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Fractional Chebyshev differential equation on a symmetric α dependent interval

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Abstract

Most of fractional differential equations are considered on a fixed interval. In this paper, we consider a typical fractional differential equation on a symmetric interval $[-\alpha, \alpha]$, where α is the order of fractional derivative. For a positive real number α we prove that the solutions are $T_{n,\alpha}(x) = (\alpha + x)^{\frac{1}{2}} Q_{n,\alpha}(x)$, where $Q_{n,\alpha}(x)$ produce a family of orthogonal polynomials with respect to the weight function $w_\alpha(x) = (\frac{\alpha+x}{\alpha-x})^{\frac{1}{2}}$ on $[-\alpha, \alpha]$. For integer case $\alpha = 1$, we show that these polynomials coincide with classical Chebyshev polynomials of the third kind. Orthogonal properties of the solutions lead to practical results in determining solutions of some fractional differential equations.

Keywords. Orthogonal polynomials, Fractional Chebyshev differential equation, Riemann-Liouville and Caputo derivatives.

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1. Introduction

Important orthogonal polynomials such as Chebyshev, Legendre, Hermite, and Laguerre are the solutions of the integer order Sturm-Liouville equation. In [8, 10, 17, 18] fractional Sturm-Liouville problems are considered and some spectral properties such as orthogonality of eigenfunctions corresponding to distinct eigenvalues are studied. Numerical solutions for fractional Sturm-Liouville problems are studied in [3, 7, 12]. Moreover, solving fractional Lagrange equation leads to fractional Sturm-Liouville problems. Fractional forms of important equations such as Legendre, Chebyshev, Lagurre and Hermite equations have been considered in [1, 9, 11, 18].

The Chebyshev equation in classical case is a second-order linear differential equation and the solutions are polynomials of the first, second, third and fourth kind Chebyshev polynomials, see [6, 14] for more details. In this paper, we define a new form of Fractinal Chebyshev Differential Equation (FCDE) of the following form which is defined on the interval $[-\alpha, \alpha]$

$$\left[{}^{c}D^{\alpha}_{\alpha^-} (\alpha^2 - x^2)^{\alpha-\frac{1}{2}} D^{\alpha}_{-\alpha^+} - \lambda_{n,\alpha} (\alpha^2 - x^2)^{-\frac{1}{2}}\right] y(x) = 0, \ x \in [-\alpha, \alpha], \tag{1.1}$$

where ${}^{c}D^{\alpha}_{\alpha^-}$ and $D^{\alpha}_{-\alpha^+}$ are Caputo and Riemann-Liouville fractional derivatives, respectively. Note that for $\alpha = 1$ the equation (1.1) is classical Chebyshev differential equation of first kind, where $\lambda_{n,1} = n^2$ for Chebyshev polynomials of first kind [15]. Our main goal in this paper is to generalize the results to FCDE by finding similar orthogonal polynomials. Fractional differential equations with non-uniform intervals depending to fractional order α appear in approximating time-dependent fractional diffrential equations by the corresponding finite difference equations.There are applications in the Chaos theory. For more details see [4, 13].

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2. Preliminaries

In this section, we present some preliminary materials of fractional calculus[5, 10, 16] and Chebyshev polynomials of the third kind [2, 15, 19]. Assume $J_n^{(\alpha,\beta)}(x)$ and $V_n(x)$ are Jacobi polynomials and Chebyshev polynomials of the third kind of degree n, respectively. An explicit form of Jacobi polynomials is defined by

$$J_n^{(\alpha,\beta)}(x) = 2^{-n} \sum_{k=0}^{n} \binom{n+\alpha}{k} \binom{n+\beta}{n-k} (x-1)^{n-k}(x+1)^k, \ \alpha > -1, \ \beta > -1. \tag{2.1}$$

Another form is defined by

$$J_n^{(\alpha,\beta)}(x) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(\alpha+\beta+n+m+1)}{2^m\Gamma(\alpha+m+1)} (x-1)^m, \ \alpha > -1, \ \beta > -1. \tag{2.2}$$

Using Eq. (2.1), the following identity is immediate

$$J_n^{(\alpha,\beta)}(-x) = (-1)^n J_n^{(\beta,\alpha)}(x). \tag{2.3}$$

The relationship between Chebyshev polynomials of the third kind and Jacobi polynomials is as follows:

$$\binom{2n}{n} V_n(x) = 2^{2n} J_n^{(-\frac{1}{2},\frac{1}{2})}(x). \tag{2.4}$$

The Chebyshev polynomials of the third kind satisfy the following recurrence relation:

$$V_n(x) = 2xV_{n-1}(x) - V_{n-2}(x), \ V_0(x) = 1, \ V_1(x) = 2x - 1. \tag{2.5}$$

Moreover, the following orthogonality property holds:

$$\int_{-1}^{1} \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} V_m(x)V_n(x)dx = \pi\delta_{m,n}. \tag{2.6}$$

By using relations (2.4), (2.3), and (2.2), the following explicit formula is obtained:

$$V_n(x) = \frac{(-1)^n 2^{2n}\Gamma(n+\frac{3}{2})}{(2n)!} \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{\Gamma(n+m+1)}{2^m\Gamma(m+\frac{3}{2})} (1+x)^m. \tag{2.7}$$

Left and right Riemann-Liouville integrals of order α are defined by

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(s)}{(x-s)^{1-\alpha}} ds, \ x > a,$$

$$I_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(s)}{(s-x)^{1-\alpha}} ds, \ x < b,$$

where α is a positive real number. If $m - 1 < \alpha < m$, where m is an integer, then left and right Riemann-Liouville and Caputo fractional derivatives are defined by

$$(D_{a^+}^{\alpha} f)(x) = D^m (I_{a^+}^{m-\alpha} f)(x), \ x > a,$$

$$(D_{b^-}^{\alpha} f)(x) = (-D)^m (I_{b^-}^{m-\alpha} f)(x), \ x < b,$$

$$({}^cD_{a^+}^{\alpha} f)(x) = (I_{a^+}^{m-\alpha} D^m f)(x), \ x > a,$$

$$({}^cD_{b^-}^{\alpha} f)(x) = (I_{b^-}^{m-\alpha} (-D)^m f)(x), \ x < b.$$

Using integration by parts it is easy to see the following equalities hold:

$$\int_a^b f(x)D_{b^-}^{\alpha} g(x)dx = \int_a^b g(x){}^cD_{a^+}^{\alpha} f(x)dx + \sum_{k=0}^{m-1} (-1)^{m-k} f^{(k)}(x) D^{m-k-1} I_{b^-}^{m-\alpha} g(x)|_{x=a}^{b}, \tag{2.8}$$

$$\int_a^b f(x)D^\alpha_{a+}g(x)dx = \int_a^b g(x)^cD^\alpha_{b-}f(x)dx + \sum_{k=0}^{m-1}(-1)^k f^{(k)}(x)D^{m-k-1}I^{m-\alpha}_{a+}g(x)|^b_{x=a}. \tag{2.9}$$

3. FCDE on symmetric interval depending on α

The classical Chebyshev differential equation of first kind is a second-order linear differential equation of the form

$$\left[(-D)(1-x^2)^{\frac{1}{2}}D - \lambda_n(1-x^2)^{-\frac{1}{2}}\right]y(x) = 0,\ x\in[-1,1], \tag{3.1}$$

where $\lambda_{n,1} = n^2$ for Chebyshev polynomials of first kind. Now we define a new FCDE of the form (1.1), where $\lambda_{n,\alpha}$ will be computed later. Using (2.8) and (2.9) on the symmetric interval $[-\alpha,\alpha]$ leads to

$$\int_{-\alpha}^{\alpha}\left[g(x).^cD^\alpha_{\alpha-}(\alpha^2-x^2)^{\alpha-\frac{1}{2}}D^\alpha_{-\alpha+}f(x) - f(x).^cD^\alpha_{\alpha-}(\alpha^2-x^2)^{\alpha-\frac{1}{2}}D^\alpha_{-\alpha+}g(x)\right]dx = 0. \tag{3.2}$$

Now we compute the orthogonal polynomials $Q_{n,\alpha}(x)$ and related eigenvalues $\lambda_{n,\alpha}$ in the following theorem. We find a recursive formula for coefficients of $Q_{n,\alpha}(x)$, where the coefficient of the leading term is $a_n = (\frac{2}{\alpha})^n$.

Theorem 3.1. *The solution of the Fractional Chebyshev differential equation (1.1) is* $T_{n,\alpha}(x) = (\alpha+x)^{\frac{1}{2}}Q_{n,\alpha}(x),\ n = 0,1,2,\cdots,$ *where* α *is a positive real number and*

$$\lambda_{n,\alpha} = \frac{(n+\frac{1}{2})\Gamma(n+\alpha+\frac{1}{2})}{\Gamma(n-\alpha+\frac{3}{2})}, \tag{3.3}$$

and

$$Q_{n,\alpha}(x) = \sum_{k=0}^{n} a_k(\alpha+x)^k. \tag{3.4}$$

The coefficients a_k *are functions of* α *and they are obtained by backward substitution starting with* $a_n = (\frac{2}{\alpha})^n$ *as follows*

$$a_k = \frac{1}{\lambda_{n,\alpha}-\lambda_{k,\alpha}}\sum_{i=k+1}^{n}(2\alpha)^{i-k}\binom{i}{i-k}\left[\frac{\Gamma(i+\frac{3}{2}).\Gamma(k+\alpha+\frac{1}{2})}{\Gamma(k+\frac{1}{2})\Gamma(i-\alpha+\frac{3}{2})} - \lambda_{n,\alpha}\right]a_i,$$
$$k = n-1, n-2, ..., 1, 0. \tag{3.5}$$

Proof. The proof is constructive and the solution is obtained by substituting $T_{n,\alpha}(x)$ in FCDE (1.1). We have

$$\begin{aligned}(\alpha^2-x^2)^{-\frac{1}{2}}T_{n,\alpha}(x) &= (\alpha-x)^{-\frac{1}{2}}\sum_{k=0}^{n}a_k(\alpha+x)^k\\ &= (\alpha-x)^{-\frac{1}{2}}\sum_{k=0}^{n}a_k(2\alpha-(\alpha-x))^k\\ &= \sum_{k=0}^{n}\sum_{j=0}^{k}a_k\binom{k}{j}(-1)^{k-j}(2\alpha)^j(\alpha-x)^{k-j-\frac{1}{2}}.\end{aligned}$$

Therefore,

$$\begin{aligned}(\alpha^2-x^2)^{\alpha-\frac{1}{2}}D^\alpha_{-\alpha+}T_{n,\alpha}(x) &= (\alpha^2-x^2)^{\alpha-\frac{1}{2}}D^\alpha_{-\alpha+}\sum_{k=0}^{n}a_k(x+\alpha)^{k+\frac{1}{2}}\\ &= (\alpha^2-x^2)^{\alpha-\frac{1}{2}}\sum_{k=0}^{n}a_k\frac{\Gamma(k+\frac{3}{2})}{\Gamma(k-\alpha+\frac{3}{2})}(x+\alpha)^{k+\frac{1}{2}-\alpha}\\ &= (\alpha-x)^{\alpha-\frac{1}{2}}\sum_{k=0}^{n}a_k\frac{\Gamma(k+\frac{3}{2})}{\Gamma(k-\alpha+\frac{3}{2})}(2\alpha-(\alpha-x))^k\end{aligned}$$

$$= \sum_{k=0}^{n}\sum_{j=0}^{k} a_k \binom{k}{j}(-1)^{k-j}(2\alpha)^j \frac{\Gamma(k+\frac{3}{2})}{\Gamma(k-\alpha+\frac{3}{2})}(\alpha-x)^{k-j+\alpha-\frac{1}{2}}. \tag{3.6}$$

Taking the right Caputo derivative of the equation (3.6) implies that

$${}^{c}D^{\alpha}_{\alpha^-}(\alpha^2-x^2)^{\alpha-\frac{1}{2}} D^{\alpha}_{-\alpha^+}T_{n,\alpha}(x) = \sum_{k=0}^{n}\sum_{j=0}^{k} a_k \binom{k}{j}(-1)^{k-j}(2\alpha)^j \frac{\Gamma(k+\frac{3}{2}).\Gamma(k-j+\alpha+\frac{1}{2})}{\Gamma(k-\alpha+\frac{3}{2}).\Gamma(k-j+\frac{1}{2})}(\alpha-x)^{k-j-\frac{1}{2}}.$$

Now substitution in FCDE (1.1) implies that

$$\sum_{k=0}^{n}\sum_{j=0}^{k} a_k \binom{k}{j}(-1)^{k-j}(2\alpha)^j \frac{\Gamma(k+\frac{3}{2}).\Gamma(k-j+\alpha+\frac{1}{2})}{\Gamma(k-\alpha+\frac{3}{2}).\Gamma(k-j+\frac{1}{2})}(\alpha-x)^{k-j-\frac{1}{2}}$$
$$-\lambda_{\alpha,n}\sum_{k=0}^{n}\sum_{j=0}^{k} a_k \binom{k}{j}(-1)^{k-j}(2\alpha)^j(\alpha-x)^{k-j-\frac{1}{2}} = 0. \tag{3.7}$$

Equating the cofficients of $(\alpha-x)^k$ to zero we find an algebraic system that computes a_k. Equating the cofficient of $(\alpha-x)^n$ to zero we find

$$a_n\frac{\Gamma(n+\frac{3}{2})\Gamma(n+\alpha+\frac{1}{2})}{\Gamma(n-\alpha+\frac{3}{2})\Gamma(n+\frac{1}{2})}(\alpha-x)^n - \lambda_{n,\alpha}a_n(\alpha-x)^n = 0.$$

Thus we find $\lambda_{n,\alpha} = \dfrac{(n+\frac{1}{2})\Gamma(n+\alpha+\frac{1}{2})}{\Gamma(n-\alpha+\frac{3}{2})}$. It is clear that a_n can be any arbitrary nonzero real number. Now we find a recursive formula to compute the coefficients a_k. Expanding the sums, choosing a_n an arbitrary real number and equating the coefficient of $(\alpha-x)^{n-1}$ to zero implies that

$$a_{n-1} = \frac{-1}{\lambda_{\alpha,n-1}-\lambda_{n,\alpha}}\binom{n}{1}(2\alpha)\left[\frac{\Gamma(n+\frac{3}{2})\Gamma(n-1+\alpha+\frac{1}{2})}{\Gamma(n-\alpha+\frac{3}{2})\Gamma(n-1+\frac{1}{2})} - \lambda_{n,\alpha}\right]a_n.$$

Similarly we find

$$\begin{aligned} a_{n-2} \quad &= \quad \frac{-1}{\lambda_{n-2,\alpha}-\lambda_{n,\alpha}}\left[\binom{n-1}{1}(2\alpha)(\frac{\Gamma(n-1+\frac{3}{2})\Gamma(n-2+\alpha+\frac{1}{2})}{\Gamma(n-1-\alpha+\frac{3}{2})\Gamma(n-2+\frac{1}{2})} - \lambda_{n,\alpha})a_{n-1}\right. \\ &+ \quad \left.\binom{n-1}{2}(2\alpha)^2(\frac{\Gamma(n+\frac{3}{2})\Gamma(n-2+\alpha+\frac{1}{2})}{\Gamma(n-\alpha+\frac{3}{2})\Gamma(n-2+\frac{1}{2})} - \lambda_{n,\alpha})a_n\right]. \end{aligned}$$

Using induction and simple calculations, we obtain the coefficient a_k given by (3.5). □

One of the most important objectives of this paper is the investigation of orthogonal properties of $T_{n,\alpha}$. The significance of this paper is to obtain the orthogonal functions on a variable interval. This is proved in the following Theorem.

Theorem 3.2. *For two distinct nonnegative integers m and n the functions $T_{m,\alpha}(x)$ and $T_{n,\alpha}(x)$ are orthogonal on the interval $[-\alpha,\alpha]$, that is*

$$\int_{-\alpha}^{\alpha}\left(\alpha^2-x^2\right)^{-\frac{1}{2}}T_{m,\alpha}(x)T_{n,\alpha}(x)dx = 0. \tag{3.8}$$

Moreover the polynomials $Q_{m,\alpha}(x)$ and $Q_{n,\alpha}(x)$ are orthogonal with different weight function as follows

$$\int_{-\alpha}^{\alpha}\left(\frac{\alpha+x}{\alpha-x}\right)^{\frac{1}{2}}Q_{m,\alpha}(x)Q_{n,\alpha}(x)dx = 0. \tag{3.9}$$

Proof. Since $T_{m,\alpha}(x)$ and $T_{n,\alpha}(x)$ are solutions of Equation (1.1) thus we have

$$\left[{}^cD^{\alpha}_{\alpha-}(\alpha^2-x^2)^{\alpha-\frac{1}{2}}D^{\alpha}_{-\alpha+}-\lambda_{n,\alpha}(\alpha^2-x^2)^{-\frac{1}{2}}\right]T_{n,\alpha}(x)=0,$$
$$\left[{}^cD^{\alpha}_{\alpha-}(\alpha^2-x^2)^{\alpha-\frac{1}{2}}D^{\alpha}_{-\alpha+}-\lambda_{m,\alpha}(\alpha^2-x^2)^{-\frac{1}{2}}\right]T_{m,\alpha}(x)=0.$$

Multiplying the first equation by $T_{m,\alpha}$ and the second equation by $T_{n,\alpha}$ and subtracting the results implies

$$T_{m,\alpha}(x){}^cD^{\alpha}_{\alpha-}(\alpha^2-x^2)^{\alpha-\frac{1}{2}}D^{\alpha}_{-\alpha+}T_{n,\alpha}(x)-T_{n,\alpha}(x){}^cD^{\alpha}_{\alpha-}(\alpha^2-x^2)^{\alpha-\frac{1}{2}}D^{\alpha}_{-\alpha+}T_{m,\alpha}(x)$$
$$=\left[\lambda_{n,\alpha}-\lambda_{m,\alpha}\right](\alpha^2-x^2)^{-\frac{1}{2}}T_{n,\alpha}(x)T_{m,\alpha}(x).$$

Now integrating over interval $[-\alpha,\alpha]$ and applying relation (3.2), we have

$$\left[\lambda_{n,\alpha}-\lambda_{m,\alpha}\right]\int_{-\alpha}^{\alpha}\left(\alpha^2-x^2\right)^{-\frac{1}{2}}T_{m,\alpha}(x)T_{n,\alpha}(x)dx=0,$$

Which completes the orthogonality relation (3.8). For orthogonality of $Q_{m,\alpha}(x)$ and $Q_{n,\alpha}(x)$ it suffices to use $T_{n,\alpha}(x)=(\alpha+x)^{\frac{1}{2}}Q_{n,\alpha}(x)$ and $T_{m,\alpha}(x)=(\alpha+x)^{\frac{1}{2}}Q_{m,\alpha}(x)$. □

Now we give an interesting relation between the polynomials $Q_{n,\alpha}(x)$ and the Chebyshev polynomials of the third kind V_n in the following theorem.

Theorem 3.3. *If n is a nonnegative integer and $x\in[-\alpha,\alpha]$ then for $a_n=(\frac{2}{\alpha})^n$, we have*

$$Q_{n,\alpha}(\alpha x)=V_n(x). \tag{3.10}$$

Proof. We use induction to prove this result. For $n=0$ the statement is true since we have

$$Q_{0,\alpha}(\alpha x)=1=V_0(x).$$

Suppose that the statement is true for all $j<n$, i.e.

$$Q_{j,\alpha}(\alpha x)=V_j(x).$$

We may write $Q_{n,\alpha}(\alpha x)$ as a linear combination as follows

$$Q_{n,\alpha}(\alpha x)=\sum_{k=0}^{n}A_k^nV_k(x)=A_n^nV_n(x)+\sum_{k=0}^{n-1}A_k^nQ_{k,\alpha}(\alpha x). \tag{3.11}$$

Using (3.9) for two different and arbitrary indices n,j and changing variables $x=\alpha u$ implies that

$$\int_{-1}^{1}(\frac{1+u}{1-u})^{\frac{1}{2}}Q_{n,\alpha}(\alpha u)Q_{j,\alpha}(\alpha u)du=0.$$

Multiplying both sides of Eq. (3.11) by $(\frac{1+x}{1-x})^{\frac{1}{2}}Q_{j,\alpha}(\alpha x)$ for $j<n$ and integrating over $[-1,1]$ implies that $A_j^n=0$, for $j=0,1,2,\cdots,n-1$. Thus by using Eq. (3.11) we find

$$Q_{n,\alpha}(\alpha x)=A_n^nV_n(x),$$

On the other hand, we have $a_n=(\frac{2}{\alpha})^n$. Since the leading coefficients of $Q_{n,\alpha}(\alpha x)$ and $V_n(x)$ are both 2^n, we conclude $A_n^n=1$ that completes the proof. □

Remark 3.4. Theorem 3.3 implies that the coefficients of polynomial $Q_{n,\alpha}(x)$ are

$$a_k=\frac{(-1)^n2^{2n}\Gamma(n+\frac{3}{2})}{(2n)!}(-1)^k\binom{n}{k}\frac{\Gamma(n+k+1)}{\alpha^k2^k\Gamma(k+\frac{3}{2})}. \tag{3.12}$$

Corollary 3.5. *Orthogonality of the polynomials $Q_{n,\alpha}(x)$ implies that*

$$\int_{-\alpha}^{\alpha} \left(\alpha^2 - x^2\right)^{-\frac{1}{2}} T_{\alpha,m}(x) T_{\alpha,n}(x) dx = \alpha\pi\delta_{mn}, \tag{3.13}$$

and

$$\int_{-\alpha}^{\alpha} \left(\frac{\alpha+x}{\alpha-x}\right)^{\frac{1}{2}} Q_{\alpha,m}(x) Q_{\alpha,n}(x) dx = \alpha\pi\delta_{mn}. \tag{3.14}$$

Proof. If $m = n$ and $x = \alpha u$ then we have

$$\int_{-\alpha}^{\alpha} \left(\alpha^2 - x^2\right)^{-\frac{1}{2}} T^2_{\alpha,n}(x) dx = \int_{-\alpha}^{\alpha} \left(\frac{\alpha+x}{\alpha-x}\right)^{\frac{1}{2}} Q^2_{\alpha,n}(x) dx = \alpha \int_{-1}^{1} \left(\frac{1+u}{1-u}\right)^{\frac{1}{2}} Q^2_{\alpha,n}(\alpha u) du.$$

Using (3.10) and (2.6) we conclude the results. □

Definition 3.6. The Chebyshev norm of the function $f(x)$ is denoted by $\|f\|_C$ and it is difined by

$$\|f\|_C = \left(\int_{-\alpha}^{\alpha} (\alpha^2 - x^2)^{-\frac{1}{2}} f(x) T_{n,\alpha}(x) dx\right)^{\frac{1}{2}}. \tag{3.15}$$

The corollary 3.5 implies that

$$\|T_{n,\alpha}\|_L^2 = \alpha\pi. \tag{3.16}$$

Now we introduce a recursive formula to compute $Q_{n,\alpha}(x)$ in the following theorem.

Lemma 3.7. *polynomials $Q_{k,\alpha}(x)$ satisfy the following recursive formula*

$$\begin{aligned} &Q_{k+1,\alpha}(x) = \frac{2x}{\alpha} Q_{k,\alpha}(x) - Q_{k-1,\alpha}(x), \;\; k \geq 1, \\ &Q_{0,\alpha}(x) = 1, \;\; Q_{1,\alpha}(x) = \tfrac{2}{\alpha}x - 1. \end{aligned} \tag{3.17}$$

Proof. Using (3.10) and (2.5), we conclude that

$$Q_{k+1,\alpha}(\alpha x) = 2x Q_{k,\alpha}(\alpha x) - Q_{k-1,\alpha}(\alpha x), \;\; k \geq 1,$$

$$Q_{0,\alpha}(\alpha x) = 1, \;\; Q_{1,\alpha}(\alpha x) = 2x - 1.$$

Changing variable $\alpha x = u$ in the last equations implies the result. □

4. Integral transform on symmetric interval $[-\alpha, \alpha]$

Integral transforms with respect to orthogonal functions are nice tools in solving classical differential equations. Some of the well-known integral transforms in classical analysis are Laplace and Fourier transforms. For fractional differential equations there are similar terminology and applications. Now we define an integral transforms corresponding to $T_{n,\alpha}$ and we define the corresponding inverse transform. Similar to the classical case, we try to apply this concept to find the solution of some nonhomogeneous fractional differential equations.

Definition 4.1. Let $F(n)$ be the integral transform of a function $f \in L_2[-\alpha, \alpha]$ in terms of $T_{n,\alpha}$ defined by

$$F(n) = T[f](n) = \int_{-\alpha}^{\alpha} (\alpha^2 - x^2)^{\frac{-1}{2}} f(x) T_{n,\alpha}(x) dx. \tag{4.1}$$

The inverse transform is defined by

$$T^{-1}[F(n)](x) = \sum_{n=0}^{\infty} \frac{1}{\alpha\pi} F(n) T_{n,\alpha}(x). \tag{4.2}$$

Lemma 4.2. *Suppose* $J_\alpha = {}^c D^\alpha_{\alpha^-} (\alpha^2 - x^2)^{\alpha - \frac{1}{2}} D^\alpha_{-\alpha^+}$. *Then*

$$T\left[(\alpha^2 - x^2)^{\frac{1}{2}} J_\alpha f\right] = \lambda_{n,\alpha} F(n). \tag{4.3}$$

Proof. Using (3.2) we have the following equality

$$\begin{aligned} T\left[(\alpha^2 - x^2)^{\frac{1}{2}} J_\alpha f(x)\right] &= \int_{-\alpha}^{\alpha} J_\alpha f(x).T_{n,\alpha}(x)dx = \int_{-\alpha}^{\alpha} f(x).J_\alpha T_{n,\alpha}(x)dx \\ &= \lambda_{n,\alpha} \int_{-\alpha}^{\alpha} (\alpha^2 - x^2)^{-\frac{1}{2}} f(x) T_{n,\alpha}(x)dx = \lambda_{n,\alpha} F(n), \end{aligned}$$

which completes the proof. □

If the integral transform of a given function $g \in L_2[-\alpha, \alpha]$ has a specific asymptotic property, then we can find the solution of a nonhomogeneous fractional differential equations of the form $\left[(\alpha^2 - x^2)^{\frac{1}{2}} J_\alpha - \lambda\right] f = g$ by using integral transform. Indeed we have the following Lemma.

Lemma 4.3. *Suppose* $\lambda \neq \lambda_{n,\alpha}$. *If the integral transform of g satisfies the following inequality*

$$|G(n)| \leq M n^\beta, \; n > n_0, \tag{4.4}$$

then the solution of fractional differential equation

$$\left[(\alpha^2 - x^2)^{\frac{1}{2}} J_\alpha - \lambda\right] y(x) = g(x), \tag{4.5}$$

for $2\alpha > \beta + 1$ *is given by the following series*

$$y(x) = \sum_{n=0}^{\infty} \frac{G(n)}{\alpha\pi(\lambda_{n,\alpha} - \lambda)} T_{n,\alpha}. \tag{4.6}$$

Proof. Taking integral transform of (4.5) and using (4.3) implies that

$$[\lambda_{\alpha,n} - \lambda]\, Y(n) = G(n),$$

which implies $Y(n) = \frac{G(n)}{\lambda_{\alpha,n} - \lambda}$. Applying inverse transform (4.2), the function $y(x)$ could be expressed in the form (4.6). For $n > n_0$ we have

$$\| \frac{G(n)}{\alpha\pi(\lambda_{n,\alpha} - \lambda)} T_{n,\alpha} \|_C \leq \frac{M n^\beta}{\sqrt{\alpha\pi} |\lambda_{n,\alpha} - \lambda|}.$$

Using the asymptotic property of the eigenvalues [8] we find

$$\lambda_{n,\alpha} \cong (n + \frac{1}{2})^{2\alpha}, \quad n \longrightarrow \infty. \tag{4.7}$$

Thus we have

$$\frac{M n^\beta}{\sqrt{\alpha\pi} |\lambda_{n,\alpha} - \lambda|} \cong \frac{M}{\sqrt{\alpha\pi} n^{2\alpha - \beta}}.$$

The assumption $2\alpha > \beta + 1$ implies uniform convergence of (4.6) on $[-\alpha, \alpha]$. □

Example 4.4. *For a fix* $m \in \mathbb{N}$, *we consider the following nonhomogeneous fractional differential equation on the interval* $[-\alpha, \alpha]$

$$\left[(\alpha^2 - x^2)^{\frac{1}{2}} J_\alpha - \lambda\right] f(x) = T_{m,\alpha}(x). \tag{4.8}$$

Taking the integral transform of (4.8) implies

$$F(m) = \frac{\alpha\pi}{\lambda_{m,\alpha} - \lambda},$$

TABLE 1. Results of Example 4.5 for $\alpha = 0.9$ and $\lambda = 8$.

N	$\|f_N - f_{18}\|_\infty$	$\|f_N - f_{18}\|_c$
4	2.7×10^{-2}	1.9×10^{-2}
6	1.0×10^{-2}	7.0×10^{-3}
8	4.3×10^{-3}	3.6×10^{-3}
10	3.0×10^{-3}	2.1×10^{-3}
12	2.2×10^{-3}	1.4×10^{-3}
14	1.3×10^{-3}	9.0×10^{-4}
16	6.1×10^{-4}	5.7×10^{-4}

TABLE 2. Results of Example 4.5 for $\alpha = 1.5$ and $\lambda = 8$.

N	$\|f_N - f_{18}\|_\infty$	$\|f_N - f_{18}\|_c$
4	2.7×10^{-3}	2.0×10^{-3}
6	9.6×10^{-4}	5.9×10^{-4}
8	4.0×10^{-4}	2.4×10^{-4}
10	1.7×10^{-4}	1.2×10^{-4}
12	9.8×10^{-5}	6.5×10^{-5}
14	5.6×10^{-5}	3.8×10^{-5}
16	2.4×10^{-5}	2.2×10^{-5}

and $F(n) = 0$ for $n \neq m$. Using relation (4.2), the particular solution of the nonhomogeneous fractional equation (4.8) is obtained as follows:

$$f(x) = \frac{1}{\lambda_{\alpha,m} - \lambda} T_{m,\alpha}(x).$$

Example 4.5. *We consider the following nonhomogeneous equation on the interval $[-\alpha, \alpha]$*

$$\left[(\alpha^2 - x^2)^{\frac{1}{2}} J_\alpha - \lambda\right] f(x) = (\alpha - x)^{\frac{1}{2}}. \tag{4.9}$$

Taking integral transform implies that

$$\begin{aligned}(\lambda_{n,\alpha} - \lambda)F(n) &= \int_{-\alpha}^{\alpha} (\alpha^2 - x^2)^{\frac{-1}{2}} (\alpha - x)^{\frac{1}{2}} T_{n,\alpha}(x)dx \\ &= \sum_{k=0}^{n} a_k \left(\int_{-\alpha}^{\alpha} (\alpha + x)^k dx\right) = \sum_{k=0}^{n} a_k \frac{(2\alpha)^{k+1}}{k+1}.\end{aligned}$$

Substituting the values of a_k from Remark 3.4, we obtain

$$F(n) = \frac{1}{(\lambda_{n,\alpha} - \lambda)} \frac{(-1)^n 2^{2n} \Gamma(n + \frac{3}{2})}{(2n)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2\alpha \frac{\Gamma(n+k+1)}{(k+1)\Gamma(k+\frac{3}{2})}.$$

Using relation (4.2), the particular solution of the nonhomogeneous fractional Equation (4.9) is obtained as follows:

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} \Gamma(n + \frac{3}{2}) T_{n,\alpha}(x)}{\pi (2n)! (\lambda_{n,\alpha} - \lambda)} \left(\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\Gamma(n+k+1)}{(k+1)\Gamma(k+\frac{3}{2})}\right).$$

We may truncate the series to approximate the solution $f(x)$ by $f_N(x)$. For $\alpha = 0.9, 1.5$ and different values of N, the graphs of $f_N(x)$ are plotted in Figure 1. The Infinity norm and Chebyshev norm of $f_N - f_{18}$ for different values of N are computed in Tables 1 and 2 .

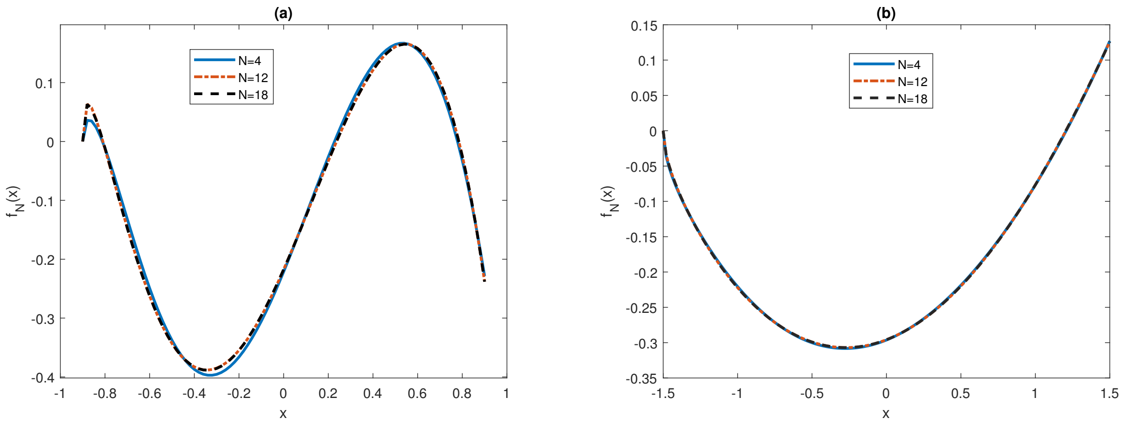

FIGURE 1. Graphs of truncated solutions $f_N(x)$ for Example 4.5 in the interval $[-\alpha, \alpha]$. (a) with $\alpha = 0.9$, $\lambda = 8$ and (b) with $\alpha = 1.5$, $\lambda = 8$.

5. Conclusions

Fractional differential equations with non-uniform intervals depending on fractional order α appear in approximating time-dependent fractional differential equations by the corresponding finite difference equations.There are applications in the Chaos theory. In this paper we define a fractional Chebyshev differential equation on a symmetric interval $[-\alpha, \alpha]$, where α is the order of FCDE. We produce a family of orthogonal polynomials $Q_{n,\alpha}(x)$ on the interval $[-\alpha, \alpha]$. For $\alpha = 1$ we prove that $Q_{n,1}(x)$ is identical to the classical Chebyshev polynomials of the third kind. Moreover, we solve some nonhomogeneous fractional differential equations by using suitable integral transforms.

References

[1] T. Aboelenen, SA. Bakr, and HM. El-Hawary, *Fractional Laguerre spectral methods and their applications to fractional differential equations on unbounded domain*, International Journal of Computer Mathematics, *94* (2017), 570–596. doi: 10.1080/00207160.2015.1119270.

[2] M. Abramowitz and A. Stegun, *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, 1970.

[3] M. Ciesielski, M. Klimek, and T. Blaszczyk, *The fractional Sturm-Liouville problem-Numerical approximation and application in fractional diffusion*, Journal of Computational and Applied Mathematics, *317* (2017), 573–588.

[4] A. Deshpande and V. Daftardar-Gejji, *Chaos in discrete fractional difference equations*, Pramana – J. Phys., *87* (2016), doi: 10.1007/s12043-016-1231-9.

[5] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer-Verlag, Berlin, 2010.

[6] E.H. Doha, W.M. Abd-Elhameed, and M.M. Alsuyuti, *On using third and fourth kinds Chebyshev polynomials for solving the integrated formsof high odd-order linear boundary value problems*, Journal of the Egyptian Mathematical Society, *23* (2015), 397–405.

[7] M.H. Heydari, M.R. Hooshmandasl, and F.M.M. Ghaini, *An efficient computational method for solving fractional biharmonic equation*, Computers and Mathematics with Applications, *68* (2014), 269-287.

[8] B. Jin and W. Rundell, *An inverse Sturm-Liouville problem with a fractional derivative*, Journal of Computational Physics, *231* (2012), 4954–4966.

[9] Z. Kavooci, K. Ghanbari, and H. Mirzaei, *New form of Laguerre Fractional Differential Equation and Applications*, Turk J Math, *46* (2022), 2998–3010. doi: 10.55730/ 1300-0098.3314.

[10] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland mathematics studies, Elsevier science, 204 (2006).

[11] M. Klimek and O.P. Agrawal, *Fractional Sturm-Liouville problem*, Computers and Mathematics with Applications, *66* (2013), 795–812.

[12] M. Klimek, M. Ciesielski, and T. Blaszczyk, *Exact and numerical solutions of the fractional Sturm-Liouville problem*, Fractional Calculus and Applied Analysis, *21* (2018), 45–71.
[13] X.T. Liu, H.G. Sun, Y. Zhang, and Z. Fu, *A scale-dependent finite difference approximation for time fractional differential equation*, Computational Mechanics, *63* (2019), 429–442.
[14] J.C. Mason, *Chebyshev polynomials of the second, third and fourth kinds in approximation, indefinite integration, and integral transforms*, Journal of Computational and Applied Mathematics, *49* (1993), 169–178.
[15] J.C. Mason and D.C. Handscomb, *Chebyshev polynomials*, Chapman and Hall, New York, 2003.
[16] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.
[17] A.K. Singh and M. Mehra, *Wavelet collocation method based on Legendre polynomials and its application in solving the stochastic fractional integro-differential equations*, Journal of Computational science, *51* (2020) 101342, 1–11.
[18] M. Zayernouri and G.E. Karniadakis, *Fractional Sturm-Liouville eigen-problems: Theory and numerical approximation* , Journal of Computational Physics, *252* (2013), 495–517.
[19] D. Funaro, *Polynomial Approximation of Differential Equations*, Springer-Verlag, New York, 1991.