\frac{t-a}{b-a}$ in Eq. (3.11), we get Eqs. (3.7), (3.8) and (3.9) respectively. □

Theorem 3.8. *We have the following integral representations:*

$$ {}^{M}\widehat{\Phi}_{p,q}^{(\alpha,\beta;m,n)}\left(\vartheta_2;\vartheta_3;z\right)=\frac{1}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_0^{1}\omega^{\vartheta_2-1}(1-\omega)^{\vartheta_3-\vartheta_2-1}\exp(z\omega) \times {}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{p}{\omega^m}\right){}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{q}{(1-\omega)^n}\right)d\omega, \tag{3.12} $$

$$ {}^{M}\widehat{\Phi}_{p,q}^{(\alpha,\beta;m,n)}\left(\vartheta_2;\vartheta_3;z\right)=\frac{1}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_0^{1}t^{\vartheta_3-\vartheta_2-1}(1-t)^{\vartheta_2-1}\exp\left(z(1-t)\right) \times {}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{p}{(1-t)^m}\right){}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{q}{t^n}\right)dt, \tag{3.13} $$

$$ {}^{M}\widehat{\Phi}_{p,q}^{(\alpha,\beta;m,n)}\left(\vartheta_2;\vartheta_3;z\right)=\frac{2}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_0^{\frac{\pi}{2}}\sin^{2\vartheta_2-1}(\theta)\cos^{2\vartheta_3-2\vartheta_2-1}(\theta)\exp\left(z\sin^2(\theta)\right) \times {}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{p}{\sin^{2m}(\theta)}\right){}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{q}{\cos^{2n}(\theta)}\right)d\theta, \tag{3.14} $$

$$
\begin{aligned}
{}^{M}\widehat{\Phi}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_2;\vartheta_3;z) &= \frac{1}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_0^\infty t^{\vartheta_2-1}(1+t)^{-\vartheta_3}\exp\left(\frac{zt}{1+t}\right)\\
&\quad\times {}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{p(1+t)^m}{t^m}\right){}_{u}^{\alpha}M_{v}^{\beta}\Big(-q(1+t)^n\Big)dt, \qquad (3.15)
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{M}\widehat{\Phi}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_2;\vartheta_3;z) &= \frac{(b-a)^{1-\vartheta_3}}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_a^b (t-a)^{\vartheta_2-1}(b-t)^{\vartheta_3-\vartheta_2-1}\exp\left(\frac{z(t-a)}{b-a}\right)\\
&\quad\times {}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{p(b-a)^m}{(t-a)^m}\right){}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{q(b-a)^n}{(b-t)^n}\right)dt. \qquad (3.16)
\end{aligned}
$$

Proof. Rewriting Eq. (3.3), we have

$$
{}^{M}\widehat{\Phi}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_2;\vartheta_3;z) = \sum_{k=0}^{\infty}\frac{{}^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_2+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\frac{z^k}{k!}. \qquad (3.17)
$$

Using Eqs. (3.1) and (3.5) in Eq. (3.17), we have

$$
\begin{aligned}
{}^{M}\widehat{\Phi}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_2;\vartheta_3;z) &= \frac{1}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_0^1 \omega^{\vartheta_2-1}(1-\omega)^{\vartheta_3-\vartheta_2-1}\exp(z\omega)\\
&\quad\times {}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{p}{\omega^m}\right){}_{u}^{\alpha}M_{v}^{\beta}\left(-\frac{q}{(1-\omega)^n}\right)d\omega, \qquad (3.18)
\end{aligned}
$$

which is Eq. (3.12). Also, taking $\omega = 1-t$, $\omega = \sin^2(\theta)$, $\omega = \frac{t}{1+t}$ and $\omega = \frac{t-a}{b-a}$ in Eq. (3.18), we get Eqs. (3.13), (3.14), (3.15), and (3.16) respectively. □

Note 3.9. The following formulas for $\Re(\vartheta_3) > \Re(\vartheta_2) > 0$ holds true:

$$
\begin{aligned}
B(\vartheta_2,\vartheta_3-\vartheta_2) &= \frac{\vartheta_3}{\vartheta_2}B(\vartheta_2+1,\vartheta_3-\vartheta_2),\\
(\vartheta_1)_{n+1} &= \vartheta_1(\vartheta_1+1)_n.
\end{aligned}
$$

These formulas are used in the proof of the theorem given below.

Theorem 3.10. *We have the following differentiation formulas:*

$$
\frac{d^r}{dz^r}\left\{{}^{M}\widehat{F}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_1,\vartheta_2;\vartheta_3;z)\right\} = \frac{(\vartheta_1)_r(\vartheta_2)_r}{(\vartheta_3)_r}\;{}^{M}\widehat{F}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_1+r,\vartheta_2+r;\vartheta_3+r;z), \qquad (3.19)
$$

$$
\frac{d^r}{dz^r}\left\{{}^{M}\widehat{\Phi}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_2;\vartheta_3;z)\right\} = \frac{(\vartheta_2)_r}{(\vartheta_3)_r}\;{}^{M}\widehat{\Phi}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_2+r;\vartheta_3+r;z). \qquad (3.20)
$$

Proof. Differentiating Eq. (3.2), we have

$$
\begin{aligned}
\frac{d}{dz}\left\{{}^{M}\widehat{F}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_1,\vartheta_2;\vartheta_3;z)\right\} &= \frac{d}{dz}\left\{\sum_{k=0}^{\infty}(\vartheta_1)_k\frac{{}^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_2+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\frac{z^k}{k!}\right\}\\
&= \sum_{k=1}^{\infty}(\vartheta_1)_k\frac{{}^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_2+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\frac{z^{k-1}}{(k-1)!}.
\end{aligned}
$$

Writing $k \to k+1$, we get

$$\frac{d}{dz}\left\{{}^{M}\widehat{F}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_1,\vartheta_2;\vartheta_3;z)\right\} = \frac{(\vartheta_1)(\vartheta_2)}{(\vartheta_3)}\sum_{k=0}^{\infty}(\vartheta_1+1)_k \frac{{}^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_2+1+k,\vartheta_3-\vartheta_2)}{B(\vartheta_2+1,\vartheta_3-\vartheta_2)}\frac{z^k}{k!}$$
$$= \frac{(\vartheta_1)(\vartheta_2)}{(\vartheta_3)}\ {}^{M}\widehat{F}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_1+1,\vartheta_2+1;\vartheta_3+1;z).$$

Using the method of induction, we obtain the more general form as follows:

$$\frac{d^r}{dz^r}\left\{{}^{M}\widehat{F}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_1,\vartheta_2;\vartheta_3;z)\right\} = \frac{(\vartheta_1)_r(\vartheta_2)_r}{(\vartheta_3)_r}\ {}^{M}\widehat{F}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_1+r,\vartheta_2+r;\vartheta_3+r;z),$$

which is Eq. (3.19). Also, we perform similar calculations for Eq. (3.3) and we obtain Eq. (3.20). □

Theorem 3.11. *We have the following transformation formulas:*

$$ {}^{M}\widehat{F}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_1,\vartheta_2;\vartheta_3;z) = (1-z)^{-\vartheta_1}\ {}^{M}\widehat{F}_{q,p}^{(\alpha,\beta;n,m)}\left(\vartheta_1,\vartheta_3-\vartheta_2;\vartheta_3;\frac{z}{z-1}\right), \tag{3.21}$$

$$ {}^{M}\widehat{\Phi}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_2;\vartheta_3;z) = \exp(z)\ {}^{M}\widehat{\Phi}_{q,p}^{(\alpha,\beta;n,m)}(\vartheta_3-\vartheta_2;\vartheta_3;-z). \tag{3.22}$$

Proof. Using equation

$$\big(1-z(1-\omega)\big)^{-\vartheta_1} = (1-z)^{-\vartheta_1}\left(1+\frac{z\omega}{1-z}\right)^{-\vartheta_1},$$

and writing $\omega \to 1-\omega$ in Eq. (3.6), we obtain

$$ {}^{M}\widehat{F}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_1,\vartheta_2;\vartheta_3;z) = \frac{(1-z)^{-\vartheta_1}}{B(\vartheta_2,\vartheta_3-\vartheta_2)}\int_0^1 \omega^{\vartheta_3-\vartheta_2-1}(1-\omega)^{\vartheta_2-1}\left(1-\frac{z\omega}{z-1}\right)^{-\vartheta_1}$$
$$\times\ {}^{\alpha}_{u}M^{\beta}_{v}\left(-\frac{p}{(1-\omega)^m}\right){}^{\alpha}_{u}M^{\beta}_{v}\left(-\frac{q}{\omega^n}\right)d\omega$$
$$= (1-z)^{-\vartheta_1}\ {}^{M}\widehat{F}_{q,p}^{(\alpha,\beta;n,m)}\left(\vartheta_1,\vartheta_3-\vartheta_2;\vartheta_3;\frac{z}{z-1}\right),$$

which is Eq. (3.21). Also, we obtain Eq. (3.22) from Eq. (3.13). □

Theorem 3.12. *We have the following double Laplace transforms:*

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\sigma,\tau)\right](s_1,s_2) = \frac{1}{s_1}\frac{1}{s_2}\ {}^{M}B_{\frac{1}{s_1},\frac{1}{s_2}}^{(\alpha,\beta;m,n)}\left[\begin{matrix}\xi_1,\ldots,\xi_u,1;\eta_1,\ldots,\eta_v\\ \kappa_1,\ldots,\kappa_u,1;\mu_1,\ldots,\mu_v\end{matrix}\middle|\sigma,\tau\right], \tag{3.23}$$

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}^{M}\widehat{F}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_1,\vartheta_2;\vartheta_3;z)\right](s_1,s_2) = \frac{1}{s_1}\frac{1}{s_2}\ {}^{M}F_{\frac{1}{s_1},\frac{1}{s_2}}^{(\alpha,\beta;m,n)}\left[\begin{matrix}\xi_1,\ldots,\xi_u,1;\eta_1,\ldots,\eta_v\\ \kappa_1,\ldots,\kappa_u,1;\mu_1,\ldots,\mu_v\end{matrix}\middle|\vartheta_1,\vartheta_2;\vartheta_3;z\right], \tag{3.24}$$

and

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}^{M}\widehat{\Phi}_{p,q}^{(\alpha,\beta;m,n)}(\vartheta_2;\vartheta_3;z)\right](s_1,s_2) = \frac{1}{s_1}\frac{1}{s_2}\ {}^{M}\Phi_{\frac{1}{s_1},\frac{1}{s_2}}^{(\alpha,\beta;m,n)}\left[\begin{matrix}\xi_1,\ldots,\xi_u,1;\eta_1,\ldots,\eta_v\\ \kappa_1,\ldots,\kappa_u,1;\mu_1,\ldots,\mu_v\end{matrix}\middle|\vartheta_2;\vartheta_3;z\right]. \tag{3.25}$$

Proof. Using Eq. (2.1), we get

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\sigma,\tau)\right](s_1,s_2) = \int_0^\infty\int_0^\infty \exp(-s_1p)\exp(-s_2q)\ {}^{M}\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\sigma,\tau)dpdq.$$

Then making the necessary calculations, we have

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}^M\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\sigma,\tau)\right](s_1,s_2)=\frac{1}{s_1}\frac{1}{s_2}\int_0^1\omega^{\sigma-1}(1-\omega)^{\tau-1}\,{}_u^\alpha M_v^\beta\left(\xi_1,\ldots,\xi_u,1;\eta_1,\ldots,\eta_v;-\frac{\frac{1}{s_1}}{\omega^m}\right)$$
$$\times\,{}_u^\alpha M_v^\beta\left(\kappa_1,\ldots,\kappa_u,1;\mu_1,\ldots,\mu_v;-\frac{\frac{1}{s_2}}{(1-\omega)^n}\right)d\omega.$$

Hence, we obtain

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}^M\widehat{B}_{p,q}^{(\alpha,\beta;m,n)}(\sigma,\tau)\right](s_1,s_2)=\frac{1}{s_1}\frac{1}{s_2}\,{}^MB_{\frac{1}{s_1},\frac{1}{s_2}}^{(\alpha,\beta;m,n)}\left[\begin{matrix}\xi_1,\ldots,\xi_u,1;\eta_1,\ldots,\eta_v\\ \kappa_1,\ldots,\kappa_u,1;\mu_1,\ldots,\mu_v\end{matrix}\middle|\sigma,\tau\right],\qquad\square$$

which is Eq. (3.23). Also, we apply Eq. (2.1) to Eqs. (1.2) and (1.3) then we obtain Eqs. (3.24) and (3.25).

4. Illustrative examples

In this section, we present the solution of fractional partial differential equations involving new generalized special functions using the double Laplace transform. We also give the beta distribution of the generalized beta function and the new generalized incomplete beta function.

Example 4.1. Let $1<\Re(\varepsilon_1),\Re(\varepsilon_2)\leq 2$. We consider the fractional partial differential equation

$${}_q^cD_{0+}^{\varepsilon_2}\,{}_p^cD_{0+}^{\varepsilon_1}y(p,q)={}^M\widehat{B}_{\varepsilon_1p,\varepsilon_2q}^{(\alpha,\beta;m,n)}(\sigma,\tau),$$

with the initial conditions

$$y(0,0)=\frac{\partial y(0,0)}{\partial p}=\frac{\partial y(0,0)}{\partial q}=\frac{\partial^2y(0,0)}{\partial q\partial p}=0,$$
$$y(p,0)=\frac{\partial y(p,0)}{\partial q}=0,$$
$$y(0,q)=\frac{\partial y(0,q)}{\partial p}=0.$$

Considering Eqs. (2.3) and (3.23) and application of Eq. (2.1) to the fractional partial differential equation gives

$$\mathfrak{L}_q\mathfrak{L}_p\left[{}_q^cD_{0+}^{\varepsilon_2}\,{}_p^cD_{0+}^{\varepsilon_1}y(p,q)\right](s_1,s_2)=\mathfrak{L}_q\mathfrak{L}_p\left[{}^M\widehat{B}_{\varepsilon_1p,\varepsilon_2q}^{(\alpha,\beta;m,n)}(\sigma,\tau)\right](s_1,s_2),$$

then

$$s_1^{\varepsilon_1}s_2^{\varepsilon_2}\Bigg(\mathfrak{L}_q\mathfrak{L}_p\left[y(p,q)\right](s_1,s_2)-s_1^{-1}\mathfrak{L}_q\left[y(0,q)\right](s_2)-s_1^{-2}\mathfrak{L}_q\left[\frac{\partial y(0,q)}{\partial p}\right](s_2)$$
$$-s_2^{-1}\mathfrak{L}_p\left[y(p,0)\right](s_1)-s_2^{-2}\mathfrak{L}_p\left[\frac{\partial y(p,0)}{\partial q}\right](s_1)+s_1^{-1}s_2^{-1}y(0,0)+s_1^{-2}s_2^{-1}\frac{\partial y(0,0)}{\partial p}$$
$$+s_1^{-1}s_2^{-2}\frac{\partial y(0,0)}{\partial q}+s_1^{-2}s_2^{-2}\frac{\partial^2y(0,0)}{\partial q\partial p}\Bigg)=\frac{1}{s_1}\frac{1}{s_2}\,{}^MB_{\frac{\varepsilon_1}{s_1},\frac{\varepsilon_2}{s_2}}^{(\alpha,\beta;m,n)}\left[\begin{matrix}\xi_1,\ldots,\xi_u,1;\eta_1,\ldots,\eta_v\\ \kappa_1,\ldots,\kappa_u,1;\mu_1,\ldots,\mu_v\end{matrix}\middle|\sigma,\tau\right].$$

Using the initial conditions, we get

$$\mathfrak{L}_q\mathfrak{L}_p\left[y(p,q)\right](s_1,s_2)=\frac{1}{s_1^{1+\varepsilon_1}}\frac{1}{s_2^{1+\varepsilon_2}}\,{}^MB_{\frac{\varepsilon_1}{s_1},\frac{\varepsilon_2}{s_2}}^{(\alpha,\beta;m,n)}\left[\begin{matrix}\xi_1,\ldots,\xi_u,1;\eta_1,\ldots,\eta_v\\ \kappa_1,\ldots,\kappa_u,1;\mu_1,\ldots,\mu_v\end{matrix}\middle|\sigma,\tau\right].$$

Application of Eq. (2.2) gives

$$y(p,q)=\frac{p^{\varepsilon_1}}{\Gamma(1+\varepsilon_1)}\frac{q^{\varepsilon_2}}{\Gamma(1+\varepsilon_2)}\,{}^MB_{p,q}^{(\alpha,\beta;m,n)}\left[\begin{matrix}\xi_1,\ldots,\xi_u,1;\eta_1,\ldots,\eta_v,1+\varepsilon_1\\ \kappa_1,\ldots,\kappa_u,1;\mu_1,\ldots,\mu_v,1+\varepsilon_2\end{matrix}\middle|\sigma,\tau\right].\qquad(4.1)$$

Example 4.2. Let $1 < \Re(\varepsilon_1), \Re(\varepsilon_2) \leq 2$. We consider the fractional partial differential equation

$${}_{q}^{c}D_{0+}^{\varepsilon_2}\ {}_{p}^{c}D_{0+}^{\varepsilon_1} y(p,q) = {}^{M}\widehat{F}_{\varepsilon_1 p, \varepsilon_2 q}^{(\alpha,\beta;m,n)}(\vartheta_1, \vartheta_2; \vartheta_3; z),$$

with the initial conditions

$$y(0,0) = \frac{\partial y(0,0)}{\partial p} = \frac{\partial y(0,0)}{\partial q} = \frac{\partial^2 y(0,0)}{\partial q \partial p} = 0,$$
$$y(p,0) = \frac{\partial y(p,0)}{\partial q} = 0,$$
$$y(0,q) = \frac{\partial y(0,q)}{\partial p} = 0.$$

Considering Eqs. (2.3) and (3.24) and application of Eq. (2.1) to the fractional partial differential equation gives

$$\mathfrak{L}_q \mathfrak{L}_p \left[{}_{q}^{c}D_{0+}^{\varepsilon_2}\ {}_{p}^{c}D_{0+}^{\varepsilon_1} y(p,q)\right](s_1,s_2) = \mathfrak{L}_q \mathfrak{L}_p \left[{}^{M}\widehat{F}_{\varepsilon_1 p, \varepsilon_2 q}^{(\alpha,\beta;m,n)}(\vartheta_1, \vartheta_2; \vartheta_3; z)\right](s_1,s_2),$$

then

$$\begin{aligned} &s_1^{\varepsilon_1} s_2^{\varepsilon_2}\Bigg(\mathfrak{L}_q \mathfrak{L}_p [y(p,q)](s_1,s_2) - s_1^{-1}\mathfrak{L}_q[y(0,q)](s_2) - s_1^{-2}\mathfrak{L}_q\left[\frac{\partial y(0,q)}{\partial p}\right](s_2) \\ &- s_2^{-1}\mathfrak{L}_p[y(p,0)](s_1) - s_2^{-2}\mathfrak{L}_p\left[\frac{\partial y(p,0)}{\partial q}\right](s_1) + s_1^{-1}s_2^{-1}y(0,0) + s_1^{-2}s_2^{-1}\frac{\partial y(0,0)}{\partial p} \\ &+ s_1^{-1}s_2^{-2}\frac{\partial y(0,0)}{\partial q} + s_1^{-2}s_2^{-2}\frac{\partial^2 y(0,0)}{\partial q \partial p}\Bigg) = \frac{1}{s_1}\frac{1}{s_2}\ {}^{M}F_{\frac{\varepsilon_1}{s_1},\frac{\varepsilon_2}{s_2}}^{(\alpha,\beta;m,n)}\left[\begin{matrix}\xi_1,\dots,\xi_u,1;\eta_1,\dots,\eta_v\\ \kappa_1,\dots,\kappa_u,1;\mu_1,\dots,\mu_v\end{matrix}\middle|\vartheta_1,\vartheta_2;\vartheta_3;z\right]. \end{aligned}$$

Using the initial conditions, we get

$$\mathfrak{L}_q \mathfrak{L}_p [y(p,q)](s_1,s_2) = \frac{1}{s_1^{1+\varepsilon_1}}\frac{1}{s_2^{1+\varepsilon_2}}\ {}^{M}F_{\frac{\varepsilon_1}{s_1},\frac{\varepsilon_2}{s_2}}^{(\alpha,\beta;m,n)}\left[\begin{matrix}\xi_1,\dots,\xi_u,1;\eta_1,\dots,\eta_v\\ \kappa_1,\dots,\kappa_u,1;\mu_1,\dots,\mu_v\end{matrix}\middle|\vartheta_1,\vartheta_2;\vartheta_3;z\right].$$

Application of Eq. (2.2) gives

$$y(p,q) = \frac{p^{\varepsilon_1}}{\Gamma(1+\varepsilon_1)}\frac{q^{\varepsilon_2}}{\Gamma(1+\varepsilon_2)}\ {}^{M}F_{p,q}^{(\alpha,\beta;m,n)}\left[\begin{matrix}\xi_1,\dots,\xi_u,1;\eta_1,\dots,\eta_v,1+\varepsilon_1\\ \kappa_1,\dots,\kappa_u,1;\mu_1,\dots,\mu_v,1+\varepsilon_2\end{matrix}\middle|\vartheta_1,\vartheta_2;\vartheta_3;z\right]. \tag{4.2}$$

Example 4.3. Let $1 < \Re(\varepsilon_1), \Re(\varepsilon_2) \leq 2$. We consider the fractional partial differential equation

$${}_{q}^{c}D_{0+}^{\varepsilon_2}\ {}_{p}^{c}D_{0+}^{\varepsilon_1} y(p,q) = {}^{M}\widehat{\Phi}_{\varepsilon_1 p, \varepsilon_2 q}^{(\alpha,\beta;m,n)}(\vartheta_2; \vartheta_3; z),$$

with the initial conditions

$$y(0,0) = \frac{\partial y(0,0)}{\partial p} = \frac{\partial y(0,0)}{\partial q} = \frac{\partial^2 y(0,0)}{\partial q \partial p} = 0,$$
$$y(p,0) = \frac{\partial y(p,0)}{\partial q} = 0,$$
$$y(0,q) = \frac{\partial y(0,q)}{\partial p} = 0.$$

Considering Eqs. (2.3) and (3.25) and application of Eq. (2.1) to the fractional partial differential equation gives

$$\mathfrak{L}_q \mathfrak{L}_p \left[{}_{q}^{c}D_{0+}^{\varepsilon_2}\ {}_{p}^{c}D_{0+}^{\varepsilon_1} y(p,q)\right](s_1,s_2) = \mathfrak{L}_q \mathfrak{L}_p \left[{}^{M}\widehat{\Phi}_{\varepsilon_1 p, \varepsilon_2 q}^{(\alpha,\beta;m,n)}(\vartheta_2; \vartheta_3; z)\right](s_1,s_2),$$

then

$$\begin{aligned} & s_1^{\varepsilon_1} s_2^{\varepsilon_2}\Bigg(\mathfrak{L}_q\mathfrak{L}_p\left[y(p,q)\right](s_1,s_2) - s_1^{-1}\mathfrak{L}_q\left[y(0,q)\right](s_2) - s_1^{-2}\mathfrak{L}_q\left[\frac{\partial y(0,q)}{\partial p}\right](s_2) \\ & - s_2^{-1}\mathfrak{L}_p\left[y(p,0)\right](s_1) - s_2^{-2}\mathfrak{L}_p\left[\frac{\partial y(p,0)}{\partial q}\right](s_1) + s_1^{-1}s_2^{-1}y(0,0) + s_1^{-2}s_2^{-1}\frac{\partial y(0,0)}{\partial p} \\ & + s_1^{-1}s_2^{-2}\frac{\partial y(0,0)}{\partial q} + s_1^{-2}s_2^{-2}\frac{\partial^2 y(0,0)}{\partial q\partial p}\Bigg) = \frac{1}{s_1}\frac{1}{s_2}\ {}^{M}\Phi^{(\alpha,\beta;m,n)}_{\frac{\varepsilon_1}{s_1},\frac{\varepsilon_2}{s_2}}\left[\begin{matrix}\xi_1,\ldots,\xi_u,1;\eta_1,\ldots,\eta_v\\ \kappa_1,\ldots,\kappa_u,1;\mu_1,\ldots,\mu_v\end{matrix}\middle|\vartheta_2;\vartheta_3;z\right].\end{aligned}$$

Using the initial conditions, we get

$$\mathfrak{L}_q\mathfrak{L}_p\left[y(p,q)\right](s_1,s_2) = \frac{1}{s_1^{1+\varepsilon_1}}\frac{1}{s_2^{1+\varepsilon_2}}\ {}^{M}\Phi^{(\alpha,\beta;m,n)}_{\frac{\varepsilon_1}{s_1},\frac{\varepsilon_2}{s_2}}\left[\begin{matrix}\xi_1,\ldots,\xi_u,1;\eta_1,\ldots,\eta_v\\ \kappa_1,\ldots,\kappa_u,1;\mu_1,\ldots,\mu_v\end{matrix}\middle|\vartheta_2;\vartheta_3;z\right].$$

Application of Eq. (2.2) gives

$$y(p,q) = \frac{p^{\varepsilon_1}}{\Gamma(1+\varepsilon_1)}\frac{q^{\varepsilon_2}}{\Gamma(1+\varepsilon_2)}\ {}^{M}\Phi^{(\alpha,\beta;m,n)}_{p,q}\left[\begin{matrix}\xi_1,\ldots,\xi_u,1;\eta_1,\ldots,\eta_v,1+\varepsilon_1\\ \kappa_1,\ldots,\kappa_u,1;\mu_1,\ldots,\mu_v,1+\varepsilon_2\end{matrix}\middle|\vartheta_2;\vartheta_3;z\right]. \tag{4.3}$$

The newly defined generalized special functions are expected to have many applications. One area where the new generalized beta function is expected to be useful is statistics. The beta distribution is a continuous probability distribution that is widely used in Bayesian statistics and in the modeling of rates and proportions. The new generalized beta function, on the other hand, is a generalization of the beta function that has found applications in various fields such as physics, engineering, and finance. We now give an illustrative example below.

Example 4.4. We describe the beta distribution of new generalized beta function by

$$F(\omega) = \begin{cases} \frac{\omega^{\sigma-1}(1-\omega)^{\tau-1}\ {}^{\alpha}_{u}M^{\beta}_{v}\left(-\frac{p}{\omega^m}\right){}^{\alpha}_{u}M^{\beta}_{v}\left(-\frac{q}{(1-\omega)^n}\right)}{{}^{M}\widehat{B}^{(\alpha,\beta;m,n)}_{p,q}(\sigma,\tau)} & ,0<\omega<1,\\ 0 & ,\text{otherwise}.\end{cases}$$

If $\lambda\in\mathbb{R}$, then for $-\infty<\sigma<\infty$, $-\infty<\tau<\infty$,

$$E\left(X^{\lambda}\right) = \frac{{}^{M}\widehat{B}^{(\alpha,\beta;m,n)}_{p,q}(\sigma+\lambda,\tau)}{{}^{M}\widehat{B}^{(\alpha,\beta;m,n)}_{p,q}(\sigma,\tau)}.$$

The variance of the distribution is

$$E\left(X^2\right) - \{E(X)\}^2 = \frac{{}^{M}\widehat{B}^{(\alpha,\beta;m,n)}_{p,q}(\sigma,\tau)\ {}^{M}\widehat{B}^{(\alpha,\beta;m,n)}_{p,q}(\sigma+2,\tau) - \left({}^{M}\widehat{B}^{(\alpha,\beta;m,n)}_{p,q}(\sigma+1,\tau)\right)^2}{\left({}^{M}\widehat{B}^{(\alpha,\beta;m,n)}_{p,q}(\sigma,\tau)\right)^2}.$$

The moment generation function of the distribution is

$$M(\omega) = \sum_{k=0}^{\infty} E\left(X^k\right)\frac{\omega^k}{k!} = \sum_{k=0}^{\infty}\frac{{}^{M}\widehat{B}^{(\alpha,\beta;m,n)}_{p,q}(\sigma+k,\tau)}{{}^{M}\widehat{B}^{(\alpha,\beta;m,n)}_{p,q}(\sigma,\tau)}\frac{\omega^k}{k!}.$$

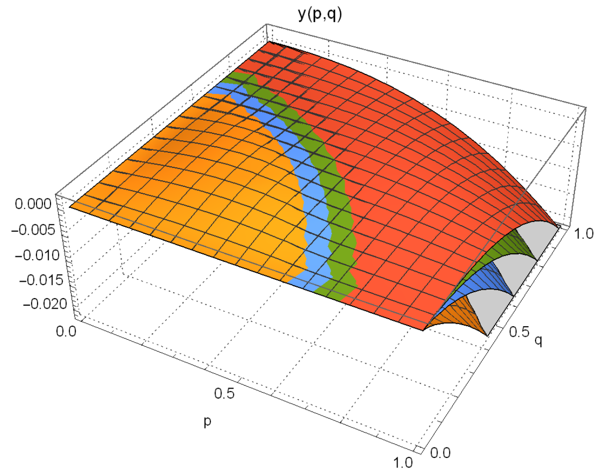
The cummulative distribution of $F(\omega)$ can be written as

$$F(X) = \frac{{}^{M}_{X}\widehat{B}^{(\alpha,\beta;m,n)}_{p,q}(\sigma,\tau)}{{}^{M}\widehat{B}^{(\alpha,\beta;m,n)}_{p,q}(\sigma,\tau)},$$

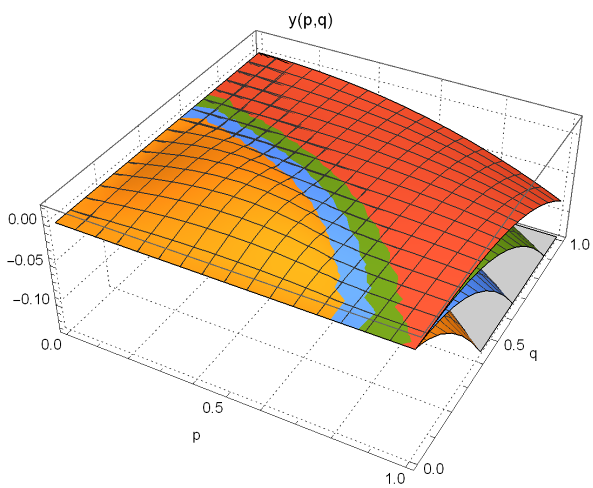
where

$${}^{M}_{X}\widehat{B}^{(\alpha,\beta;m,n)}_{p,q}(\sigma,\tau) = \int_0^X \omega^{\sigma-1}(1-\omega)^{\tau-1}\ {}^{\alpha}_{u}M^{\beta}_{v}\left(-\frac{p}{\omega^m}\right){}^{\alpha}_{u}M^{\beta}_{v}\left(-\frac{q}{(1-\omega)^n}\right)d\omega,$$

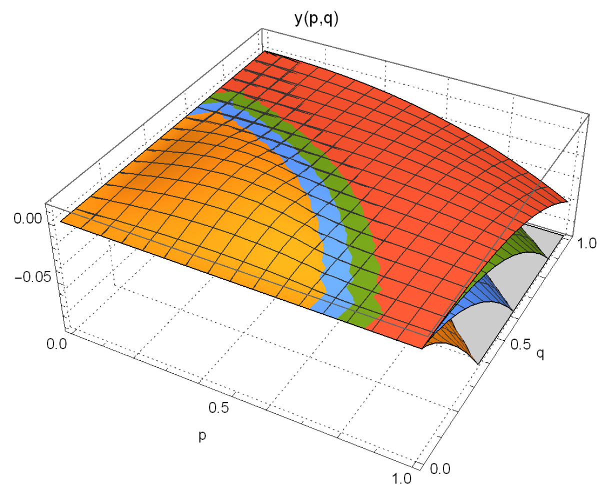
is new generalized incomplete beta function.



(A) The graphs of (4.1) for the values $u = v = \xi = \eta = \kappa = \mu = \alpha = \beta = m = n = 1$, $\sigma = \tau = 2$, $0 < p, q < 1$, generalized M-series indexs $i, j = 0, 1$ and $\varepsilon_1 = \varepsilon_2 = 1.2$ (yellow), $\varepsilon_1 = \varepsilon_2 = 1.4$ (blue), $\varepsilon_1 = \varepsilon_2 = 1.6$ (green), $\varepsilon_1 = \varepsilon_2 = 1.8$ (red).



(B) The graphs of (4.2) for the values $u = v = \xi = \eta = \kappa = \mu = \alpha = \beta = m = n = 1$, $z = 0.5$, $\vartheta_1 = 3$, $\vartheta_2 = 2$, $\vartheta_3 = 4$, $0 < p, q < 1$, Gauss hypergeometric series index $k = 0, 1$, generalized M-series indexs $i, j = 0, 1$ and $\varepsilon_1 = \varepsilon_2 = 1.2$ (yellow), $\varepsilon_1 = \varepsilon_2 = 1.4$ (blue), $\varepsilon_1 = \varepsilon_2 = 1.6$ (green), $\varepsilon_1 = \varepsilon_2 = 1.8$ (red).



(C) The graphs of (4.3) for the values $u = v = \xi = \eta = \kappa = \mu = \alpha = \beta = m = n = 1$, $z = 0.5$, $\vartheta_2 = 2$, $\vartheta_3 = 4$, $0 < p, q < 1$, confluent hypergeometric series index $k = 0, 1$, generalized M-series indexs $i, j = 0, 1$ and $\varepsilon_1 = \varepsilon_2 = 1.2$ (yellow), $\varepsilon_1 = \varepsilon_2 = 1.4$ (blue), $\varepsilon_1 = \varepsilon_2 = 1.6$ (green), $\varepsilon_1 = \varepsilon_2 = 1.8$ (red).

FIGURE 1. Approximate behaviors of the solutions of Examples (4.1), (4.2) and (4.3).

Note 4.5. In our forthcoming studies about the beta distribution of this new generalized beta function, we hope to gain a deeper understanding of its properties and potential applications as well as its relationship to other well-known distributions such as the gamma distribution and the beta prime distribution.

5. Conclusion

In this paper, we defined generalized beta, Gauss hypergeometric, and confluent hypergeometric functions including two generalized M-series at their kernels and gave their basic properties. As examples, we obtained the solutions of fractional partial differential equations involving these new generalized special functions by means of the double Laplace transform and then presented the graphs of their approximate behavior in Figure 1. We also introduced the beta distribution of the generalized beta function and the generalized incomplete beta function.

Since the generalized M-series has a more general form than most special functions, it becomes a special case of many generalized special functions in the literature. Therefore, we conclude this paper by presenting the relationship between the special functions defined in this paper and other special functions found in the literature below.

Mubeen et al. [19]:

$$ {}^{M}B_{p,q}^{(1,1;1,1)}\left[\begin{matrix}\xi;\eta\\ \xi;\eta\end{matrix}\middle|\sigma,\tau\right]=B_{p,q}^{\xi,\eta}(\sigma,\tau). $$

Rahman et al. [23]:

$$ {}^{M}B_{p,q}^{(\alpha,1;1,1)}\left[\begin{matrix}\xi;\xi\\ \eta;\eta\end{matrix}\middle|\sigma,\tau\right]=B_{p,q}^{\alpha}(\sigma,\tau), $$

$$ {}^{M}F_{p,q}^{(\alpha,1;1,1)}\left[\begin{matrix}\xi;\xi\\ \eta;\eta\end{matrix}\middle|\vartheta_1,\vartheta_2;\vartheta_3;z\right]=F_{p,q}^{\alpha}(\vartheta_1,\vartheta_2;\vartheta_3;z), $$

$$ {}^{M}\Phi_{p,q}^{(\alpha,1;1,1)}\left[\begin{matrix}\xi;\xi\\ \eta;\eta\end{matrix}\middle|\vartheta_2;\vartheta_3;z\right]=\Phi_{p,q}^{\alpha}(\vartheta_2;\vartheta_3;z). $$

Şahin et al. [29]:

$$ {}^{M}B_{p,q}^{(1,1;m,n)}\left[\begin{matrix}\xi;\xi\\ \eta;\eta\end{matrix}\middle|\sigma,\tau\right]=B_{p,q}^{(m,n)}(\sigma,\tau), $$

$$ {}^{M}F_{p,q}^{(1,1;m,n)}\left[\begin{matrix}\xi;\xi\\ \eta;\eta\end{matrix}\middle|\vartheta_1,\vartheta_2;\vartheta_3;z\right]=F_{p,q}^{(m,n)}(\vartheta_1,\vartheta_2;\vartheta_3;z), $$

$$ {}^{M}\Phi_{p,q}^{(1,1;m,n)}\left[\begin{matrix}\xi;\xi\\ \eta;\eta\end{matrix}\middle|\vartheta_2;\vartheta_3;z\right]=\Phi_{p,q}^{(m,n)}(\vartheta_2;\vartheta_3;z). $$

Classic functions [2]:

$$ {}^{M}B_{0,0}^{(\alpha,1;m,n)}\left[\begin{matrix}\xi_1,\ldots,\xi_u;\eta_1,\ldots,\eta_v\\ \kappa_1,\ldots,\kappa_u;\mu_1,\ldots,\mu_v\end{matrix}\middle|\sigma,\tau\right]=B(\sigma,\tau), $$

$$ {}^{M}F_{0,0}^{(\alpha,1;m,n)}\left[\begin{matrix}\xi_1,\ldots,\xi_u;\eta_1,\ldots,\eta_v\\ \kappa_1,\ldots,\kappa_u;\mu_1,\ldots,\mu_v\end{matrix}\middle|\vartheta_1,\vartheta_2;\vartheta_3;z\right]=F(\vartheta_1,\vartheta_2;\vartheta_3;z), $$

$$ {}^{M}\Phi_{0,0}^{(\alpha,1;m,n)}\left[\begin{matrix}\xi_1,\ldots,\xi_u;\eta_1,\ldots,\eta_v\\ \kappa_1,\ldots,\kappa_u;\mu_1,\ldots,\mu_v\end{matrix}\middle|\vartheta_2;\vartheta_3;z\right]=\Phi(\vartheta_2;\vartheta_3;z). $$

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