*Research Paper*
*Computational Methods for Differential Equations*
http://cmde.tabrizu.ac.ir
Vol. 12, No. 2, 2024, pp. 392-412
DOI:10.22034/cmde.2023.53478.2254

# Novel traveling wave solutions of generalized seventh-order KdV equation and related equation

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**Abstract**

In this paper, we acquire novel traveling wave solutions of the generalized seventh-order Korteweg–de Vries equation and the seventh-order Kawahara equation as a special case with physical interest. Primarily, we use the advanced $\exp(-\varphi(\xi))$-expansion method to find new exact solutions of the first equation, by considering two auxiliary equations. Then, we attain some exact solutions of the seventh-order Kawahara equation by using this method with another auxiliary equation, and also using the modified $(G^{'}/G)$-expansion method, where G satisfies a second-order linear ordinary differential equation. Additionally, utilizing the recent scientific instruments, the 2D, 3D, and contour plots are displayed. The solutions obtained in this paper include bright solitons, dark solitary wave solutions, and multiple dark solitary wave solutions. It is shown that these two methods provide an effective mathematical tool for solving nonlinear evolution equations arising in mathematical physics and engineering.

**Keywords.** Nonlinear evolution equation, Generalized seventh-order Korteweg-de Vries equation, Traveling wave solution, $exp(-\varphi(\xi))$-expansion method, $(G^{'}/G)$-expansion method.

**2010 Mathematics Subject Classification.** 35Q53, 35G20, 35C07, 35C08.

## 1. Introduction

Nonlinear evolution equations (NLEEs) are useful for describing a wide range of phenomena in many fields, including fluid dynamics, fiber optics, plasma waves, mathematical biology, etc. Finding the exact traveling wave solutions to these equations are crucial for the study of many nonlinear scientific fields. These solutions aid physicists and engineers in better comprehending the mechanisms at work and addressing physical issues. Analytical solutions of nonlinear partial differential equations (PDEs) enable the deciphering of underlying mechanisms behind many nonlinear complex phenomena. They provide insights into various aspects such as the absence or abundance of steady states under different necessary conditions, the existence of peak regimes, and the spatial localization of transfer processes.

Over the past ten years, numerous effective techniques have been developed and extended for finding traveling wave solutions to NLEEs. Some of these techniques include the modified simple equation method, the simplified Hirota's method, the modified $(G^{'}/G)$-expansion method, the $(1/G^{'})$-expansion method, the generalized $(G^{'}/G)$-expansion method, the Jacobi elliptic function expansion method, the modified exponential function method, the improved $\tan(\frac{\varphi(\xi)}{2})$-expansion method, Painlevé transformation, Kudryashov method, the extended trial equation method, the modified extended direct algebraic method, and others (see [1, 18, 19, 21–23, 29, 38, 41–44]). Furthermore, significant advancements have been made in the search for analytical solitary wave solutions for PDEs. Some notable references in this area are [4, 6, 8, 13, 14, 24, 25, 33].

Among the most efficient and reliable techniques are the $\exp(-\varphi(\xi))$-expansion method and the $(G^{'}/G)$-expansion method. The former was first introduced by Zhao and Li [45] and has since been extensively used and expanded upon in various studies [2, 7, 11, 12, 15–17, 26, 27] to derive different variations. The latter was initially introduced by

Received: 14 September 2022 ; Accepted: 07 July 2023.
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Wang et al. [39] and has been utilized and extended in several works [3, 20, 31, 36, 37, 44, 46]. It has been shown that these methods provide many explicit solutions to the NLEEs.

In this study, we use two integration schemes known as the advanced $\exp(-\varphi(\xi))$-expansion method and the modified $(G'/G)$-expansion method to attempt to find new exact solutions of the generalized seventh-order Korteweg–de Vries equation.

The advanced $\exp(-\varphi(\xi))$-expansion method is based on the assumption that the traveling wave solutions can be described by a polynomial in $\exp(-\varphi(\xi))$. This method takes into account three ordinary differential equations (ODEs) as auxiliary equations:

$$\varphi'(\xi) = \exp(-\varphi(\xi)) + \mu \exp(\varphi(\xi)) + \lambda,$$

$$\varphi'(\xi) = -\lambda \exp(\varphi(\xi)) - \mu \exp(-\varphi(\xi)),$$

$$\text{and } \varphi'(\xi) = -\sqrt{\lambda + \mu \left(\exp(-\varphi(\xi))\right)^2}.$$

While the modified $(G'/G)$-expansion method is predicated on the idea that the traveling wave solutions can be expressed by a polynomial in $(G'/G)$, where $G(\xi)$ fulfills the second-order linear ODE:

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0.$$

The generalized seventh-order Korteweg–de Vries (gsKdV) equation can be expressed as:

$$u_t + auu_x + bu_{3x} + cu_{5x} + du_{7x} = 0, \tag{1.1}$$

where $a,\ b,\ c,\ d$ are real constants, $d \neq 0$, and $u_{nx} = \frac{\partial^n u}{\partial x^n}$.

This equation plays an important role in mathematical physics, engineering, and applied sciences. The case $a = 6,\ b = 1,\ c = -1$, and $d = \alpha$ corresponds to the seventh-order Kawahara (sKawahara) equation. It describes various phenomena, such as the dynamics of long waves in a viscous fluid, shallow water waves with surface tension, and magneto-acoustic waves in plasmas.

Model equation (1.1) was initially introduced by Pomeau et al. [35] to study the stability of a solution to the KdV equation when higher-order spatial derivative terms are added. Ma [28] was the first to find an explicit solution to (1.1) with a $sech^6$-term, which, unfortunately, does not satisfy this equation as pointed out by Duffy and Parkes [10]. They corrected Ma's solution and found another solution, Parkes et al. [34] surveyed the generalized $(2m+1)$-order KdV equation, but the solutions were not written explicitly. Mohyud-Din et al. [32] studied the solitary wave solutions of (1.1) using He's polynomials. Their suggested iterative scheme leads to the desired solution. Mancas and Hereman [30] investigated the applications of (1.1) using an elliptic function method, which calculated hump-type solitary waves and cnoidal wave solutions. Several researchers have studied a special case of this equation using analytical and numerical methods [5, 9, 40].

The rest of this paper is arranged as follows: In Section 2, we present the $\exp(-\varphi(\xi))$-expansion method combined with three auxiliary equations and the modified $(G'/G)$-expansion method. In Section 3, these methods are implemented to the gsKdV equation and related equation. Results and discussion are given in Section 4. Finally, a brief conclusion is given in Section 5.

## 2. Description of the methods

Let us consider the NLEE as follows:

$$P\left(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, ...\right) = 0, \tag{2.1}$$

where $P$ is a polynomial and $u(x,t)$ is an unknown function.

Firstly, suppose $u(x,t) = u(\xi)$, where $\xi = x \pm \omega t$. Then, Eq. (2.1) reduces to a nonlinear ODE for $u = u(\xi)$:

$$Q\left(u, u', u'', u''', .......\right) = 0, \tag{2.2}$$

where $Q$ is a function of $u$ and its derivatives.

2.1. **Description of the advanced** $\exp(-\varphi(\xi))$**-expansion method.** This method is performed by the following steps:

Step 1 . Let us consider the solution of Eq. (2.2) to be of the form:

$$u(\xi) = \sum_{i=0}^{m} \alpha_i \left(\exp(-\varphi(\xi))\right)^i, \ \alpha_m \neq 0, \tag{2.3}$$

where $m$ is a positive integer, $\alpha_i$ $(0 \leq i \leq m)$ are constants to be evaluated and $\varphi = \varphi(\xi)$ satisfies one of the following three ODEs:

$$\varphi'(\xi) = \exp(-\varphi(\xi)) + \mu \exp(\varphi(\xi)) + \lambda, \tag{2.4}$$

$$\varphi'(\xi) = -\lambda \exp(\varphi(\xi)) - \mu \exp(-\varphi(\xi)), \tag{2.5}$$

$$\varphi'(\xi) = -\sqrt{\lambda + \mu \left(\exp(-\varphi(\xi))\right)^2}. \tag{2.6}$$

Here, $\lambda$ and $\mu$ are arbitrary constants. It is important to note that the solutions of Eqs. (2.4), (2.5), and (2.6) can be found in [15].

Step 2 . Calculate $m$ by using the homogeneous balance principle in Eq. (2.2).

Step 3 . By substituting Eq. (2.3) into Eq. (2.2) and using Eq. (2.4) or (2.5) or (2.6) recursively, we obtain a system of algebraic equations for $\alpha_i, \lambda, \mu$, and $\omega$. In this way, the exact solutions of Eq. (2.1) can be found.

2.2. **Description of the modified** $(G'/G)$**-expansion method.** The following steps are used to carry out this method:

Step 1 . Let's say the solution to Eq. (2.2) can be stated as follows:

$$u(\xi) = \sum_{i=-m}^{m} \alpha_i \left(\frac{G'}{G}\right)^i, \ \alpha_{-m} \neq 0, \ \alpha_m \neq 0, \tag{2.7}$$

where $m$ is a positive integer, $\alpha_i$ $(-m \leq i \leq m)$ are constants to be evaluated, and $G = G(\xi)$ satisfies the following second-order linear ODE:

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{2.8}$$

where $\lambda$ and $\mu$ are arbitrary constants.

Step 2 . Apply the homogeneous balancing principle to Eq. (2.2) to obtain $m$.

Step 3 . By substituting Eq. (2.7) into Eq. (2.2) and using Eq. (2.8), we obtain a system of algebraic equations for $\alpha_i, \lambda, \mu$, and $\omega$. Since we are aware of the general solutions to Eq. (2.8) depending on the sign of $\lambda^2 - 4\mu$, we can find the exact solutions of Eq. (2.1).

## 3. Applications of the proposed methods

3.1. **Application of the advanced** $\exp(-\varphi(\xi))$**-expansion method to the gsKdV equation.** The general form of the gsKdV equation is

$$u_t + auu_x + bu_{3x} + cu_{5x} + du_{7x} = 0. \tag{3.1}$$

By using the change of variables $u(x,t) = u(\xi)$ and $\xi = x - \omega t$, this equation reduces to a nonlinear ODE:

$$-\omega u' + auu' + bu''' + cu''''' + du''''''' = 0. \tag{3.2}$$

Integrating Eq. (3.2) with respect to $\xi$, and taking the integration constants as zero, we obtain:

$$-\omega u + \frac{a}{2}u^2 + bu'' + cu'''' + du'''''' = 0. \tag{3.3}$$

By applying homogeneous balance, we obtain $m = 6$. Therefore, the solution to Eq. (3.3) can be expressed as follows:

$$u(\xi) = \sum_{i=0}^{6}\alpha_i(\exp(-\varphi(\xi)))^i,\ \alpha_6 \neq 0. \tag{3.4}$$

where $\varphi(\xi)$ satisfies one of the ODEs (2.5), ( 2.6), and (2.4). The constants $\alpha_i$ are unknown constants that need to be determined.

Substituting Eq. (3.4) into Eq. (3.3) and using the auxiliary equation (2.5), we can equate the coefficients of $(\exp(-\varphi(\xi))^i$ to zero. This leads to a system of algebraic equations. For the sake of convenience, this system has been overlooked. By solving the resulting system, we obtain the following sets of solutions:

**Set 1**

$$b = \frac{769}{4\cdot 5^4}\frac{c^2}{d}, \qquad \lambda = \frac{1}{200}\frac{c}{d\mu}, \qquad \omega = \pm\frac{18}{5^4}\frac{c^3}{d^2},$$
$$\alpha_0 = -\frac{639}{8\cdot 5^5}\frac{c^3}{ad^2} \quad \text{or} \quad \alpha_0 = -\frac{2079}{8\cdot 5^5}\frac{c^3}{ad^2}, \qquad \alpha_1 = 0, \qquad \alpha_2 = -\frac{6237}{5^3}\frac{c^2\mu^2}{ad},$$
$$\alpha_3 = 0, \qquad \alpha_4 = -\frac{49896}{5}\frac{c\mu^4}{a}, \qquad \alpha_5 = 0, \qquad \alpha_6 = -665280\frac{d\mu^6}{a}.$$

**Set 2**

$$b = \frac{2159}{10^4}\frac{c^2}{d}, \qquad \lambda = \frac{1}{400}\frac{c}{d\mu}, \qquad \omega = \pm\frac{71}{10^4}\frac{c^3}{d^2},$$
$$\alpha_0 = -\frac{659}{10^5}\frac{c^3}{ad^2} \quad \text{or} \quad \alpha_0 = -\frac{2079}{10^5}\frac{c^3}{ad^2}, \qquad \alpha_1 = 0, \qquad \alpha_2 = -\frac{2079}{10^2}\frac{c^2\mu^2}{ad},$$
$$\alpha_3 = 0, \qquad \alpha_4 = -\frac{33264}{5}\frac{c\mu^4}{a}, \qquad \alpha_5 = 0, \qquad \alpha_6 = -665280\frac{d\mu^6}{a}.$$

In accordance with Set 1 and Set 2, the solutions of the gsKdV equation are obtained in the following form:

- For *Set 1*, we have $b = \frac{769}{4\cdot 5^4}\frac{c^2}{d}$.

**Case 1.1** When $\lambda\mu > 0$,

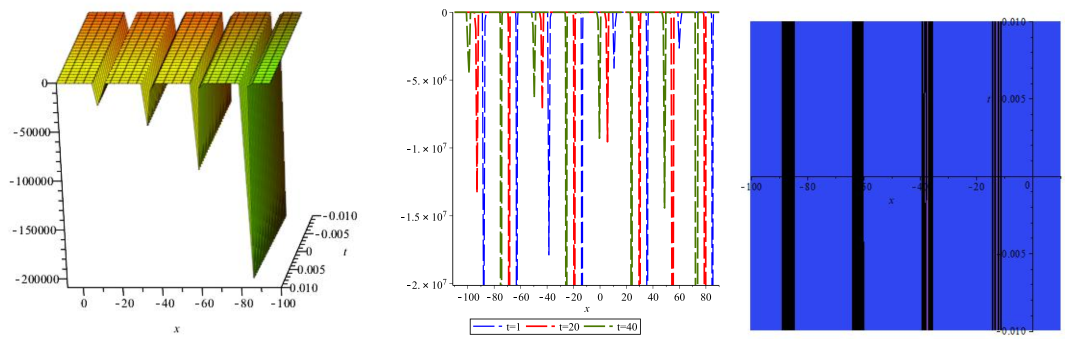
$$\begin{aligned} u_1(\zeta) &= -\frac{\rho c^3}{ad^2}\left(\varrho + 3\tan^2(\zeta) + 3\tan^4(\zeta) + \tan^6(\zeta)\right), \\ u_2(\zeta) &= -\frac{\rho c^3}{ad^2}\left(\varrho + 3\cot^2(\zeta) + 3\cot^4(\zeta) + \cot^6(\zeta)\right), \end{aligned} \tag{3.5}$$

where $\rho = \frac{2079}{8\cdot 5^5}$, $\varrho = \frac{71}{231}$, $\zeta = \frac{1}{10}\sqrt{\frac{c}{2d}}(\xi + k)$, $\xi = x - \frac{18}{625}\frac{c^3}{d^2}t$ , and $k$ is a constant of integration.

Or

$$\begin{aligned} u_3(\zeta) &= -\frac{\rho c^3}{ad^2}\left(1 + 3\tan^2(\zeta) + 3\tan^4(\zeta) + \tan^6(\zeta)\right), \\ u_4(\zeta) &= -\frac{\rho c^3}{ad^2}\left(1 + 3\cot^2(\zeta) + 3\cot^4(\zeta) + \cot^6(\zeta)\right), \end{aligned} \tag{3.6}$$

where $\xi = x + \frac{18}{625}\frac{c^3}{d^2}t$.




(a) $u_3$ in 3D with $-100 \le x \le 8$, $-0.01 \le t \le 0.01$

(b) $u_3$ in 2D with $-110 \le x \le 90$, $t = 1,\ t = 20,\ t = 40$

(c) $u_3$ in contourplot: $-100 \le x \le 10,\ -0.01 \le t \le 0.01$

Figure 1: Multiple dark solitary wave solution of $u_3$ for $a = 6$, $b = 1$, $c = 1$, $d = \frac{769}{4 \cdot 5^4}$, $\lambda\mu = \frac{25}{1538}$, and $k = 1$.

**Case 1.2** When $\lambda\mu < 0$,

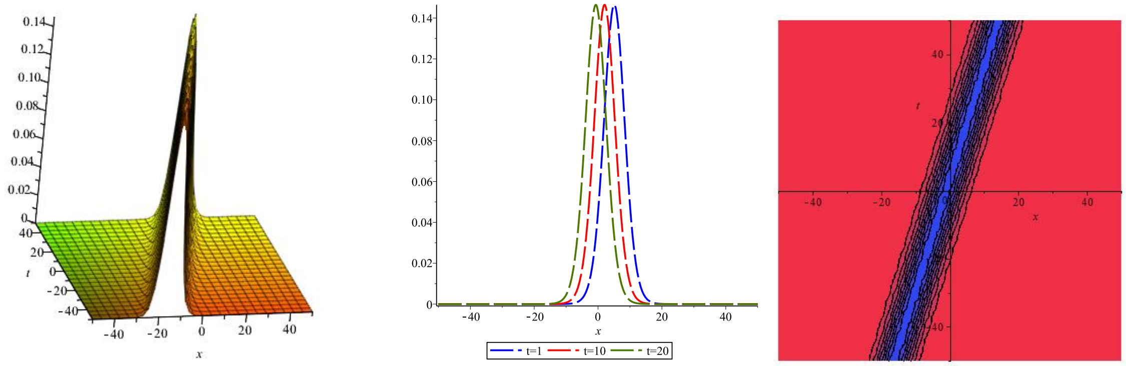
$$
\begin{aligned}
u_5(\zeta) &= -\frac{\rho c^3}{ad^2}\left(\varrho - 3\tanh^2(\zeta) + 3\tanh^4(\zeta) - \tanh^6(\zeta)\right),\\
u_6(\zeta) &= -\frac{\rho c^3}{ad^2}\left(\varrho - 3\coth^2(\zeta) + 3\coth^4(\zeta) - \coth^6(\zeta)\right),
\end{aligned}
\tag{3.7}
$$

where $\rho = \dfrac{2079}{8 \cdot 5^5}$, $\varrho = \dfrac{71}{231}$, $\zeta = \dfrac{1}{10}\sqrt{\dfrac{-c}{2d}}(\xi + k)$, and $\xi = x - \dfrac{18}{625}\dfrac{c^3}{d^2}t$.

Or

$$
\begin{aligned}
u_7(\zeta) &= -\frac{\rho c^3}{ad^2}\left(1 - 3\tanh^2(\zeta) + 3\tanh^4(\zeta) - \tanh^6(\zeta)\right),\\
u_8(\zeta) &= -\frac{\rho c^3}{ad^2}\left(1 - 3\coth^2(\zeta) + 3\coth^4(\zeta) - \coth^6(\zeta)\right),
\end{aligned}
\tag{3.8}
$$

where $\xi = x + \dfrac{18}{625}\dfrac{c^3}{d^2}t$.




(a) $u_7$ in 3D with $-50 \leq x,\ t \leq 50$

(b) $u_7$ in 2D with $-50 \leq x \leq 50$, $t = 1,\ t = 10,\ t = 20$

(c) $u_7$ in contourplot: $-50 \leq x,\ t \leq 50$

Figure 2: Bright soliton solution of $u_7$ for $a = 6$, $b = 1$, $c = -1$, $d = \frac{769}{4 \cdot 5^4}$, $\lambda\mu = \frac{-25}{1538}$, and $k = 1$.

- For *Set 2*, we have $b = \dfrac{2159}{10^4}\dfrac{c^2}{d}$.

**Case 2.1** When $\lambda\mu > 0$,

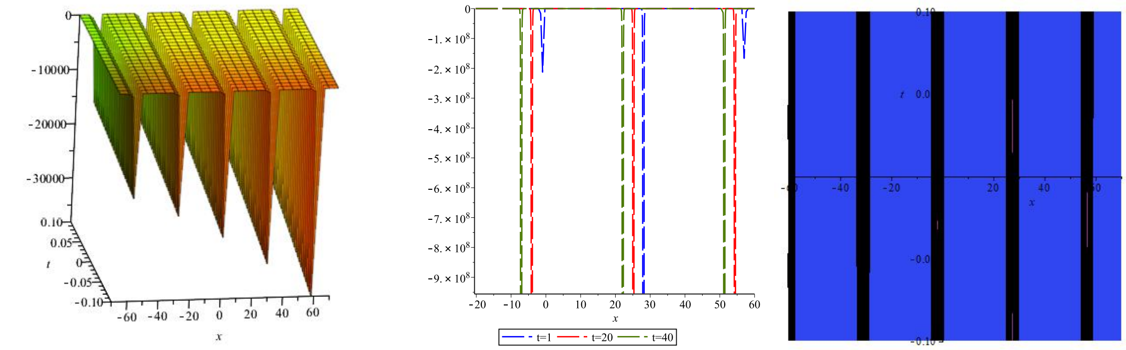
$$
\begin{aligned}
u_9(\zeta) &= -\frac{\rho c^3}{ad^2}\left(\varrho + 5\tan^2(\zeta) + 4\tan^4(\zeta) + \tan^6(\zeta)\right),\\
u_{10}(\zeta) &= -\frac{\rho c^3}{ad^2}\left(\varrho + 5\cot^2(\zeta) + 4\cot^4(\zeta) + \cot^6(\zeta)\right),
\end{aligned} \tag{3.9}
$$

where $\rho = \dfrac{2079}{2 \cdot 10^5}$, $\varrho = \dfrac{1318}{2079}$, $\zeta = \dfrac{1}{20}\sqrt{\dfrac{c}{d}}(\xi + k)$, and $\xi = x - \dfrac{71}{10^4}\dfrac{c^3}{d^2}t$.

Or

$$
\begin{aligned}
u_{11}(\zeta) &= -\frac{\rho c^3}{ad^2}\left(2 + 5\tan^2(\zeta) + 4\tan^4(\zeta) + \tan^6(\zeta)\right),\\
u_{12}(\zeta) &= -\frac{\rho c^3}{ad^2}\left(2 + 5\cot^2(\zeta) + 4\cot^4(\zeta) + \cot^6(\zeta)\right),
\end{aligned} \tag{3.10}
$$

where $\xi = x + \dfrac{71}{10^4}\dfrac{c^3}{d^2}t$.




(a) $u_{12}$ in 3D with $-70 \leq x \leq 70$, $-0.1 \leq t \leq 0.1$

(b) $u_{12}$ in 2D with $-20 \leq x \leq 60$, $t = 1,\ t = 20,\ t = 40$

(c) $u_{12}$ in contourplot: $-60 \leq x \leq 70$, $-0.1 \leq t \leq 0.1$

Figure 3: Multiple dark solitary wave solution of $u_{12}$ for $a = 6$, $b = 1$, $c = 1$, $d = \frac{2159}{10^4}$, $\lambda\mu = \frac{25}{2159}$, and $k = 1$.

**Case 2.2** When $\lambda\mu < 0$,

$$\begin{aligned} u_{13}(\zeta) &= -\frac{\rho c^3}{ad^2}\left(\varrho - 5\tanh^2(\zeta) + 4\tanh^4(\zeta) - \tanh^6(\zeta)\right), \\ u_{14}(\zeta) &= -\frac{\rho c^3}{ad^2}\left(\varrho - 5\coth^2(\zeta) + 4\coth^4(\zeta) - \coth^6(\zeta)\right), \end{aligned} \tag{3.11}$$

where $\rho = \dfrac{2079}{2 \cdot 10^5}$, $\varrho = \dfrac{1318}{2079}$, $\zeta = \dfrac{1}{20}\sqrt{\dfrac{-c}{d}}(\xi + k)$, and $\xi = x - \dfrac{71}{10^4}\dfrac{c^3}{d^2}t$.

Or

$$\begin{aligned} u_{15}(\zeta) &= -\frac{\rho c^3}{ad^2}\left(2 - 5\tanh^2(\zeta) + 4\tanh^4(\zeta) - \tanh^6(\zeta)\right), \\ u_{16}(\zeta) &= -\frac{\rho c^3}{ad^2}\left(2 - 5\coth^2(\zeta) + 4\coth^4(\zeta) - \coth^6(\zeta)\right), \end{aligned} \tag{3.12}$$

where $\xi = x + \dfrac{71}{10^4}\dfrac{c^3}{d^2}t$.

Again, by substituting Eq. (3.4) into Eq. (3.3) and using the auxiliary equation (2.6), and by equating the coefficients of $(\exp(-\varphi(\xi))^i$ to zero, we obtain a system of algebraic equations that is overlooked for convenience. Solving the obtaining system, we obtain the following sets of solutions:

**Set 3**

$$b = \frac{769}{4 \cdot 5^4}\frac{c^2}{d}, \qquad \lambda = -\frac{1}{200}\frac{c}{d}, \qquad \omega = \pm\frac{18}{5^4}\frac{c^3}{d^2},$$

$$\alpha_0 = \frac{36}{5^4}\frac{c^3}{ad^2} \qquad \text{or} \qquad \alpha_0 = 0, \qquad \alpha_1 = 0, \qquad \alpha_2 = 0, \qquad \alpha_3 = 0,$$

$$\alpha_4 = 0, \qquad \alpha_5 = 0, \qquad \alpha_6 = -\frac{665280\ d\mu^3}{a}.$$

**Set 4**

$$b = \frac{2159}{10^4}\frac{c^2}{d}, \qquad \lambda = -\frac{1}{400}\frac{c}{d}, \qquad \omega = \pm\frac{71}{10^4}\frac{c^3}{d^2},$$
$$\alpha_0 = \frac{71}{8{\cdot}5^4}\frac{c^3}{ad^2} \quad \text{or} \quad \alpha_0 = 0, \quad \alpha_1 = 0, \quad \alpha_2 = 0, \quad \alpha_3 = 0,$$
$$\alpha_4 = -\frac{8316}{5}\frac{c\mu^2}{a}, \qquad \alpha_5 = 0, \qquad \alpha_6 = -\frac{665280}{a}\frac{d\mu^3}{}.$$

According to Set 3 and Set 4, the solutions of the gsKdV equation are derived in the following form:

- For *Set 3*, we have $b = \frac{769}{4{\cdot}5^4}\frac{c^2}{d}$.

**Case 3.1** When $\lambda > 0,\ \mu > 0$,

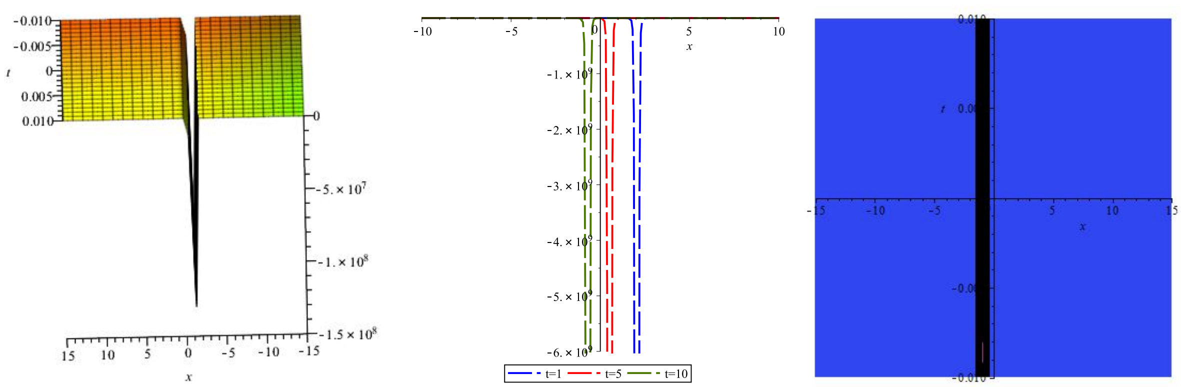
$$u_{17}(\zeta) = \frac{\rho c^3}{ad^2}\left(\varrho + \operatorname{csch}^6(\zeta)\right), \tag{3.13}$$

where $\rho = \frac{2079}{8{\cdot}5^5}$, $\varrho = \frac{160}{231}$, $\zeta = \frac{1}{10}\sqrt{\frac{-c}{2d}}(\xi + k)$, and $\xi = x - \frac{18}{625}\frac{c^3}{d^2}t$.

Or

$$u_{18}(\zeta) = \frac{\rho c^3}{ad^2}\operatorname{csch}^6(\zeta), \tag{3.14}$$

where $\xi = x + \frac{18}{625}\frac{c^3}{d^2}t$.



(a) $u_{18}$ in 3D with $-15 \le x \le 15$, $-0.01 \le t \le 0.01$

(b) $u_{18}$ in 2D with $-10 \le x \le 10$, $t = 1,\ t = 5,\ t = 10$

(c) $u_{18}$ in contourplot: $-15 \le x \le 15$, $-0.01 \le t \le 0.01$

Figure 4: Dark solitary wave solution of $u_{18}$ for $a = 6$, $b = 1$, $c = -1$, $d = \frac{769}{4{\cdot}5^4}$, $\lambda = \frac{25}{1538}$, and $k = 1$.

**Case 3.2** When $\lambda < 0,\ \mu > 0$,

$$u_{19}(\zeta) = \frac{\rho c^3}{ad^2}\left(\varrho - \sec^6(\zeta)\right), \tag{3.15}$$

where $\rho = \frac{2079}{8{\cdot}5^5}$, $\varrho = \frac{160}{231}$, $\zeta = \frac{1}{10}\sqrt{\frac{c}{2d}}(\xi + k)$, and $\xi = x - \frac{18}{625}\frac{c^3}{d^2}t$.

Or

$$u_{20}(\zeta) = -\frac{\rho c^3}{ad^2}\sec^6(\zeta), \tag{3.16}$$

where $\xi = x + \dfrac{18}{625}\dfrac{c^3}{d^2}t$.

**Case 3.3** When $\lambda > 0,\ \mu < 0,$

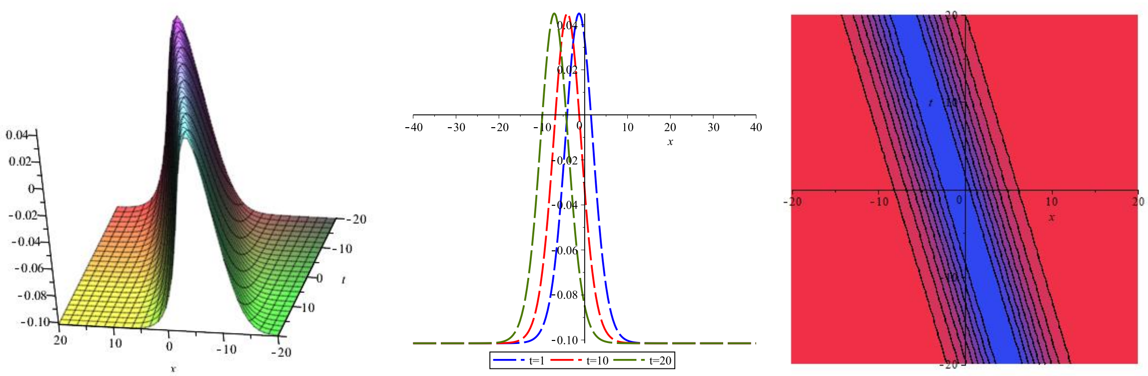
$$u_{21}(\zeta) = \frac{\rho c^3}{ad^2}\left(\varrho - \operatorname{sech}^6(\zeta)\right), \tag{3.17}$$

where $\rho = \dfrac{2079}{8\cdot 5^5},\ \varrho = \dfrac{160}{231},\ \zeta = \dfrac{1}{10}\sqrt{\dfrac{-c}{2d}}(\xi + k)$, and $\xi = x - \dfrac{18}{625}\dfrac{c^3}{d^2}t$.

Or

$$u_{22}(\zeta) = -\frac{\rho c^3}{ad^2}\operatorname{sech}^6(\zeta), \tag{3.18}$$

where $\xi = x + \dfrac{18}{625}\dfrac{c^3}{d^2}t$.




(a) $u_{21}$ in 3D with $-20 \leq x,\ t \leq 20$

(b) $u_{21}$ in 2D with $-40 \leq x \leq 40$, $t = 1,\ t = 10,\ t = 20$

(c) $u_{21}$ in contourplot:$-20 \leq x,\ t \leq 20$

Figure 5: Bright soliton solution of $u_{21}$ for $a = 6,\ b = 1,\ c = -1,\ d = \frac{769}{4\cdot 5^4},\ \lambda = \frac{25}{1538}$, and $k = 1$.

**Case 3.4** When $\lambda < 0,\ \mu < 0,$

$$u_{23}(\zeta) = \frac{\rho c^3}{ad^2}\left(\varrho - \csc^6(\zeta)\right), \tag{3.19}$$

where $\rho = \dfrac{2079}{8\cdot 5^5},\ \varrho = \dfrac{160}{231},\ \zeta = \dfrac{1}{10}\sqrt{\dfrac{c}{2d}}(\xi + k)$, and $\xi = x - \dfrac{18}{625}\dfrac{c^3}{d^2}t$.

Or

$$u_{24}(\zeta) = -\frac{\rho c^3}{ad^2}\csc^6(\zeta), \tag{3.20}$$

where $\xi = x + \dfrac{18}{625}\dfrac{c^3}{d^2}t$.

- For *Set 4*, we have $b = \dfrac{2159}{10^4}\dfrac{c^2}{d}$.

**Case 4.1** When $\lambda > 0,\ \mu > 0$,

$$u_{25}(\zeta) = \frac{\rho c^3}{ad^2}\left(\varrho - \operatorname{csch}^4(\zeta) + \operatorname{csch}^6(\zeta)\right), \tag{3.21}$$

where $\rho = \dfrac{2079}{2\cdot 10^5}$, $\varrho = \dfrac{2840}{2079}$, $\zeta = \dfrac{1}{20}\sqrt{\dfrac{-c}{d}}(\xi + k)$, and $\xi = x - \dfrac{71}{10^4}\dfrac{c^3}{d^2}t$.

Or

$$u_{26}(\zeta) = -\frac{\rho c^3}{ad^2}\left(\operatorname{csch}^4(\zeta) - \operatorname{csch}^6(\zeta)\right), \tag{3.22}$$

where $\xi = x + \dfrac{71}{10^4}\dfrac{c^3}{d^2}t$.

**Case 4.2** When $\lambda < 0,\ \mu > 0$,

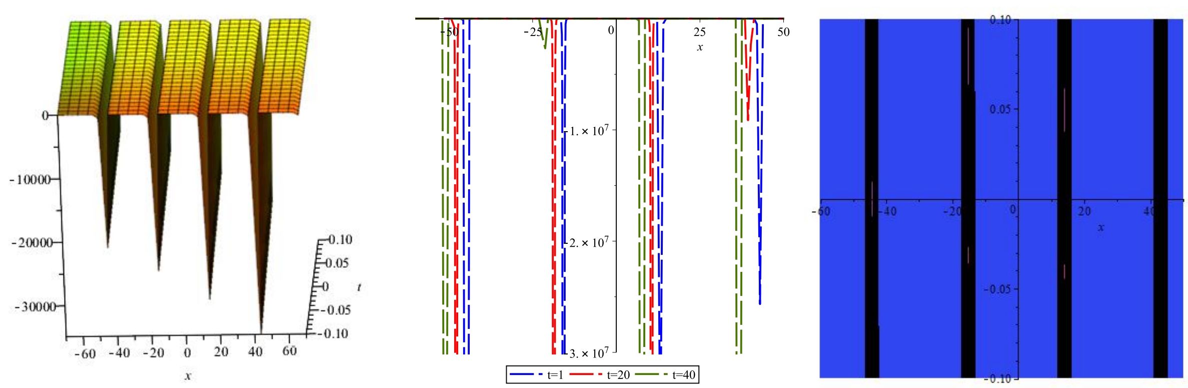
$$u_{27}(\zeta) = \frac{\rho c^3}{ad^2}\left(\varrho - \sec^4(\zeta) - \sec^6(\zeta)\right), \tag{3.23}$$

where $\rho = \dfrac{2079}{2\cdot 10^5}$, $\varrho = \dfrac{2840}{2079}$, $\zeta = \dfrac{1}{20}\sqrt{\dfrac{c}{d}}(\xi + k)$, and $\xi = x - \dfrac{71}{10^4}\dfrac{c^3}{d^2}t$.

Or

$$u_{28}(\zeta) = -\frac{\rho c^3}{ad^2}\left(\sec^4(\zeta) + \sec^6(\zeta)\right), \tag{3.24}$$

where $\xi = x + \dfrac{71}{10^4}\dfrac{c^3}{d^2}t$.




(a) $u_{28}$ in 3D with $-70 \le x \le 70$, $-0.1 \le t \le 0.1$

(b) $u_{28}$ in 2D with $-60 \le x \le 50$, $t = 1,\ t = 20,\ t = 40$

(c) $u_{28}$ in contourplot: $-60 \le x \le 50$, $-0.1 \le t \le 0.1$

Figure 6: Multiple dark solitary wave solution of $u_{28}$ for $a = 6$, $b = 1$, $c = 1$, $d = \frac{2159}{10^4}$, $\lambda = \frac{-25}{2159}$, and $k = 1$.

**Case 4.3** When $\lambda > 0,\ \mu < 0$,

$$u_{29}(\zeta) = \frac{\rho c^3}{ad^2}\left(\varrho - \operatorname{sech}^4(\zeta) - \operatorname{sech}^6(\zeta)\right), \tag{3.25}$$

where $\rho = \dfrac{2079}{2\cdot 10^5}$, $\varrho = \dfrac{2840}{2079}$, $\zeta = \dfrac{1}{20}\sqrt{\dfrac{-c}{d}}(\xi + k)$, and $\xi = x - \dfrac{71}{10^4}\dfrac{c^3}{d^2}t$.

Or

$$u_{30}(\zeta) = -\frac{\rho c^3}{ad^2}\left(\operatorname{sech}^4(\zeta) + \operatorname{sech}^6(\zeta)\right), \tag{3.26}$$

where $\xi = x + \dfrac{71}{10^4}\dfrac{c^3}{d^2}t$.

**Case 4.4** When $\lambda < 0,\ \mu < 0$,

$$u_{31}(\zeta) = \frac{\rho c^3}{ad^2}\left(\varrho - \csc^4(\zeta) - \csc^6(\zeta)\right), \tag{3.27}$$

where $\rho = \dfrac{2079}{2\cdot 10^5}$, $\varrho = \dfrac{2840}{2079}$, $\zeta = \dfrac{1}{20}\sqrt{\dfrac{c}{d}}(\xi + k)$, and $\xi = x - \dfrac{71}{10^4}\dfrac{c^3}{d^2}t$.

Or

$$u_{32}(\zeta) = -\frac{\rho c^3}{ad^2}\left(\csc^4(\zeta) + \csc^6(\zeta)\right), \tag{3.28}$$

where $\xi = x + \dfrac{71}{10^4}\dfrac{c^3}{d^2}t$.

Again, by substituting Eq. (3.4) into Eq. (3.3) and using the auxiliary equation (2.4), we obtain a system of algebraic equations that is very difficult to solve using symbolic computation. Consequently, we restricted our study to the special case of the gsKdV equation, specifically the Kawahara equation.

3.2. **Application of the advanced $\exp(-\varphi(\xi))$-expansion method to the sKawahara equation.** In Eq. (3.1), when we have $a = 6,\ b = 1,\ c = -1$, and $d = \alpha$, it corresponds to the standard form of the seventh-order Kawahara equation [40]:

$$u_t + 6uu_x + u_{3x} - u_{5x} + \alpha u_{7x} = 0. \tag{3.29}$$

This equation can be reduced to a nonlinear ODE:

$$-\omega u + 3u^2 + u'' - u'''' + \alpha u'''''' = 0. \tag{3.30}$$

By substituting Eq. (3.4) into Eq. (3.30) and using the auxiliary equation (2.4), we can equate the coefficients of $(\exp(-\varphi(\xi))^i$ to zero, which leads to a system of algebraic equations. However, for convenience, we neglect this system and proceed to solve the obtained system, obtaining the following sets of solutions:

**Set 5**

$$\alpha = \frac{769}{4\cdot 5^4}, \qquad \lambda = \pm\sqrt{4\mu + \frac{50}{769}}, \qquad \omega = \pm\frac{18\cdot 10^4}{769^2},$$
$$\alpha_0 = -\frac{4263336}{125}\mu^3 \quad \text{or} \quad \alpha_0 = -\frac{4263336}{125}\mu^3 - \frac{6\cdot 10^4}{591361}, \quad \alpha_1 = -\frac{12790008}{125}\lambda\mu^2,$$
$$\alpha_2 = -\frac{12790008}{25}\mu^2 - \frac{33264}{5}\mu, \quad \alpha_3 = -\frac{11088}{25}\lambda(769\mu + 5),$$
$$\alpha_4 = -\frac{12790008}{25}\mu - \frac{33264}{5}, \quad \alpha_5 = -\frac{12790008}{125}\lambda, \quad \alpha_6 = -\frac{4263336}{125}.$$

**Set 6**

$$
\begin{aligned}
&\alpha = \frac{2159}{10^4}, \qquad \lambda = \pm\sqrt{4\mu + \frac{100}{2159}}, \qquad \omega = \pm\frac{71\cdot 10^4}{2159^2},\\
&\alpha_0 = -\frac{1386}{5}\mu^2(\frac{2159}{25}\mu - 1) \quad \text{or} \quad \alpha_0 = -\frac{2992374}{125}\mu^3 + \frac{1386}{5}\mu^2 - \frac{71\cdot 10^4}{13983843},\\
&\alpha_1 = -\frac{1386}{125}\lambda\mu(6477\mu - 50), \quad \alpha_2 = -\frac{8977122}{25}\mu^2 - \frac{8316}{5}\mu + \frac{27720}{2159},\\
&\alpha_3 = -\frac{2772}{25}\lambda(2159\mu + 5), \quad \alpha_4 = -\frac{8977122}{25}\mu - \frac{15246}{5},\\
&\alpha_5 = -\frac{8977122}{125}\lambda, \quad \alpha_6 = -\frac{2992374}{125}.
\end{aligned}
$$

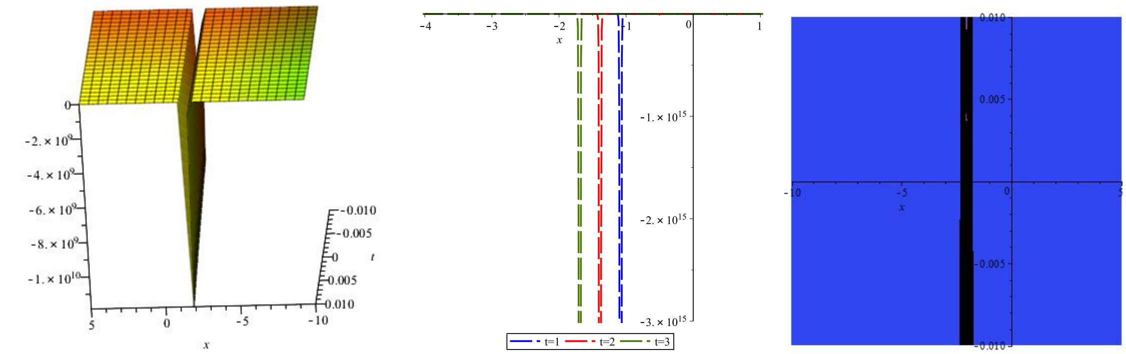
Depending on Set 5 and Set 6, and given that $\lambda^2 - 4\mu > 0$, the solutions of the Kawahara equation are obtained in the following form:

- For *Set 5*, we have $\alpha = \dfrac{769}{4\cdot 5^4}$.

**Case 5.1** When $\mu \neq 0$,

$$
\begin{aligned}
u_{33}(\zeta) =& -\rho\mu^3 + \frac{6\rho\lambda\mu^3}{\psi_1(\zeta)} - \frac{60\,(\rho\mu+\sigma)\,\mu^3}{\psi_1^2(\zeta)} + \frac{8\sigma(769\mu+5)\lambda\mu^3}{\psi_1^3(\zeta)}\\
&- \frac{240\,(\rho\mu+\sigma)\,\mu^4}{\psi_1^4(\zeta)} + \frac{96\rho\lambda\mu^5}{\psi_1^5(\zeta)} - \frac{64\rho\mu^6}{\psi_1^6(\zeta)},\\
u_{34}(\zeta) =& -\rho\mu^3 + \frac{6\rho\lambda\mu^3}{\psi_2(\zeta)} - \frac{60\,(\rho\mu+\sigma)\,\mu^3}{\psi_2^2(\zeta)} + \frac{8\sigma(769\mu+5)\lambda\mu^3}{\psi_2^3(\zeta)}\\
&- \frac{240\,(\rho\mu+\sigma)\,\mu^4}{\psi_2^4(\zeta)} + \frac{96\rho\lambda\mu^5}{\psi_2^5(\zeta)} - \frac{64\rho\mu^6}{\psi_2^6(\zeta)},
\end{aligned}
\tag{3.31}
$$

where $\rho = \dfrac{4263336}{125}$, $\sigma = \dfrac{11088}{25}$, $\psi_1(\zeta) = \dfrac{10}{\sqrt{1538}}\tanh(\zeta) + \lambda$, $\psi_2(\zeta) = \dfrac{10}{\sqrt{1538}}\coth(\zeta) + \lambda$, $\zeta = \dfrac{5}{\sqrt{1538}}(\xi + k)$, and $\xi = x - \dfrac{18\cdot 10^4}{769^2}t$.




(a) $u_{34}$ in 3D with $-10 \leq x \leq 5$, $-0.01 \leq t \leq 0.01$

(b) $u_{34}$ in 2D with $-4 \leq x \leq 1$, $t=1,\ t=2,\ t=3$

(c) $u_{34}$ in contourplot: $-10 \leq x \leq 5,\ -0.01 \leq t \leq 0.01$

Figure 7: Dark solitary wave solution of $u_{34}$ for $\alpha = \frac{769}{4 \cdot 5^4}$, $\mu = 1$, and $k = 1$.

Or

$$
\begin{aligned}
u_{35}(\zeta) =& -\rho\mu^3 - \frac{6 \cdot 10^4}{(769)^2} + \frac{6\rho\lambda\mu^3}{\psi_1(\zeta)} - \frac{60\,(\rho\mu+\sigma)\,\mu^3}{\psi_1^2(\zeta)} + \frac{8\sigma(769\mu+5)\lambda\mu^3}{\psi_1^3(\zeta)} \\
&- \frac{240\,(\rho\mu+\sigma)\,\mu^4}{\psi_1^4(\zeta)} + \frac{96\rho\lambda\mu^5}{\psi_1^5(\zeta)} - \frac{64\rho\mu^6}{\psi_1^6(\zeta)}, \\
u_{36}(\zeta) =& -\rho\mu^3 - \frac{6 \cdot 10^4}{(769)^2} + \frac{6\rho\lambda\mu^3}{\psi_2(\zeta)} - \frac{60\,(\rho\mu+\sigma)\,\mu^3}{\psi_2^2(\zeta)} + \frac{8\sigma(769\mu+5)\lambda\mu^3}{\psi_2^3(\zeta)} \\
&- \frac{240\,(\rho\mu+\sigma)\,\mu^4}{\psi_2^4(\zeta)} + \frac{96\rho\lambda\mu^5}{\psi_2^5(\zeta)} - \frac{64\rho\mu^6}{\psi_2^6(\zeta)},
\end{aligned}
\tag{3.32}
$$

where $\xi = x + \dfrac{18 \cdot 10^4}{769^2} t$.

**Case 5.2** When $\mu = 0$,

$$
u_{37}(\zeta) = -\frac{5\sigma\lambda^4 \exp(3\zeta)}{(\exp(\zeta)-1)^6}, \tag{3.33}
$$

where $\lambda^2 = \dfrac{50}{769}$, $\sigma = \dfrac{11088}{25}$, $\zeta = \lambda(\xi + k)$, and $\xi = x - 72\lambda^4 t$.

Or

$$
u_{38}(\zeta) = -24\lambda^4 - \frac{5\sigma\lambda^4 \exp(3\zeta)}{(\exp(\zeta)-1)^6}, \tag{3.34}
$$

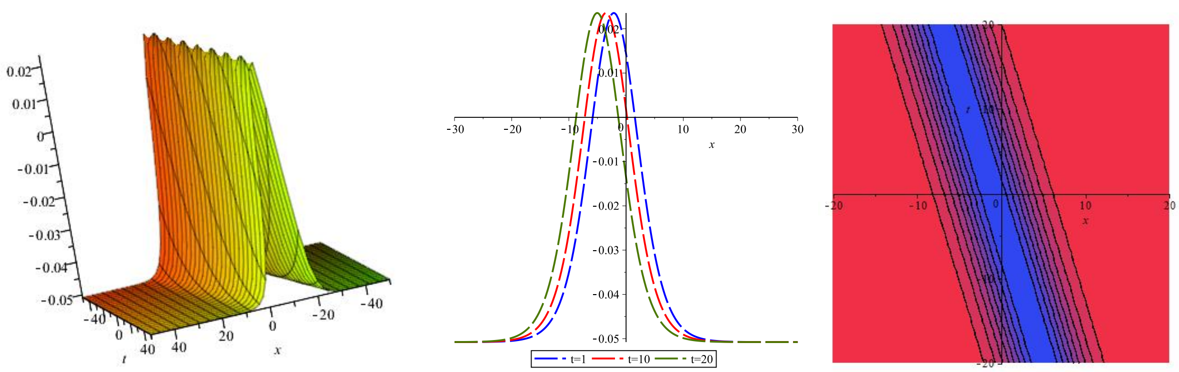
where $\xi = x + 72\lambda^4 t$.

- For *Set 6*, we have $\alpha = \dfrac{2159}{10^4}$.

**Case 6.1** When $\mu \neq 0$,

$$\begin{aligned}
u_{39}(\zeta) =& -\rho\mu^3 + \frac{5}{2}\sigma\mu^2 + \frac{\sigma(6477\mu-50)\lambda\mu^2}{5\psi_3(\zeta)} - \frac{60\left(\rho\mu^2+\sigma\mu-\gamma_1\right)\mu^2}{\psi_3^2(\zeta)} \\
&+ \frac{8\sigma(2159\mu+5)\lambda\mu^3}{\psi_3^3(\zeta)} - \frac{240\left(\rho\mu+\gamma_2\right)\mu^4}{\psi_3^4(\zeta)} + \frac{96\rho\lambda\mu^5}{\psi_3^5(\zeta)} - \frac{64\rho\mu^6}{\psi_3^6(\zeta)}, \\
u_{40}(\zeta) =& -\rho\mu^3 + \frac{5}{2}\sigma\mu^2 + \frac{\sigma(6477\mu-50)\lambda\mu^2}{5\psi_4(\zeta)} - \frac{60\left(\rho\mu^2+\sigma\mu-\gamma_1\right)\mu^2}{\psi_4^2(\zeta)} \\
&+ \frac{8\sigma(2159\mu+5)\lambda\mu^3}{\psi_4^3(\zeta)} - \frac{240\left(\rho\mu+\gamma_2\right)\mu^4}{\psi_4^4(\zeta)} + \frac{96\rho\lambda\mu^5}{\psi_4^5(\zeta)} - \frac{64\rho\mu^6}{\psi_4^6(\zeta)},
\end{aligned} \tag{3.35}$$

where $\rho = \frac{2992374}{125}$, $\sigma = \frac{2772}{25}$, $\gamma_1 = \frac{1848}{2159}$, $\gamma_2 = \frac{5082}{25}$, $\psi_3(\zeta) = \frac{10}{\sqrt{2159}}\tanh(\zeta) + \lambda$,
$\psi_4(\zeta) = \frac{10}{\sqrt{2159}}\coth(\zeta) + \lambda$, $\zeta = \frac{5}{\sqrt{2159}}(\xi + k)$, and $\xi = x - \frac{71\cdot 10^4}{2159^2}t$.



(a) $u_{41}$ in 3D with $-50 \leq x,\ t \leq 50$

(b) $u_{41}$ in 2D with $-30 \leq x \leq 30$, $t = 1,\ t = 10,\ t = 20$

(c) $u_{41}$ in contourplot:$-50 \leq x,\ t \leq 50$

Figure 8: Bright soliton solution of $u_{41}$ for $\alpha = \frac{2159}{10^4}$, $\mu = 1$, and $k = 1$.

Or

$$\begin{aligned}
u_{41}(\zeta) =& -\rho\mu^3 + \frac{5}{2}\sigma\mu^2 - \frac{71\cdot 10^4}{3\cdot 2159^2} + \frac{\sigma(6477\mu-50)\lambda\mu^2}{5\psi_3(\zeta)} - \frac{60\left(\rho\mu^2+\sigma\mu-\gamma_1\right)\mu^2}{\psi_3^2(\zeta)} \\
&+ \frac{8\sigma(2159\mu+5)\lambda\mu^3}{\psi_3^3(\zeta)} - \frac{240\left(\rho\mu+\gamma_2\right)\mu^4}{\psi_3^4(\zeta)} + \frac{96\rho\lambda\mu^5}{\psi_3^5(\zeta)} - \frac{64\rho\mu^6}{\psi_3^6(\zeta)}, \\
u_{42}(\zeta) =& -\rho\mu^3 + \frac{5}{2}\sigma\mu^2 - \frac{71\cdot 10^4}{3\cdot 2159^2} + \frac{\sigma(6477\mu-50)\lambda\mu^2}{5\psi_4(\zeta)} - \frac{60\left(\rho\mu^2+\sigma\mu-\gamma_1\right)\mu^2}{\psi_4^2(\zeta)} \\
&+ \frac{8\sigma(2159\mu+5)\lambda\mu^3}{\psi_4^3(\zeta)} - \frac{240\left(\rho\mu+\gamma_2\right)\mu^4}{\psi_4^4(\zeta)} + \frac{96\rho\lambda\mu^5}{\psi_4^5(\zeta)} - \frac{64\rho\mu^6}{\psi_4^6(\zeta)},
\end{aligned} \tag{3.36}$$

where $\xi = x + \frac{71\cdot 10^4}{2159^2}t$.

**Case 6.2** When $\mu = 0$,

$$u_{43}(\zeta) = \frac{5\sigma\lambda^4}{2}\left(\frac{\exp(4\zeta) - 6\exp(3\zeta) + \exp(2\zeta)}{(\exp(\zeta) - 1)^6}\right), \tag{3.37}$$

where $\lambda^2 = \dfrac{100}{2159}$, $\sigma = \dfrac{2772}{25}$, $\zeta = \lambda(\xi + k)$, and $\xi = x - 71\lambda^4 t$.

Or

$$u_{44}(\zeta) = -\frac{71\lambda^4}{3} + \frac{5\sigma\lambda^4}{2}\left(\frac{\exp(4\zeta) - 6\exp(3\zeta) + \exp(2\zeta)}{(\exp(\zeta) - 1)^6}\right), \tag{3.38}$$

where $\xi = x + 71\lambda^4 t$.

3.3. **Application of the modified $(G'/G)$-expansion method to the gsKdV equation.** As the second method, we use the modified $(G'/G)$-expansion method for Eq. (3.3), which is given as:

$$-\omega u + \frac{a}{2}u^2 + bu'' + cu'''' + du'''''' = 0. \tag{3.39}$$

Considering the homogenous balance, we obtain $m = 6$. So the solution of (3.39) can be described as:

$$u(\xi) = \sum_{i=-6}^{6}\alpha_i\left(\frac{G'}{G}\right)^i, \ \alpha_{-6} \neq 0, \ \alpha_6 \neq 0. \tag{3.40}$$

where $G(\xi)$ satisfies the second-order linear differential equation (2.8), and $\alpha_i$ are unknown constants to be identified.

Substituting Eq. (3.40) into Eq. (3.39) and using the auxiliary equation (2.8), we can equate the coefficients of $\left(\frac{G'}{G}\right)^i$ to zero. However, this leads to an algebraic system that proves to be difficult to solve using symbolic arithmetic. Similarly to the previous method utilizing the auxiliary equation (2.4), the study of the sKawahara equation was consequently limited.

3.4. **Application of the modified $(G'/G)$-expansion method to the sKawahara equation.** We use the modified $(G'/G)$-expansion method for Eq. (3.30) which is given as:

$$-\omega u + 3u^2 + u'' - u'''' + \alpha u'''''' = 0. \tag{3.41}$$

By substituting Eq. (3.40) into Eq. (3.41) and using the auxiliary equation (2.8), and by equating the coefficients of $\left(\frac{G'}{G}\right)^i$ to zero, we obtain a system of algebraic equations. For the sake of convenience, this system is disregarded. Solving the obtained system, we obtain the following sets of solutions:

**Set 7**

$$\begin{aligned}
&\alpha = \frac{769}{4\cdot 5^4}, \quad \lambda = 0, \quad \mu = -\frac{25}{769\cdot 2^3}, \quad \omega = \mp\frac{18\cdot 10^4}{769^2},\\
&\alpha_{-6} = -\frac{693\cdot 5^9}{769^5\cdot 2^{15}}, \quad \alpha_{-5} = 0, \quad \alpha_{-4} = \frac{2079\cdot 5^7}{769^4\cdot 2^{11}}, \quad \alpha_{-3} = 0, \quad \alpha_{-2} = -\frac{2079\cdot 5^6}{769^3\cdot 2^9},\\
&\alpha_{-1} = 0, \quad \alpha_0 = -\frac{843\cdot 5^4}{769^2\cdot 2^4} \ \text{ or } \ \alpha_0 = \frac{693\cdot 5^4}{769^2\cdot 2^4}, \quad \alpha_1 = 0, \quad \alpha_2 = -\frac{51975}{769\cdot 2^3},\\
&\alpha_3 = 0, \quad \alpha_4 = \frac{4158}{5}, \quad \alpha_5 = 0, \quad \alpha_6 = -\frac{4263336}{125}.
\end{aligned}$$

**Set 8**

$$\begin{aligned}
&\alpha = \frac{2159}{10^4}, \qquad \lambda = 0, \qquad \mu = -\frac{25}{2159 \cdot 2^2}, \qquad \omega = \mp \frac{71 \cdot 10^4}{2159^2}, \\
&\alpha_{-6} = -\frac{693 \cdot 5^9}{2159^5 \cdot 2^{11}}, \quad \alpha_{-5} = 0, \quad \alpha_{-4} = \frac{693 \cdot 5^8}{2159^4 \cdot 2^8}, \quad \alpha_{-3} = 0, \quad \alpha_{-2} = -\frac{21483 \cdot 5^5}{2159^3 \cdot 2^7}, \\
&\alpha_{-1} = 0, \quad \alpha_0 = -\frac{22571 \cdot 5^3}{24 \cdot 2159^2} \quad \text{or} \quad \alpha_0 = \frac{7623 \cdot 5^3}{2159^2 \cdot 2^3}, \quad \alpha_1 = 0, \quad \alpha_2 = -\frac{107415}{2159 \cdot 2^3}, \\
&\alpha_3 = 0, \qquad \alpha_4 = 693, \qquad \alpha_5 = 0, \qquad \alpha_6 = -\frac{2992374}{125}.
\end{aligned}$$

Given that $\lambda = 0$ and $\mu < 0$, we can infer for $\lambda^2 - 4\mu > 0$ that

$$\begin{aligned}
\frac{G'}{G} &= \sqrt{-\mu}\left(\frac{C_1 \sinh(\sqrt{-\mu}\ \xi) + C_2 \cosh(\sqrt{-\mu}\ \xi)}{C_1 \cosh(\sqrt{-\mu}\ \xi) + C_2 \sinh(\sqrt{-\mu}\ \xi)}\right) \\
&= \sqrt{-\mu} \tanh\left(\sqrt{-\mu}\ (\xi + \xi_0)\right),
\end{aligned} \tag{3.42}$$

where $\tanh(\sqrt{-\mu}\ \xi_0) = \dfrac{C_2}{C_1}$.

Depending on Set 7 and Set 8, the solutions of the sKawahara equation are obtained in the following form:

- For *Set 7*, we have $\alpha = \dfrac{769}{4 \cdot 5^4}$.

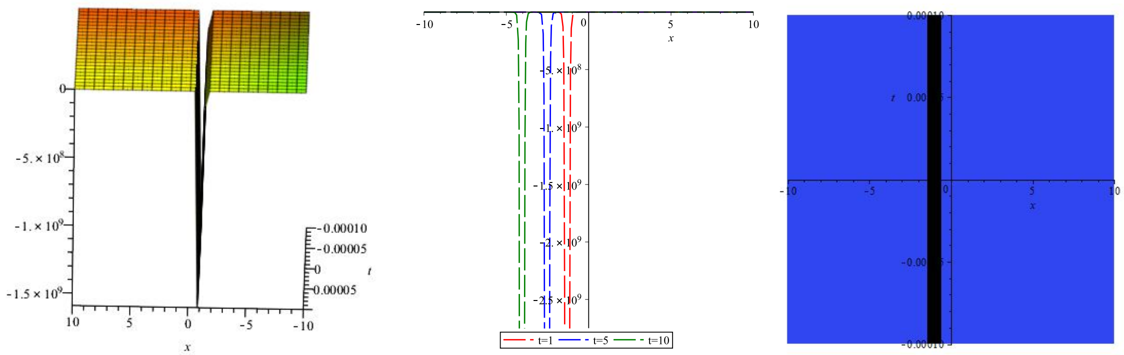
$$\begin{aligned}
u_{45}(\zeta) = &-\frac{5620}{231}\beta - 15\beta \tanh^2(\zeta) + 6\beta \tanh^4(\zeta) - \beta \tanh^6(\zeta) \\
&- 15\beta \coth^2(\zeta) + 6\beta \coth^4(\zeta) - \beta \coth^6(\zeta),
\end{aligned} \tag{3.43}$$

where $\beta = \dfrac{693 \cdot 5^3}{2^6 \cdot 769^2}$, $\zeta = \dfrac{5}{2\sqrt{1538}}(\xi + \xi_0)$, and $\xi = x + \dfrac{18 \cdot 10^4}{769^2} t$.

Or

$$\begin{aligned}
u_{46}(\zeta) = &20\beta - 15\beta \tanh^2(\zeta) + 6\beta \tanh^4(\zeta) - \beta \tanh^6(\zeta) \\
&- 15\beta \coth^2(\zeta) + 6\beta \coth^4(\zeta) - \beta \coth^6(\zeta),
\end{aligned} \tag{3.44}$$

where $\xi = x - \dfrac{18 \cdot 10^4}{769^2} t$.




(a) $u_{45}$ in 3D with $-10 \le x \le 10$, $-0.0001 \le t \le 0.0001$

(b) $u_{45}$ in 2D with $-10 \le x \le 10$, $t = 1,\ t = 5,\ t = 10$

(c) $u_{45}$ in contourplot: $-10 \le x \le 10$, $-0.0001 \le t \le 0.0001$

Figure 9: Dark solitary wave solution of $u_{45}$ for $\alpha = \frac{769}{4\cdot 5^4}$, and $\xi_0 = 1$.

- For *Set 8*, we have $\alpha = \dfrac{2159}{10^4}$.

$$
\begin{aligned}
u_{47}(\xi) = &-\frac{90284}{2079}\beta - 31\beta\tanh^2(\zeta) + 10\beta\tanh^4(\zeta) - \beta\tanh^6(\zeta) \\
&- 31\beta\coth^2(\zeta) + 10\beta\coth^4(\zeta) - \beta\coth^6(\zeta),
\end{aligned}
\tag{3.45}
$$

where $\beta = \dfrac{693\cdot 5^3}{2^5\cdot 2159^2}$, $\zeta = \dfrac{5}{2\sqrt{2159}}(\xi + \xi_0)$, and $\xi = x + \dfrac{71\cdot 10^4}{2159^2}t$.

Or

$$
\begin{aligned}
u_{48}(\xi) = &44\beta - 31\beta\tanh^2(\zeta) + 10\beta\tanh^4(\zeta) - \beta\tanh^6(\zeta) \\
&- 31\beta\coth^2(\zeta) + 10\beta\coth^4(\zeta) - \beta\coth^6(\zeta),
\end{aligned}
\tag{3.46}
$$

where $\xi = x - \dfrac{71\cdot 10^4}{2159^2}t$.

**Remark 3.1.** All of the solutions found in this study were checked with Maple by reinserting them into the original equations, and they were found to be correct.

## 4. Results and discussion

In this paper, two analytical techniques are employed to investigate the generalized seventh-order KdV equation with constant coefficients (1.1). The specific components of this parametric equation describe various phenomena, including magneto-acoustic, hydrodynamic, and electrical impulses in plasmas. This study enabled the calculation of bright solitons, dark solitary wave solutions, and multiple dark solitary wave solutions. These findings are crucial for understanding practical physical problems and can serve as references for numerical solvers and perturbation methods. It was observed that the coefficients cannot be chosen arbitrarily; instead, they satisfy specific algebraic relations. The results of these analyses are highly intriguing and offer appealing insights. Furthermore, a broader and significant number of exact traveling wave solutions were obtained.

While many of the solutions in this study are brand new, several of them show results that have already been documented in the literature. To the best of our knowledge, every solution that came from Sets 1 and 2 derived from the solutions of Eq. (2.5), and a large number of solutions that came from Sets 3 and 4 derived from the solutions of Eq. (2.6) are all novel solutions. Solution (3.18) is equivalent to a solution reported by Ma [28], Duffy and Parkes [10], and by Mancas and Hereman [30]. Actually, the solution (25) on page 114002-4 in [30] that reads as follows:

$$u(x,t) = -\frac{154\cdot 3^3\cdot 5^3\ b^2}{769^2\ ac}\operatorname{sech}^6\left(\frac{5}{\sqrt{1538}}\sqrt{-\frac{b}{c}}\left(x+\frac{18\cdot 10^4\ b^2}{769^2\ c}t\right)\right). \tag{4.1}$$

To convert Eq. (3.18) into Eq. (4.1), we set $k = 0$ and substitute the value of $d$ as $d = \dfrac{769}{4\cdot 5^4}\dfrac{c^2}{b}$.
Solution (3.26) was initially overlooked by Ma [28], but later computed by Duffy and Parkes [10], and subsequently confirmed by Mancas and Hereman [30]. The solution (21) on page 114002-3 in [30] is given by

$$\begin{aligned} u(x,t) = -\frac{385\cdot 3^3\cdot 10^2\ b^2}{17^2\cdot 127^2\ ac}&\operatorname{sech}^4\left(\frac{5}{\sqrt{2159}}\sqrt{-\frac{b}{c}}\left(x+\frac{71\cdot 10^4\ b^2}{17^2\cdot 127^2\ c}t\right)\right)\\ &\times\left[1+\operatorname{sech}^2\left(\frac{5}{\sqrt{2159}}\sqrt{-\frac{b}{c}}\left(x+\frac{71\cdot 10^4\ b^2}{17^2\cdot 127^2\ c}t\right)\right)\right]. \end{aligned} \tag{4.2}$$

To convert Eq. (3.26) into Eq. (4.2), we set $k = 0$ and substitute the value of $d$ as $d = \dfrac{2159}{10^4}\dfrac{c^2}{b}$.

Regarding the comparison of the two methods, we would like to point out that the reliability, applicability, and validity of both methods have been tested. Additionally, due to the utilization of different auxiliary equations, the solutions generated by both methods exhibit distinct characteristics. It is important to note that the advanced $\exp(-\varphi(\xi))$-expansion method successfully identified novel solutions to the gsKdV equation using the two auxiliary equations (2.5) and (2.6), in contrast to the auxiliary equation (2.4). However, the modified $(G'/G)$-expansion method was unsuccessful in solving the gsKdV equation. On the other hand, when it comes to the sKawahara equation, both methods performed well and provided us with some solutions.

Three fascinating types of traveling wave solutions for solitary wave theory are presented in this study. Some solutions are described and presented graphically. We created 2D, 3D, and contour plot views in specific finite fields using Maple's computational tools. The bright soliton-type solution is presented in Figures 2, 5, and 8, corresponding to $u_7$, $u_{21}$, and $u_{41}$, respectively, for the fixed values of the parameters indicated in the captions. Figures 4, 7, and 9 show the dark solitary wave solution corresponding to $u_{18}$, $u_{34}$, and $u_{45}$, respectively. Figures 1, 3, and 6 show the multiple dark solitary wave solution corresponding to $u_3$, $u_{12}$, and $u_{28}$, respectively, with fixed parameters indicated in the captions of the figures.

## 5. Conclusion

In summary, we have successfully obtained more explicit and exact traveling wave solutions to the gsKdV equation (1.1) and the sKawahara equation (3.29) as a special case. The advanced $\exp(-\varphi(\xi))$-expansion method and the modified $(G'/G)$-expansion method have been effectively used to achieve the goals of this work. These methods not only reproduce known solutions but also uncover new solutions that were previously missed by other researchers. The results indicate that the mentioned equations admit exact traveling wave solutions with different values of arbitrary coefficients. These solutions represent three types of traveling wave solutions: bright solitons, dark solitary wave solutions, and multiple dark solitary wave solutions. The results demonstrate that the advanced $\exp(-\varphi(\xi))$-expansion method is an efficient and reliable approach for discovering new exact solutions for NLEEs in various scientific and physical fields. The obtained solutions can be utilized by physicists and engineers for more in-depth analysis in different cases.

## Acknowledgements

The authors appreciate the anonymous referees' attentive reading of the paper and helpful criticism, which greatly contributed to its improvement.

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