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Adaptive-grid technique for the numerical solution of a class of fractional boundary-value-problems

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Abstract

In this study, we numerically solve a class of two-point boundary-value-problems with a Riemann-Liouville-Caputo fractional derivative, where the solution might contain a weak singularity. Using the shooting technique based on the secant iterative approach, the boundary value problem is first transformed into an initial value problem, and the initial value problem is then converted into an analogous integral equation. The functions contained in the fractional integral are finally approximated using linear interpolation. An adaptive mesh is produced by equidistributing a monitor function in order to capture the singularity of the solution. A modified Gronwall inequality is used to establish the stability of the numerical scheme. To show the effectiveness of the suggested approach over an equidistributed grid, two numerical examples are provided.

Keywords. Fractional differential equation, Riemann-Liouville-Caputo fractional derivative, Shooting method, Stability estimate.

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1. Introduction

Any positive order integrals and derivatives are permitted in fractional calculus, the term fractional is kept only for historical reasons. In recent decades, it has become increasingly popular and significant across a range of science and engineering sectors. It appears in various fields such as elasticity [23], rheology [12], quantitative biology [15], porous or fractured media [2], transport theory [6], etc. The Riemann-Liouville-Caputo (RLC) fractional derivative [10, 13], also known as the Patie-Simon fractional derivative [14, 18], was first introduced by Pierre Patie and Thomas Simon in the study of asymmetric α-stable Lévy processes. This derivative is used as the infinitesimal generator of the spectrally positive (resp. spectrally negative) α-stable Lévy process reflected at its running supremum.

This article analyzes the following class of steady-state fractional boundary-value-problems (BVPs) with the highest order derivative of RLC fractional type:

$$\begin{cases} -D^{\alpha}_{RLC}u(t) + (bu)'(t) + (cu)(t) = f(t), \quad t \in \Omega = (0,T), \\ D^{\alpha-1}_{C}u(0) = \gamma, \quad u(T) + \beta_1 u'(T) = \gamma_1, \end{cases} \tag{1.1}$$

where the functions b, c, f satisfy (1.3) and the constants $\beta_1 \geq 0, \gamma, \gamma_1 \geq 0$ are given, and for $1 < \alpha < 2$,

$$D^{\alpha}_{RLC}u(t) := \frac{d}{dt}\left(D^{\alpha-1}_{C}u(t)\right) = \frac{d}{dt}\left(\int_{s=0}^{t} \frac{(t-s)^{1-\alpha}u'(s)}{\Gamma(2-\alpha)}ds\right), \quad t > 0. \tag{1.2}$$

For each positive integer r and $-\infty < \mu < 1$, let us define the space

$$\mathcal{C}^{r,\mu}(0,T] := \{y \in \mathcal{C}(\overline{\Omega}) : y^{(r)} \in \mathcal{C}(0,T] \quad \text{with} \quad \|y\|_{r,\mu} < \infty\},$$

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where

$$\|y\|_{r,\mu} := \sup_{0<t\leq T} |y(t)| + \sum_{i=1}^{r} \sup_{0<t\leq T} [t^{i-(1-\mu)}|y^{(i)}(t)|],$$

and $\mathcal{C}^n(\overline{\Omega})$ is the space of n times continuously differentiable functions defined on $\overline{\Omega}$.

We assume that for $r \in \mathbb{Z}^+$ and some $\nu \in (-\infty, 1)$,

$$b, c, f \in \mathcal{C}^{r,\nu}(0,T], \quad c + b' \geq 0, \tag{1.3}$$

and under the assumptions given in (1.3), the solution u to the problem (1.1) exists.

Gracia et al. [10], in their study of the BVP (1.1) with $D_C^{\alpha-1}u(0) = 0$, noted that $D_C^{\alpha-1}u(0) \neq 0$ may result in singularity in the problem's solution. We have employed an adaptive grid based on grid equidistribution with a length curve monitor function to overcome the singularity in the numerical solution. Here, the key concept behind using this equidistribution grid for numerical solution is that there should be more grid points where the singularity occurs in the solution curve than other locations in the domain in order to increase the accuracy of the numerical solution.

Definition 1.1. *The Riemann-Liouville fractional integral of order $\beta \geq 0$ for any function $f \in L^p(a,b)\,(1 \leq p \leq \infty)$ is defined by*

$$(I_t^{\beta} f)(t) = \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} f(s)ds, \quad t > a,$$

and the Riemann-Liouville fractional derivative of order $\omega \in (0,1)$ for any function $f \in AC[a,b]$ (the space of functions which are absolutely continuous on $[a,b]$), is defined by

$$D_{RL}^{\omega} f(t) = \frac{d}{dt}\left(I_t^{1-\omega} f\right)(t) = \frac{1}{\Gamma(1-\omega)} \frac{d}{dt}\left(\int_a^t (t-s)^{-\omega} f(s)ds\right).$$

The finite difference approach was most likely first used to examine the heat conduction equation with fractional derivative and boundary condition by Shkhanukov et al. [22]. Shkhanukov proposed the finite difference approach for Dirichlet BVP for ordinary differential equations (ODEs) with Riemann-Liouville fractional derivative in [21]. An extrapolation method was devised by Diethelm and Walz [8] to solve ordinary differential equations numerically using the so-called Caputo fractional derivative. Panda et al. [17] studied the Adomian decomposition method and Homotopy perturbation method for solving time-fractional partial integro-differential equation. Santra et al. [19] developed a novel approach for solving multi-term time fractional partial integro-differential equations. Oqielat et al. [16] studied numerical solution technique for time-fractional nonlinear water wave partial differential equation. Extremal solutions for multi-term nonlinear fractional differential equations are obtained in Fazli et al. [9]. Shallal et al. [20] established a numerical technique for the time-fractional gas dynamics equation using a finite element approach with cubic hermit element.

We suggest a numerical approach to solve the BVP (1.1) in this article. More specifically, first, we translate the BVP into an equivalent initial-value-problem (IVP) by using the shooting technique. The IVP is afterward transformed into an equivalent integral equation. Then, using an adaptive mesh (by equidistributing a monitor function), we accurately approximate the integral equation. The process is then repeated with the initial condition modified until the solution satisfies the required right boundary condition with the desired accuracy. Using a modified Gronwall inequality, we establish the discrete stability of the proposed method. The effectiveness and accuracy of the current method are illustrated by numerical examples.

The rest of the paper is organized as follows: we propose the numerical method for solving the BVP (1.1) over an equidistributed grid in section 2. Section 3 deals with the stability estimate of the proposed scheme. Finally, numerical experiments are carried out in section 4.

2. Proposed Numerical Method

In this section, we will discuss some sequential approximation methods for solving the BVP numerically instead of solving it analytically.

2.1. **Shooting Technique.** First, by using the shooting technique, we convert the given BVP (1.1) into the following IVP:

$$\begin{cases} -D^{\alpha}_{RLC}u(t) + (bu)'(t) + (cu)(t) = f(t), \quad t \in \Omega, \\ u(0) = \eta, \quad D^{\alpha-1}_{C}u(0) = \gamma, \end{cases} \tag{2.1}$$

where η is an initial guess that will be modified to ensure that the solution fulfills the given condition at $t = T$, *i.e.*, $u(T) + \beta_1 u'(T) = \gamma_1$. The secant iterative method serves as the foundation of this shooting technique.

In the literature, the shooting method was applied to solve the fractional BVPs, for example, one can refer to Diethelm and Ford [7], Al-Mdallal et al. [1], Huang et al. [11] and Cen et al. [3–5].

Instead of solving (2.1) directly, we will transform it into an equivalent integral equation, by assuming some regularity condition on u. The following theorem yields the equivalence of the equation (2.1) and an integral equation.

Theorem 2.1. *Let us assume the conditions given in (1.3) hold true. If $u \in \mathcal{C}^1(\overline{\Omega})$, then $u(t)$ satisfies the equation (2.1) if, and only if, $u(t)$ satisfies the Volterra integral equation with a weakly singular kernel:*

$$\begin{cases} u(t) = u(0) + \left(\dfrac{D^{\alpha-1}_{C}u(0) - b(0)u(0)}{\Gamma(\alpha)}\right) t^{\alpha-1} + \dfrac{1}{\Gamma(\alpha-1)} \displaystyle\int_0^t (t-s)^{\alpha-2}(bu)(s)ds \\ \qquad + \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1}(cu-f)(s)ds, \quad t \in \Omega, \\ u(0) = \eta, \quad D^{\alpha-1}_{C}u(0) = \gamma. \end{cases} \tag{2.2}$$

In general, it is quite difficult to find out the exact value of η, we have to look at some approximation method for finding η in such a way that $u(T;\eta) + \beta_1 u'(T;\eta) - \gamma_1 = 0$, where $u(t;\eta)$ be the solution of (2.2) for each $u(0) = \eta$. We use the secant method to compute η iteratively, by finding the solution of $\phi(\eta) = 0$, where

$$\phi(\eta) = u(T;\eta) + \beta_1 u'(T;\eta) - \gamma_1. \tag{2.3}$$

First, we take some initial values η_0 and η_1, and iteratively compute η_i $(i \geq 2)$ as

$$\eta_{i+1} = \eta_i - \frac{\eta_i - \eta_{i-1}}{\phi(\eta_i) - \phi(\eta_{i-1})}\phi(\eta_i), \quad i \geq 1. \tag{2.4}$$

Let $\varepsilon_i = \eta_i - \eta$ be the error due to this secant method, where η is such that $\phi(\eta) = 0$ and η_i is defined in (2.4). Then by a simple calculation, one can show that

$$\varepsilon_{i+1} = -\frac{1}{2}\frac{\phi''(\xi_i)}{\phi'(\xi_i')}\varepsilon_i\varepsilon_{i-1},\ \xi_i \in int(\eta, \eta_{i-1}, \eta_i) \text{ and } \xi_i' \in int(\eta_{i-1}, \eta_i), \tag{2.5}$$

where $int(a,b)$ = the interval formed by a and b and $int(a,b,c)$ = the interval formed by a, b and c.

In the following theorem, we will show the convergence of the shooting method.

Theorem 2.2. *Let us consider the conditions given in (1.3). Then the function ϕ defined by equation (2.3) satisfies the following:*

$$\left|\frac{\phi''(\xi_i)}{\phi'(\xi_i')}\right| \leq C, \tag{2.6}$$

and hence the Secant method is convergent, i.e., $\eta_i \to u(0)$ with the order of convergence 1.618.

2.2. **Discretization Scheme.** A nonuniform mesh with the discretization parameter N being a positive integer is defined as $\overline{\Omega}_N = \{0 = t_0 < t_1 < \cdots < t_N = T\}$. Let $\tau_i = t_i - t_{i-1}$, for $i = 1, \cdots, N$ and $\widehat{\tau} = \max_i \tau_i$. Then, in order to approximate the problem (2.2), the following integral discretization technique is employed:

$$\begin{aligned}
u_k^N &\approx u(0) + \frac{D_C^{\alpha-1}u(0) - b(0)u(0)}{\Gamma(\alpha)} t_k^{\alpha-1} + \sum_{i=1}^{k} \int\limits_{t_{i-1}}^{t_i} \frac{(t_k - s)^{\alpha-2}}{\Gamma(\alpha-1)} \left[\frac{t_i - s}{\tau_i} b_{i-1}u_{i-1}^N + \frac{s - t_{i-1}}{\tau_i} b_i u_i^N\right] ds \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{k} \int\limits_{t_{i-1}}^{t_i} (t_k - s)^{\alpha-1} \left[\frac{t_i - s}{\tau_i}(c_{i-1}u_{i-1}^N - f_{i-1}) + \frac{s - t_{i-1}}{\tau_i}(c_i u_i^N - f_i)\right] ds \\
&= \eta + \frac{\gamma - \eta b_0}{\Gamma(\alpha)} t_k^{\alpha-1} + \sum_{i=1}^{k} b_{i-1}u_{i-1}^N \left[\frac{\psi_{k,i-1}^{\alpha-1}}{\Gamma(\alpha)} - \frac{\psi_{k,i-1}^{\alpha} - \psi_{k,i}^{\alpha}}{\tau_i \Gamma(\alpha+1)}\right] + \sum_{i=1}^{k} b_i u_i^N \left[\frac{\psi_{k,i-1}^{\alpha} - \psi_{k,i}^{\alpha}}{\tau_i \Gamma(\alpha+1)} - \frac{\psi_{k,i}^{\alpha-1}}{\Gamma(\alpha)}\right] \\
&+ \sum_{i=1}^{k} (c_{i-1}u_{i-1}^N - f_{i-1}) \left[\frac{\psi_{k,i-1}^{\alpha}}{\Gamma(\alpha+1)} - \frac{\psi_{k,i-1}^{\alpha+1} - \psi_{k,i}^{\alpha+1}}{\tau_i \Gamma(\alpha+2)}\right] \\
&+ \sum_{i=1}^{k} (c_i u_i^N - f_i) \left[\frac{\psi_{k,i-1}^{\alpha+1} - \psi_{k,i}^{\alpha+1}}{\tau_i \Gamma(\alpha+2)} - \frac{\psi_{k,i}^{\alpha}}{\Gamma(\alpha+1)}\right],
\end{aligned} \tag{2.7}$$

where, $\psi_{i,k}^{\alpha} = (t_k - t_i)^{\alpha}$ and $k = 1, 2, \cdots, N$.

To create an adapted mesh, we develop a solution-adapted algorithm. We employ the notion of equidistribution of a specified positive monitor function, $M(t) = \sqrt{1 + [(\widehat{u}^N(t))']^2}$, to obtain such a mesh, where $\widehat{u}^N(t)$ is continuous on $[0, T]$, linear on every $[t_{k-1}, t_k]$, and

$$\widehat{u}^N(t) = u_k^N + (t - t_k)\frac{u_k^N - u_{k-1}^N}{\tau_k}, \quad t \in (t_{k-1}, t_k),\ k = 1, 2, \cdots, N.$$

By figuring out the following equidistribution problem, an adapted mesh can be generated:

$$M_k \tau_k = \frac{1}{N}\sum_{j=1}^{N} M_j \tau_j, \quad k = 1, \cdots, N.$$

We develop the subsequent iteration algorithm to address the aforementioned equidistribution problem:

3. Discrete stability estimate

By using the generalized discrete Gronwall inequality, which will be given in the following lemma, we perform a discrete stability analysis of the discretization scheme (2.7) of the computed solution in this section.

Lemma 3.1. *(Generalized discrete Gronwall inequality, [11])*
Assume that μ, D_0 are positive constants. Let g_0 be positive and the sequence $\{\phi_i\}_{i=0}^N$ satisfy

$$\begin{cases} \phi_0 \leq g_0, \\ \phi_i \leq \sum\limits_{k=0}^{i-1} a_{k,i}\phi_k + g_0, \end{cases} \tag{3.1}$$

where, $a_{0,i} = D_0\tau_1 t_i^{\mu-1}$, $\quad a_{k,i} = D_0\tau_{k+1}(t_i - t_k)^{\mu-1}$ or $\quad a_{k,i} = D_0\tau_k(t_i - t_{k-1})^{\mu-1}$ for $k \geq 1$ with $\tau_k = t_k - t_{k-1}$, then $\phi_i \leq D_0 g_0$, $\quad i = 1, 2, \cdots, N$.

In the next lemma, we present the stability analysis of the discrete scheme (2.7) on the mesh $\overline{\Omega}_N$.

Numerical Algorithm

Let C_0 be a chosen constant by the user, where $C_0 > 1$.
Set the maximum number (N_∞) of iterations allowed in the equidistribution loop.
Initialize the mesh by using the uniform mesh $\overline{\Omega}_N^{(0)} = \{t_k^{(0)} : t_k^{(0)} = Tk/N,\ 0 \le k \le N\}$ as the starting point.

for $j = 0$ *to* N_∞ *(Equidistribution Iteration)* **do**

On the mesh $\overline{\Omega}_N^{(j)} = \{t_k^{(j)} : 0 \le k \le N\}$ with mesh size $\tau_k^{(j)} = t_k^{(j)} - t_{k-1}^{(j)}$, resolve the discrete problem (2.7) for $\{u_k^{N,(j)} : 0 \le k \le N\}$.

Set

$$\ell_k^{(j)} = \tau_k^{(j)} \sqrt{1 + \left(\frac{u_k^{N,(j)} - u_{k-1}^{N,(j)}}{\tau_k^{(j)}} \right)^2}, \quad I^{(j)} = \sum_{k=1}^{N} \ell_k^{(j)}.$$

If the iteration algorithm's termination criterion is $\max_{1 \le k \le N} \{\ell_k^{(j)}\} < C_0 I^{(j)}/N$ is satisfied then break the iteration.

Develop a new mesh $\overline{\Omega}_N^{(j+1)}$ in such a way that

$$\tau_k^{(j+1)} \sqrt{1 + \left(\frac{u_k^{N,(j)} - u_{k-1}^{N,(j)}}{\tau_k^{(j)}} \right)^2} = I^{(j)}/N, \quad k = 1, 2, \cdots, N.$$

end

Take $\overline{\Omega}_N^{*} = \overline{\Omega}_N^{(j)}$ as the final mesh and set $\{u_k^{N,*} : 0 \le k \le N\} = \{u_k^{N,(j)} : 0 \le k \le N\}$.

Set $\{u_k^{N,*} : 0 \le k \le N\}$ as the required solution at the final time step.

Lemma 3.2. *(Discrete stability estimate)*
Assume the conditions given in (1.3) are true. Let $\{u_i^N\}_{i=0}^N$ be the solution of the scheme (2.7) on an arbitrary mesh $\overline{\Omega}_N$. Then $\{u_i^N\}_{i=0}^N$ satisfies

$$\|u^N\|_{\overline{\Omega}_N} \le C(|\eta| + |\gamma| + \|f\|_{\overline{\Omega}_N}),$$

where, C is a positive constant and $\|\cdot\|_{\overline{\Omega}_N}$ is the discrete maximum norm over $\overline{\Omega}_N$.

Proof. From (2.7), we have

$$\begin{cases} |u_1^N| \le d_1 \rho_1, \\ |u_k^N| \le d_k |\rho_k| + d_k \sum\limits_{i=1}^{k-1} w_{i,k} u_i^N, \quad k = 2, \cdots, N, \end{cases} \tag{3.2}$$

where, for $1 \leq k \leq N$, $\psi_{i,k}^{\alpha} = (t_k - t_i)^{\alpha}$,

$$\begin{aligned} d_k &= \left|1 - \frac{\tau_k^{\alpha-1} b_k}{\Gamma(\alpha+1)} - \frac{\tau_k^{\alpha} c_k}{\Gamma(\alpha+2)}\right|^{-1}, \qquad (3.3)\\ \rho_k &= \eta + \frac{\gamma - b_0\eta}{\Gamma(\alpha)} t_k^{\alpha-1} - \frac{f_k \tau_k^{\alpha}}{\Gamma(\alpha+2)} + \frac{b_0\eta}{\Gamma(\alpha)}\left[\psi_{k,0}^{\alpha-1} - \frac{\psi_{k,0}^{\alpha} - \psi_{k,1}^{\alpha}}{\alpha\tau_1}\right] \\ &\quad + \frac{c_0\eta - f_0}{\Gamma(\alpha+1)}\left[\psi_{k,0}^{\alpha} - \frac{\psi_{k,0}^{\alpha+1} - \psi_{k,1}^{\alpha+1}}{(\alpha+1)\tau_1}\right] - \sum_{i=1}^{k-1} \frac{f_i}{\Gamma(\alpha+2)}\left[\frac{\psi_{k,i-1}^{\alpha+1} - \psi_{k,i}^{\alpha+1}}{\tau_i} - \frac{\psi_{k,i}^{\alpha+1} - \psi_{k,i+1}^{\alpha+1}}{\tau_{i+1}}\right], \qquad (3.4) \end{aligned}$$

and for $2 \leq k \leq N$, $1 \leq i \leq k-1$,

$$w_{i,k} = \frac{b_i}{\Gamma(\alpha+1)}\left[\frac{\psi_{k,i-1}^{\alpha} - \psi_{k,i}^{\alpha}}{\tau_i} - \frac{\psi_{k,i}^{\alpha} - \psi_{k,i+1}^{\alpha}}{\tau_{i+1}}\right] + \frac{c_i}{\Gamma(\alpha+2)}\left[\frac{\psi_{k,i-1}^{\alpha+1} - \psi_{k,i}^{\alpha+1}}{\tau_i} - \frac{\psi_{k,i}^{\alpha+1} - \psi_{k,i+1}^{\alpha+1}}{\tau_{i+1}}\right]. \qquad (3.5)$$

Now by using the mean-value theorem, we have

$$\begin{aligned} |\rho_k| &\leq |\eta| + \frac{|\gamma| + |b_0||\eta|}{\Gamma(\alpha)}|t_k|^{\alpha-1} + \frac{|f_k|\tau_k^{\alpha}}{\Gamma(\alpha+2)} + \frac{|b_0||\eta|}{\Gamma(\alpha-1)}\tau_1(t_k - t_1)^{\alpha-2} \\ &\quad + \frac{|c_0||\eta| + |f_0|}{\Gamma(\alpha)}(t_k - t_1)^{\alpha-1}\tau_1 + \sum_{i=1}^{k-1}\frac{|f_i|}{\Gamma(\alpha)}(t_k - t_{i-1})^{\alpha-1}(\tau_i + \tau_{i+1}) \\ &\leq C(|\eta| + |\gamma| + \|f\|_{\overline{\Omega}_N}), \qquad (3.6) \end{aligned}$$

where C is a constant, and from (3.5), we have

$$w_{i,k} \leq \frac{\|b\|_{\overline{\Omega}_N}}{\Gamma(\alpha)}\left[(t_k - t_{i-1})^{\alpha-1} - (t_k - t_{i+1})^{\alpha-1}\right] + \frac{\|c\|_{\overline{\Omega}_N}}{\Gamma(\alpha+1)}\left[(t_k - t_{i-1})^{\alpha} - (t_k - t_{i+1})^{\alpha}\right]. \qquad (3.7)$$

By using the assumptions on b and c given in (1.3), for some positive constant C_1 and sufficiently large N, we have

$$|d_k| \leq \left|1 - \frac{\|b\|_{\overline{\Omega}_N}}{\Gamma(\alpha)}\widehat{\tau}^{\alpha-1} - \frac{\|c\|_{\overline{\Omega}_N}}{\Gamma(\alpha+1)}\widehat{\tau}^{\alpha}\right|^{-1} \leq C_1. \qquad (3.8)$$

Case 1: For the case $\tau_i \leq \tau_{i+1}$ with $t_k - t_i \geq 2\tau_{i+1}$, from (3.7) we have

$$\begin{aligned} w_{i,k} &\leq \frac{\|b\|_{\overline{\Omega}_N}}{\Gamma(\alpha-1)}(t_k - t_{i+1})^{\alpha-2}(\tau_i + \tau_{i+1}) + \frac{\|c\|_{\overline{\Omega}_N}}{\Gamma(\alpha)}(t_k - t_{i-1})^{\alpha-1}(\tau_i + \tau_{i+1}) \\ &\leq \frac{2\|b\|_{\overline{\Omega}_N}}{\Gamma(\alpha-1)}\tau_{i+1}(t_k - t_{i+1})^{\alpha-2} + \frac{2\|c\|_{\overline{\Omega}_N}}{\Gamma(\alpha)}\tau_{i+1}(t_k - t_{i-1})^{\alpha-1} \\ &\leq C\tau_{i+1}(t_k - t_i)^{\alpha-2}, \qquad (3.9) \end{aligned}$$

where, we have used the mean-value theorem and the fact

$$\left(\frac{t_k - t_{i+1}}{t_k - t_i}\right)^{\alpha-2} = \left(1 + \frac{\tau_{i+1}}{t_k - t_{i+1}}\right)^{2-\alpha} \leq \left(\frac{3}{2}\right)^{2-\alpha}.$$

Case 2: For the case $\tau_i \le \tau_{i+1}$ with $t_k - t_i < 2\tau_{i+1}$, from (3.7) we have

$$\begin{aligned}
w_{i,k} &\le \frac{\|b\|_{\overline{\Omega}_N}}{\Gamma(\alpha)}(t_k - t_{i-1})^{\alpha-1} + \frac{\|c\|_{\overline{\Omega}_N}}{\Gamma(\alpha+1)}(t_k - t_{i-1})^{\alpha} \\
&\le \frac{\|b\|_{\overline{\Omega}_N}}{\Gamma(\alpha)}(t_k - t_i + \tau_{i+1})^{\alpha-1} + \frac{\|c\|_{\overline{\Omega}_N}}{\Gamma(\alpha+1)}(t_k - t_i + \tau_{i+1})^{\alpha} \\
&< \frac{3^{\alpha-1}\|b\|_{\overline{\Omega}_N}}{\Gamma(\alpha)}\tau_{i+1}^{\alpha-1} + \frac{3^{\alpha}\|c\|_{\overline{\Omega}_N}}{\Gamma(\alpha+1)}\tau_{i+1}^{\alpha} \\
&< \left(\frac{3^{\alpha-1}\|b\|_{\overline{\Omega}_N}}{\Gamma(\alpha)} + \frac{3^{\alpha}L\|c\|_{\overline{\Omega}_N}}{\Gamma(\alpha+1)}\right) 2^{2-\alpha}\tau_{i+1}(t_k - t_i)^{\alpha-2} \\
&\le C\tau_{i+1}(t_k - t_i)^{\alpha-2}. \qquad (3.10)
\end{aligned}$$

Case 3: For the case $\tau_i > \tau_{i+1}$ with $t_k - t_i \ge 2\tau_i$, from (3.7) we have

$$\begin{aligned}
w_{i,k} &\le \frac{\|b\|_{\overline{\Omega}_N}}{\Gamma(\alpha-1)}(t_k - t_{i+1})^{\alpha-2}(\tau_i + \tau_{i+1}) + \frac{\|c\|_{\overline{\Omega}_N}}{\Gamma(\alpha)}(t_k - t_{i-1})^{\alpha-1}(\tau_i + \tau_{i+1}) \\
&< \frac{2\|b\|_{\overline{\Omega}_N}}{\Gamma(\alpha-1)}\tau_i(t_k - t_{i+1})^{\alpha-2} + \frac{2\|c\|_{\overline{\Omega}_N}}{\Gamma(\alpha)}\tau_i(t_k - t_{i-1})^{\alpha-1} \\
&\le C\tau_i(t_k - t_{i-1})^{\alpha-2}, \qquad (3.11)
\end{aligned}$$

where, we have used the mean-value theorem and the fact

$$\left(\frac{t_k - t_{i+1}}{t_k - t_{i-1}}\right)^{\alpha-2} = \left(1 + \frac{\tau_{i+1}}{t_k - t_{i+1}}\right)^{2-\alpha}\left(1 + \frac{\tau_i}{t_k - t_i}\right)^{2-\alpha} \le \left(\frac{3}{2}\right)^{4-2\alpha}. \qquad (3.12)$$

Case 4: For the case $\tau_i > \tau_{i+1}$ with $t_k - t_i < 2\tau_i$, from (3.7) we have

$$\begin{aligned}
w_{i,k} &\le \frac{\|b\|_{\overline{\Omega}_N}}{\Gamma(\alpha)}(t_k - t_{i-1})^{\alpha-1} + \frac{\|c\|_{\overline{\Omega}_N}}{\Gamma(\alpha+1)}(t_k - t_{i-1})^{\alpha} \\
&= \frac{\|b\|_{\overline{\Omega}_N}}{\Gamma(\alpha)}(t_k - t_i + \tau_i)^{\alpha-1} + \frac{\|c\|_{\overline{\Omega}_N}}{\Gamma(\alpha+1)}(t_k - t_i + \tau_i)^{\alpha} \\
&< \frac{3^{\alpha-1}\|b\|_{\overline{\Omega}_N}}{\Gamma(\alpha)}\tau_i^{\alpha-1} + \frac{3^{\alpha}\|c\|_{\overline{\Omega}_N}}{\Gamma(\alpha+1)}\tau_i^{\alpha} \\
&< \left(\frac{3\|b\|_{\overline{\Omega}_N}}{\Gamma(\alpha)} + \frac{9L\|c\|_{\overline{\Omega}_N}}{\Gamma(\alpha+1)}\right)\tau_i(t_k - t_{i-1})^{\alpha-2} \\
&\le C\tau_i(t_k - t_{i-1})^{\alpha-2}. \qquad (3.13)
\end{aligned}$$

Thus, by using the Lemma 3.1 in (3.2) combined with the estimations (3.6)-(3.13), we are able to get the required result. □

Remark 3.3. *A rigorous analysis to determine the convergence order will be very difficult to carry out because of the nature of adaptive grids. We can conclude that the suggested numerical method maintains its convergence features based on the results of next section.*

4. Numerical Experiments

This section provides the efficiency of the proposed method over an equidistributed grid by experimenting with two numerical test problems with numerical results.

Example 4.1. *Consider the following problem for $1 < \alpha < 2$:*

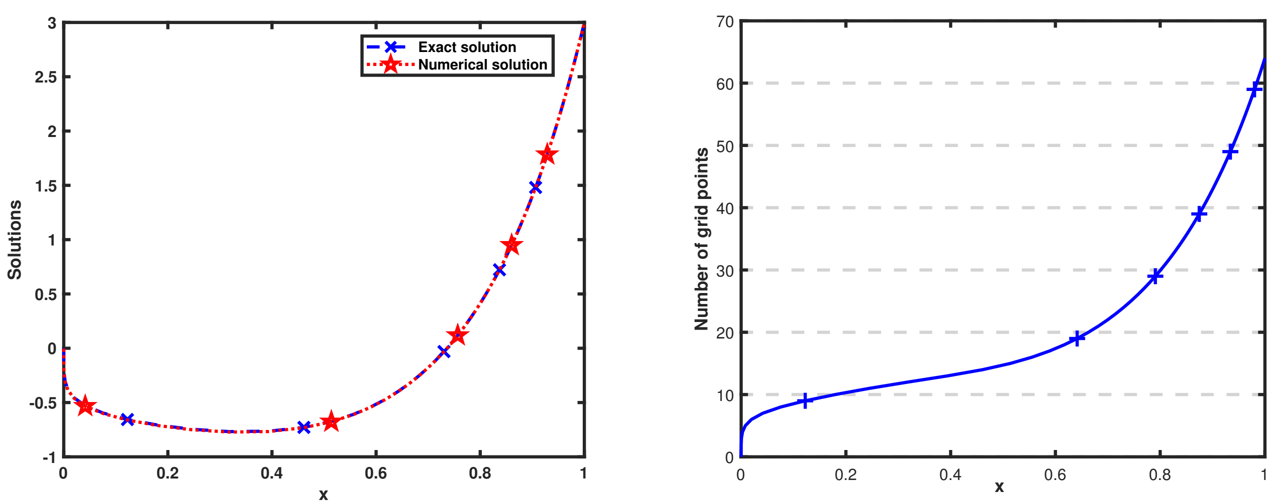
$$\begin{cases} -D^{\alpha}_{RLC}u(t) + u(t) = f(t), \quad t \in (0,1), \\ {}^{C}D_0^{\alpha-1}u(0) = -\Gamma(\alpha), \quad u(1) = 3. \end{cases} \tag{4.1}$$

The source term $f(t)$ in (4.1) is chosen in such a way that the solution to this problem is $u(t) = -t^{\alpha-1} + 3t^{\frac{\alpha+7}{2}} + t^8$.

The maximum error in $\{u_j^N\}_{j=0}^N$ and the order of convergence are respectively determined by

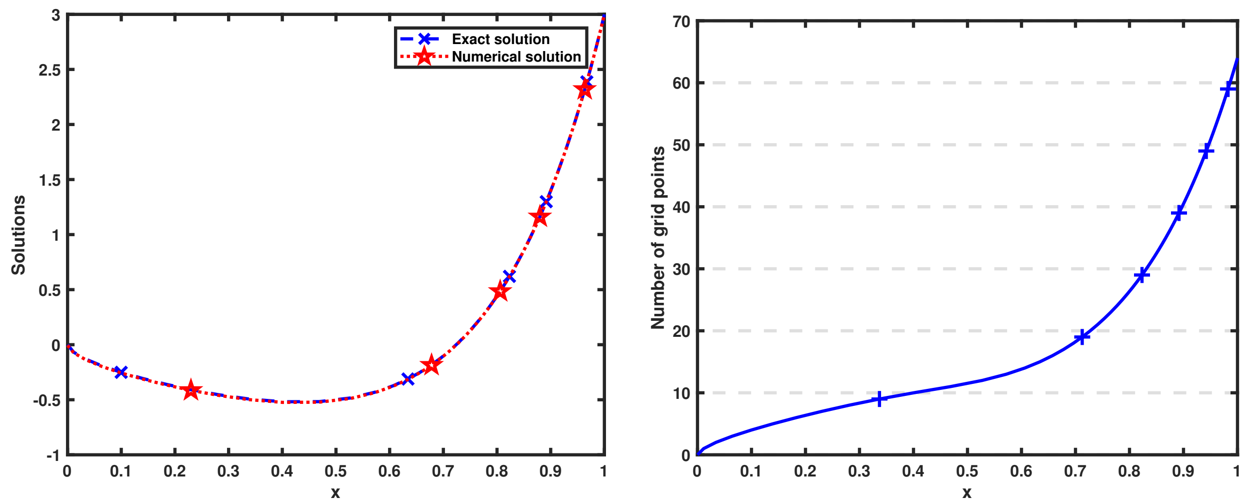
$$E_N := \|u^N - u\|_{\Omega^N} = \max_{0\leq j\leq N} |u_j^N - u(x(j))|, \quad p_N := \log_2\left(\frac{E_N}{E_{2N}}\right).$$

From Figure 1 (a) and Figure 2 (a), one can observe that the solution of Example 4.1 has singularities at the boundary points, which will be resolved by the equidistributed grid. In Figure 1 (b) and Figure 2 (b), we have shown the grid distribution over the domain for Example 4.1. Figure 3 displays the error and log-log plot for Example 4.1 for various values of α. Table 1 and Table 2 show the maximum error and order convergence for Example 4.1 of the proposed method over the uniform grid and equidistributed grid, respectively, from where one can observe that the proposed method is more efficient over the equidistributed grid than over the uniform grid.



(a) *Exact and numerical solutions.* (b) *Equidistributed grid.*

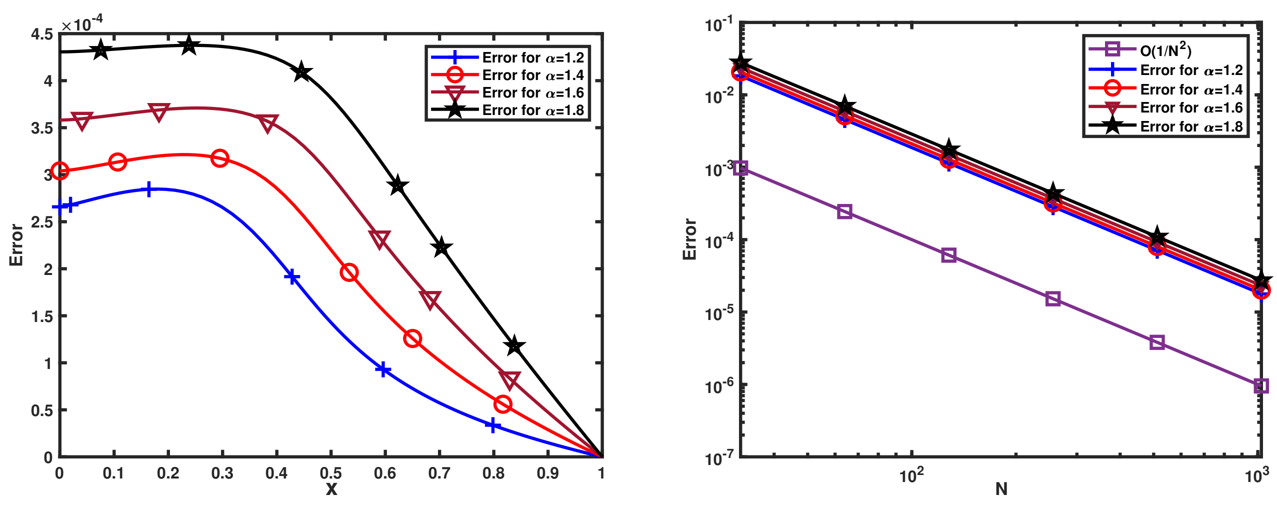
Figure 1. *Solutions and equidistributed grid for Example 4.1 with $\alpha = 1.2$ and $N = 64$.*



(a) *Exact and numerical solutions.*

(b) *Equidistributed grid.*

FIGURE 2. *Solutions and equidistributed grid for Example 4.1 with $\alpha = 1.6$ and $N = 64$.*



(a) *Errors for various α*

(b) *Log-log plot corresponding to Table 2.*

FIGURE 3. *Plots of errors to the Example 4.1 with $\alpha = 1.2$ and $N = 64$.*

Table 1. Numerical results for Example 4.1 over uniform grid.

$\alpha \downarrow$	$N \rightarrow$	32	64	128	256	512	1024
1.2	E_N	1.340e-03	7.086e-04	3.712e-04	1.930e-04	9.972e-05	5.126e-05
	p_N	0.9187	0.9326	0.9437	0.9527	0.9600	
1.4	E_N	1.814e-03	9.311e-04	4.748e-04	2.409e-04	1.218e-04	6.141e-05
	p_N	0.9619	0.9716	0.9788	0.9841	0.9880	
1.6	E_N	1.882e-03	9.514e-04	4.793e-04	2.408e-04	1.208e-04	6.053e-05
	p_N	0.9839	0.9893	0.9929	0.9953	0.9969	
1.8	E_N	1.715e-03	8.607e-04	4.313e-04	2.159e-04	1.081e-04	5.405e-05
	p_N	0.9948	0.9969	0.9981	0.9989	0.9993	

Table 2. Numerical results for Example 4.1 over equidistribution grid.

$\alpha \downarrow$	$N \rightarrow$	32	64	128	256	512	1024
1.2	E_N	1.829e-02	4.557e-03	1.139e-03	2.847e-04	7.119e-05	1.780e-05
	p_N	2.0052	2.0003	2.0000	2.0000	2.0000	
1.4	E_N	2.069e-02	5.142e-03	1.286e-03	3.213e-04	8.034e-05	2.009e-05
	p_N	2.0084	1.9997	2.0003	1.9999	2.0000	
1.6	E_N	2.377e-02	5.935e-03	1.484e-03	3.709e-04	9.273e-05	2.318e-05
	p_N	2.0018	1.9994	2.0005	2.0001	2.0000	
1.8	E_N	2.785e-02	7.008e-03	1.751e-03	4.376e-04	1.094e-04	2.735e-05
	p_N	1.9909	2.0010	2.0004	2.0001	2.0000	

Example 4.2. *Consider the following problem for* $1 < \alpha < 2$*:*

$$\begin{cases} -D^{\alpha}_{RLC}u(t) + 3tu(t) = 1, \quad t \in (0,1), \\ {}^{C}D_0^{\alpha-1}u(0) = 0, \quad u(1) = 0. \end{cases} \tag{4.2}$$

The exact solution of Equation (4.2) is unknown, therefore, we use the two-mesh principle to estimate the maximum two-mesh differences and order of convergence. Let $\{u_k^N\}_{k=0}^N$ and $\{v_k^{2N}\}_{k=0}^{2N}$ are computed solutions of the equation (4.2) on the meshes $\{t_k\}_{k=0}^N$ and $\{y_k\}_{k=0}^{2N}$ respectively, where $\{y_k\}_{k=0}^{2N}$ is constructed by bisecting the interval in $\{t_k\}_{k=0}^N$. The maximum two-mesh differences D_N and the order of convergence q_N are respectively calculated by

$$D_N := \max_{0 \le j \le N} |u_j^N - v_j^{2N}| \quad \text{and} \quad q_N := \log_2\left(\frac{D_N}{D_{2N}}\right).$$

The maximum two-mesh differences and order convergence of the method given in [10] over the uniform grid and the proposed method over the equidistributed grid for Example 4.2 are given in Tables 3 and 4, respectively.

Table 3. Numerical results for Example 4.2 with method given in [10].

$\alpha \downarrow$	$N \rightarrow$	16	32	64	128	256	512
1.2	D_N	2.5339e-02	1.2925e-02	6.5306e-03	3.2836e-03	1.6467e-03	8.2471e-04
	q_N	0.9713	0.9848	0.9919	0.9957	0.9977	
1.4	D_N	2.3763e-02	1.2262e-02	6.2453e-03	3.1572e-03	1.5892e-03	7.9786e-04
	q_N	0.9546	0.9733	0.9841	0.9904	0.9941	
1.6	D_N	2.1316e-02	1.1138e-02	5.7355e-03	2.9261e-03	1.4839e-03	7.4948e-04
	q_N	0.9364	0.9576	0.9709	0.9796	0.9854	
1.8	D_N	1.7693e-02	9.2971e-03	4.8284e-03	2.4876e-03	1.2744e-03	6.5015e-04
	q_N	0.9284	0.9452	0.9568	0.9649	0.9710	

Table 4. Numerical results for Example 4.2 over equidistribution grid with the present method.

$\alpha \downarrow$	$N \rightarrow$	16	32	64	128	256	512
1.2	D_N	8.4231e-03	2.0812e-03	5.1825e-04	1.2938e-04	3.2326e-05	8.0796e-06
	q_N	2.0169	2.0057	2.0021	2.0008	2.0003	
1.4	D_N	3.5419e-03	8.7702e-04	2.1858e-04	5.4592e-05	1.3643e-05	3.4104e-06
	q_N	2.0138	2.0044	2.0014	2.0005	2.0002	
1.6	D_N	1.7200e-03	4.2726e-04	1.0662e-04	2.6641e-05	6.6591e-06	1.6647e-06
	q_N	2.0093	2.0026	2.0008	2.0002	2.0001	
1.8	D_N	9.0986e-04	2.2650e-04	5.6562e-05	1.4136e-05	3.5338e-06	8.8343e-07
	q_N	2.0061	2.0016	2.0004	2.0001	2.0000	

5. Conclusions

In this paper, we have solved a two-point boundary-value problem with the highest order RLC fractional derivative, by using a shooting technique based on secant method and transforming it into an integral equation. We established the stability of the proposed numerical scheme by using a modified Gronwall inequality. The numerical results demonstrate the efficiency and accuracy of the present method with an equidistribution grid over the uniform grid.

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