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Existence and uniqueness results for generalized fractional integrodifferential equations with nonlocal terminal condition

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Abstract

In this study, we give results on the existence and uniqueness of solutions for generalized fractional integrodifferential equations with a nonlocal terminal condition. We have proved the existence of solutions to the problem proposed using the Schauder fixed point theorem and the uniqueness of its solutions is proved using the Banach fixed point theorem. At the end, we discussed the examples to support our results.

Keywords. Generalized fractional derivative, Fractional integrodifferential equation, Terminal condition, Schauder fixed point theorem.

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1. Introduction

Fractional calculus is the branch of mathematics that has grown a lot in the last five decades through the study of various fractional differential equations and fractional integrodifferential equations via initial conditions, boundary conditions, nonlocal conditions, etc. It is very natural to describe many phenomena occurring in science, engineering such as biology, geology, elasticity, etc. made it quite popular among researchers and motivated to develop numerous models using different fractional derivatives.

In 1991, Byszewski initiated the study of fractional differential equations with nonlocal conditions, see ([2–4]). After that, many researchers have turned to the fractional differential equations with nonlocal conditions. To name a few, authors studied fractional integrodifferential equations with Caputo fractional derivative with nonlocal conditions in Banach space and studied existence results for the problem in [9], also see [5, 8, 10–15, 23, 24]. Fractional terminal value problems have been found interesting by many researchers. One may refer to [17, 18, 21, 22].

Motivated by all the aforementioned work, in this article, we propose generalized fractional integrodifferential equation with terminal condition of the type

$$\begin{cases} ({}^{\delta}D^{\mu,\eta}_{a+}u)(t) &= g(t,u(t),H(u(t))), \quad 0<\mu<1,\ 0\le\eta\le 1,\ t\in(a,T], \\ ({}^{\delta}I^{1-\zeta}u)(T) &= \sum_{i=1}^{n}\lambda_i u(\xi_i), \quad \mu\le\zeta=\mu+\eta(1-\mu),\ \xi_i\in(a,T], \end{cases} \tag{1.1}$$

where

$$H(u(t)) = \int_0^t k(t,p)u(p)dp, \tag{1.2}$$

and $\mu\in(0,1),\ \eta\in[0,1],\ \delta>0,\ \mu\le\zeta=\mu+\eta(1-\mu),\ \xi_i\in(a,T]$ and ${}^{\delta}D^{\mu,\eta}_{a+}$ and ${}^{\delta}I^{1-\zeta}$ denote generalized Katugampola fractional derivative of order μ and Katugampola fractional integral of order $1-\zeta$, function $g:(a,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a given function, ξ_i are pre-fixed points satisfying $0<a<\xi_1\le\xi_2\le\cdots\le\xi_n<T$ and $\lambda_i=1,2,\cdots,n$ are real numbers.

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2. Preliminaries

In this section, we see some important definitions and results that we use in the paper. Beta and Gamma functions are defined by

$$\Gamma(\mu)=\int_0^\infty t^{\mu-1}e^{-t}dt,\ B(\mu,\eta)=\int_0^1(1-t)^{\mu-1}t^{\eta-1}dt,\ \mu,\ \eta>0.$$

Definition 2.1. [16] The space $X_c^q(a,T), c\in\mathbb{R}, q\geq 1$ consists of all real valued Lebesgue measurable functions g on (a,T) for which $||g||_{X_c^q}<\infty$, where

$$||g||_{X_c^q}=\Big(\int_a^b|t^cg(t)|^q\frac{dt}{t}\Big)^{1/q},\ q\geq 1,\ \text{and}\ ||g||_{X_c^\infty}=\sup_{a\leq t\leq T}|t^cg(t)|.$$

In particular, when $c=\frac{1}{q}$ we get $X_{1/q}^c(a,T)=L_q(a,T)$.

Definition 2.2. [19] We denote by $C[a,T]$, a space of continuous functions g on $(a,T]$ with the norm

$$||g||_C=\max_{t\in[a,T]}|g(t)|.$$

The weighted space

$$C_{\zeta,\delta}[a,T]=\Big\{g:(a,T]\to\mathbb{R}:\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^\zeta g(t)\in C[a,T]\Big\},\tag{2.1}$$

with the norm

$$||g||_{C_{\zeta,\delta}}=\parallel\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^\zeta g(t)\parallel=\max_{t\in[a,T]}\Big|\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^\zeta g(t)\Big|\ \text{and}\ C_{0,\delta}[a,T]=C[a,T].$$

Definition 2.3. [19] Let $\Delta_\delta=\Big(t^{\delta-1}d/dt\Big),\ 0\leq\zeta<1$. Also denote $C^n[a,T]$ the Banach space of functions g which are continuously differentiable, with Δ_δ on $[a,T]$ upto order $(n-1)$ and have derivative $\Delta_\delta^k g$ on $(a,T]$ such that $\Delta_\delta^n g\in C_{\zeta,\delta}[a,T]$.

$$C^n_{\Delta_\delta,\zeta}[a,T]=\Big\{\Delta_\delta^k g\in C[a,T],k=0,1,\cdots,n-1,\Delta_\delta^n g\in C_{\zeta,\delta}[a,T]\Big\},\ n\in\mathbb{N},$$

with the norm given by,

$$||g||_{C^n_{\Delta_\delta,\zeta}}=\sum_{k=0}^{n-1}||\Delta_\delta^k g||_C+||\Delta_\delta^n g||_{C_{\zeta,\delta}},\ ||g||_{C^n_{\Delta_\delta,\zeta}}=\sum_{k=0}^{n}\max_{t\in(a,T]}|\Delta_\delta^k g(t)|.$$

In particular, for $n=0$, we get $C^0_{\Delta_\delta,\zeta}[a,T]=C_{\zeta,\delta}[a,T]$.

Definition 2.4. [6] Let $\mu>0$ and $g\in X_c^q(a,T)$, where X_c^q is as defined in Definition 2.1. Then the left-sided Katugampola fractional integral ${}^\delta I^\mu_{a+}$ of order μ is defined as

$${}^\delta I^\mu_{a+}g(t)=\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta-p^\delta}{\delta}\Big)^{\mu-1}g(p)dp,\ t>a.\tag{2.2}$$

Definition 2.5. [7] Let $\mu\in\mathbb{R}^+-\mathbb{N}$ and $n=[\mu]+1$, where $[\mu]$ is the integer part of μ. The left sided Katugampola fractional derivative ${}^\delta D^\mu_{a+}$ is defined as

$${}^\delta D^\mu_{a+}g(t)=\Delta_\delta^n({}^\delta I^{n-\mu}_{a+}g(p))(t)\tag{2.3}$$

$$=\Big(t^{\mu-1}\frac{d}{dt}\Big)^n\frac{1}{\Gamma(n-\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta-p^\delta}{\delta}\Big)^{n-\mu-1}g(p)dp.\tag{2.4}$$

Definition 2.6. [19] The left-sided generalized Katugampola fractional derivative ${}^{\delta}D^{\mu,\eta}_{a+}$ of order $0 < \mu < 1$ and type $0 \leq \eta \leq 1$ is defined as,

$$({}^{\delta}D^{\mu,\eta}_{a+}g)(t) = ({}^{\delta}I^{\eta(1-\mu)}_{a+}\Delta^{\delta}_{\delta}I^{(1-\eta)(1-\mu)}_{a+}g)(t), \tag{2.5}$$

for the functions for which the right-hand side expression exists.

Lemma 2.7. *[20] Suppose that $\mu > 0$, $\eta > 0$, $q \geq 1$ and δ, $c \in \mathbb{R}$ such that $\delta \geq c$. Then for $g \in X^q_c(a,T)$, the semigroup property of Katugampola integral is valid. i.e.*

$$({}^{\delta}I^{\mu}_{a+})({}^{\delta}I^{\eta}_{a+})g(t) =^{\delta} I^{\mu+\eta}_{a+}g(t). \tag{2.6}$$

Lemma 2.8. *[7] Suppose that $\mu > 0$, $0 \leq \zeta < 1$ and $g \in C_{\zeta,\delta}[a,T]$. Then for $t \in (a,T]$,*

$$({}^{\delta}D^{\mu}_{a+})({}^{\delta}I^{\mu}_{a+})g(t) = g(t). \tag{2.7}$$

Lemma 2.9. *[7] Suppose that $\mu > 0$, $0 \leq \zeta < 1$ and $g \in C_{\zeta,\delta}[a,T]$. and ${}^{\delta}I^{1-\mu}_{a+}g \in C^1_{\zeta,\delta}[a,T]$. Then,*

$$({}^{\delta}I^{\mu}_{a+})({}^{\delta}D^{\mu}_{a+})g(t) = g(t) - \frac{{}^{\delta}I^{1-\mu}_{a+}g(a)}{\Gamma(\mu)}\Big(\frac{t^{\delta}-a^{\delta}}{\delta}\Big)^{\mu-1}. \tag{2.8}$$

Lemma 2.10. *[1] If ${}^{\delta}I^{\mu}_{a+}$ and ${}^{\delta}D^{\mu}_{a+}$ are defined as in Definition 2.4 and 2.5 respectively, then*

$$\begin{aligned}
&{}^{\delta}I^{\mu}_{a+}\Big(\frac{t^{\delta}-a^{\delta}}{\delta}\Big)^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\sigma+1)}\Big(\frac{t^{\delta}-a^{\delta}}{\delta}\Big)^{\sigma+\mu-1}, \ \mu \geq 0, \ \sigma > 0, \ t > a,\\
&{}^{\delta}D^{\mu}_{a+}\Big(\frac{t^{\delta}-a^{\delta}}{\delta}\Big)^{\mu-1} = 0, \ 0 < \mu < 1.
\end{aligned}$$

Remark 2.11. [1] For $0 < \mu < 1$, $0 \leq \eta \leq 1$, the generalized Katugampola fractional derivative ${}^{\delta}D^{\mu,\eta}_{a+}$ can be written in terms of Katugampola fractional derivative as,

$${}^{\delta}D^{\mu,\eta}_{a+} = ({}^{\delta}I^{\eta(1-\mu)}_{a+})\Delta_{\delta}({}^{\delta}I^{1-\mu}_{a+}) = ({}^{\delta}I^{\eta(1-\mu)}_{a+})({}^{\delta}D^{\zeta}_{a+}), \ \zeta = \mu + \eta(1-\mu).$$

Lemma 2.12. *[19] Let $\mu > 0$, $0 < \zeta \leq 1$ and $g \in C_{1-\zeta,\delta}[a,b]$. If $\mu > \zeta$, then*

$${}^{\delta}I^{\mu}_{a+}g(a) = \lim_{t\to a+}({}^{\delta}I^{\mu}_{a+}g)(t) = 0.$$

To discuss the main results, we need following spaces.

$$C^{\mu,\eta}_{1-\zeta,\delta}[a,T] = \Big\{g \in C_{1-\zeta,\delta}[a,T] :^{\delta} D^{\mu,\eta}_{a+}g \in C_{1-\zeta,\delta}[a,T]\Big\}, \ 0 < \zeta \leq 1, \tag{2.9}$$

and

$$C^{\zeta}_{1-\zeta,\delta}[a,T] = \Big\{g \in C_{1-\zeta,\delta}[a,T] :^{\delta} D^{\zeta}_{a+}g \in C_{1-\zeta,\delta}[a,T]\Big\}, \ 0 < \zeta \leq 1,$$

as ${}^{\delta}D^{\mu,\eta}_{a+}g = ({}^{\delta}I^{\eta(1-\mu)}_{a+})({}^{\delta}D^{\zeta}_{a+})g$, it is clear that $C^{\zeta}_{1-\zeta,\delta}[a,T] \subset C^{\mu,\eta}_{1-\zeta,\delta}[a,T]$.

Lemma 2.13. *[7] Let $\mu > 0$, $\eta > 0$ and $\zeta = \mu + \eta - \mu\eta$. If $g \in C^{\zeta}_{1-\zeta,\delta}[a,T]$, then*

$$({}^{\delta}I^{\zeta}_{a+})({}^{\delta}D^{\zeta}_{a+})g(t) = ({}^{\delta}I^{\mu}_{a+})({}^{\delta}D^{\mu,\eta}_{a+})g(t) =^{\delta} D^{\eta(1-\mu)}_{a+}g(t).$$

Lemma 2.14. *[1] Let $0 < \mu < 1$, $0 \leq \eta \leq 1$, $\zeta = \mu + \eta - \mu\eta$. If $g : (a,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that $g(\cdot,\ u(\cdot),\ H(u(\cdot))) \in C_{1-\zeta,\delta}[a,T]$ for any $u \in C_{1-\zeta,\delta}[a,T]$ for any $u \in C^{\zeta}_{1-\zeta,\delta}[a,T]$ satisfies Terminal Value Problem (TVP)*

$$\begin{cases} ({}^{\delta}D^{\mu,\eta}_{a+}u)(t) &= g(t,u(t)), \qquad 0 < \mu < 1, \ 0 \leq \eta \leq 1, \ t \in (a,T], \\ ({}^{\delta}I^{1-\zeta}u)(T) &= \sum_{i=1}^{n}\lambda_i u(\xi_i), \quad \mu \leq \zeta = \mu + \eta(1-\mu), \ \xi_i \in (a,T], \end{cases} \tag{2.10}$$

iff u *satisfies the mixed-type nonlinear Volterra integral equation*

$$u(t) = \frac{K}{\Gamma(\mu)}\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{\zeta-1}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1} g(p,u(p))dp + \frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1} g(p,u(p))dp, \tag{2.11}$$

where

$$K = \left(\Gamma(\zeta) - \sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta - a^\delta}{\delta}\Big)^{\zeta-1}\right)^{-1}. \tag{2.12}$$

Theorem 2.15. *Let* $0<\mu<1,\ 0\le\eta\le1,\ \zeta=\mu+\eta-\mu\eta$. *If* $g:(a,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *is a function such that* $g(\cdot,\ u(\cdot),\ H(u(\cdot)))\in C_{1-\zeta,\delta}[a,T]$ *for any* $u\in C_{1-\zeta,\delta}[a,T]$ *for any* $u\in C^{\zeta}_{1-\zeta,\delta}[a,T]$ *satisfies TVP (1.1) iff* u *satisfies the mixed-type nonlinear Volterra integral equation*

$$u(t) = \frac{K}{\Gamma(\mu)}\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{\zeta-1}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1} g(p,u(p),H(u(p)))dp + \frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1} g(p,u(p),H(u(p)))dp, \tag{2.13}$$

where

$$K = \left(\Gamma(\zeta) - \sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta - a^\delta}{\delta}\Big)^{\zeta-1}\right)^{-1}. \tag{2.14}$$

Proof. The proof follows from the Lemma 2.14. □

3. Main Results

In this section, we prove the existence and uniqueness results for our problem. For this, we state some hypotheses:

(H_1) $g:(a,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a function such that $g(\cdot,u(\cdot),v(\cdot))\in C^{\eta(1-\mu)}_{1-\zeta,\delta}[a,T]$ for any $u\in C_{1-\zeta,\delta}$ and there exists a positive constant $M,\ N>0$ such that for all $u_1,\ v_1,\ u_2,\ v_2\in(a,T]$,

$$|g(t,u_1,u_2)-g(t,v_1,v_2)|\le M|u_1-v_1|+N|u_2-v_2|. \tag{3.1}$$

(H_2) The constants

$$\Omega_1 = \frac{\Gamma(\zeta)}{\Gamma(\zeta+\mu)}\left(|K|\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta - a^\delta}{\delta}\Big)^{\mu+\zeta-1} + \Big(\frac{T^\delta - a^\delta}{\delta}\Big)^{\mu}\right), \tag{3.2}$$

and

$$\Omega_2 = \frac{T^\zeta N k_T}{\Gamma(\mu+1)}\left[|K|\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta - a^\delta}{\delta}\Big)^{\mu} + \Big(\frac{T^\delta - a^\delta}{\delta}\Big)^{\mu-\zeta+1}\right], \tag{3.3}$$

are such that

$$M\Omega_1+\Omega_2<1. \tag{3.4}$$

(H_3) The function $g:(a,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is such that $g(\cdot,u(\cdot),H(u(\cdot)))\in C^{\eta(1-\mu)}_{1-\zeta,\delta}$, for any $u\in C_{1-\zeta,\delta}$ and $\forall\ t\in(a,T]$, there exist constants $M,\ N>0$ and $L\ge0$ such that

$$|g(t,u,\bar{u})|\le M|u|+N|\bar{u}|+L. \tag{3.5}$$

Theorem 3.1. *If the hypotheses* $(H_1)-(H_3)$ *holds then the problem (1.1) has at least one solution in* $C^{\zeta}_{1-\zeta,\delta}[a,T]\subset C^{\mu,\eta}_{1-\zeta,\delta}[a,T]$.

Proof. Define $B_\epsilon = \{u \in C_{1-\zeta,\delta}[a,T] : ||u||_{C_{1-\zeta,\delta}} \le \epsilon\}$ with $\epsilon \ge \frac{\bar{L}\ \Omega_2}{(1-(M\Omega_1+\Omega_2))}$, $\bar{L} = \frac{L}{NT^{\zeta}k_T}$. Define an operator $\mathcal{F} : C_{1-\zeta,\delta}[a,T] \to C_{1-\zeta,\delta}[a,T]$ by

$$
\begin{aligned}
(\mathcal{F}u)(t) &= \frac{K}{\Gamma(\mu)}\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{\zeta-1}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1} g\Big(p, u(p), H(u(p))\Big)dp \\
&+ \frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1} g\Big(p, u(p), H(u(p))\Big)dp. \qquad (3.6)
\end{aligned}
$$

Claim 1: $\mathcal{F}(B_\epsilon) \subset B_\epsilon$. Consider

$$
\begin{aligned}
&\Big|\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}(\mathcal{F}u)(t)\Big| \le \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1}\Big|g\Big(p, u(p), H(u(p))\Big)\Big|dp \\
&+ \Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1}\Big|g\Big(p, u(p), H(u(p))\Big)\Big|dp \\
&\le \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1}\Big[M|u(p)| + N|H(u(p))| + L\Big]dp \\
&+ \Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1}\Big[M|u(p)| + N|H(u(p))| + L\Big]dp \\
&\le \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{\zeta-1}\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\Big[M|u(p)| + N|H(u(p))| + L\Big]dp \\
&+ \Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{\zeta-1}\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\Big[M|u(p)| + N|H(u(p))| + L\Big]dp \\
&\le \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{\zeta-1}\Big[M||u||_{C_{1-\zeta,\delta}} + L\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\Big]dp \\
&+ N\frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1}k_T\int_0^p\Big(\frac{\tau^\delta - a^\delta}{\delta}\Big)^{\zeta-1}||u||_{C_{1-\zeta,\delta}}d\tau\ dp \\
&+ \Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{\zeta-1}\Big[M||u||_{C_{1-\zeta,\delta}} + L\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\Big]dp \\
&+ N\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1}k_T\int_0^p\Big(\frac{\tau^\delta - a^\delta}{\delta}\Big)^{\zeta-1}||u||_{C_{1-\zeta,\delta}}d\tau\ dp
\end{aligned}
$$

$$\leq ||u||_{C_{1-\zeta,\delta}} \frac{M|K|}{\Gamma(\mu)} \sum_{i=1}^{n} \lambda_i \int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{\zeta-1} dp$$
$$+ L\frac{|K|}{\Gamma(\mu)} \sum_{i=1}^{n} \lambda_i \int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1} dp + T^\zeta N k_T ||u||_{C_{1-\zeta,\delta}} \frac{|K|}{\Gamma(\mu)} \sum_{i=1}^{n} \lambda_i \int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1} dp$$
$$+ M||u||_{C_{1-\zeta,\delta}}\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta} \frac{1}{\Gamma(\mu)} \int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta - a^\delta}{\delta}\Big)^{\zeta-1} dp$$
$$+ L\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta} \frac{1}{\Gamma(\mu)} \int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1} dp$$
$$+ T^\zeta N k_T ||u||_{C_{1-\zeta,\delta}}\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta} \frac{1}{\Gamma(\mu)} \int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1} dp$$

$$\leq ||u||_{C_{1-\zeta,\delta}} \frac{M|K|\Gamma(\zeta)}{\Gamma(\zeta+\mu)} \sum_{i=1}^{n} \lambda_i \Big(\frac{\xi_i^\delta - a^\delta}{\delta}\Big)^{\mu+\zeta-1} + \frac{L|K|}{\Gamma(\mu+1)} \sum_{i=1}^{n} \lambda_i \Big(\frac{\xi_i^\delta - a^\delta}{\delta}\Big)^{\mu}$$
$$+ \frac{T^\zeta N K_T |K|}{\Gamma(\mu+1)} \sum_{i=1}^{n} \lambda_i \Big(\frac{\xi_i^\delta - a^\delta}{\delta}\Big)^{\mu} + ||u||_{C_{1-\zeta,\delta}} \frac{M\Gamma(\zeta)}{\Gamma(\zeta+\mu)}\Big(\frac{T^\delta - a^\delta}{\delta}\Big)^{\mu}$$
$$+ \frac{L}{\Gamma(\mu+1)}\Big(\frac{T^\delta - a^\delta}{\delta}\Big)^{\mu-\zeta+1} + ||u||_{C_{1-\zeta,\delta}} \frac{T^\zeta k_T N}{\Gamma(\mu+1)}\Big(\frac{T^\delta - a^\delta}{\delta}\Big)^{\mu-\zeta+1}$$
$$\leq M\frac{\Gamma(\zeta)}{\Gamma(\zeta+\mu)}\Big[|K| \sum_{i=1}^{n} \lambda_i \Big(\frac{\xi_i^\delta - a^\delta}{\delta}\Big)^{\mu+\zeta-1} + \Big(\frac{T^\delta - a^\delta}{\delta}\Big)^{\mu}\Big] ||u||_{C_{1-\zeta,\delta}}$$
$$+ N\frac{T^\zeta k_T}{\Gamma(\mu+1)}\Big[|K| \sum_{i=1}^{n} \lambda_i \Big(\frac{\xi_i^\delta - a^\delta}{\delta}\Big)^{\mu} + \Big(\frac{T^\delta - a^\delta}{\delta}\Big)^{\mu-\zeta+1}\Big] ||u||_{C_{1-\zeta,\delta}}$$
$$+ L\frac{1}{\Gamma(\mu+1)}\Big[|K| \sum_{i=1}^{n} \lambda_i \Big(\frac{\xi_i^\delta - a^\delta}{\delta}\Big)^{\mu} + \Big(\frac{T^\delta - a^\delta}{\delta}\Big)^{\mu-\zeta+1}\Big]$$
$$\leq (M\Omega_1 + \Omega_2)\epsilon + \frac{L}{NT^\zeta k_T}\Omega_2$$
$$\leq (M\Omega_1 + \Omega_2)\epsilon + \bar{L}\ \Omega_2 \leq \epsilon.$$

That is,

$$||\mathcal{F}u||_{C_{1-\zeta,\delta}} \leq \epsilon,$$

which gives $\mathcal{F}(B_\epsilon) \subset B_\epsilon$.

Claim 2: $\mathcal{F}$ is completely continuous. Let $\{x_n\}$ be a sequence such that $x_n \to x$ in B_ϵ. Then for every $t \in (a, T]$, consider

$$\Big|\Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta}\Big((\mathcal{F}u_n)(t) - (\mathcal{F}u)(t)\Big)\Big|$$
$$\leq \frac{|K|}{\Gamma(\mu)} \sum_{i=1}^{n} \lambda_i \int_a^{\xi_i} p^{\delta-1}\Big(\frac{\xi_i^\delta - p^\delta}{\delta}\Big)^{\mu-1} |g(t, u_n(t), H(u_n(p))) - g(t, u, H(u(p)))| dp$$
$$+ \Big(\frac{t^\delta - a^\delta}{\delta}\Big)^{1-\zeta} \frac{1}{\Gamma(\mu)} \int_a^t p^{\delta-1}\Big(\frac{t^\delta - p^\delta}{\delta}\Big)^{\mu-1} |g(t, u_n(t), H(u_n(p))) - g(t, u, H(u(p)))| dp$$

$$\leq \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i}p^{\delta-1}\Big(\frac{\xi_i^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\times||g(\cdot,\ u_n(\cdot),\ H(u_n(\cdot)))-g(\cdot,\ u(\cdot),\ H(u(\cdot)))||_{C_{1-\zeta,\delta}}dp$$
$$+\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\times||g(\cdot,\ u_n(\cdot),\ H(u_n(\cdot)))-g(\cdot,\ u(\cdot),\ H(u(\cdot)))||_{C_{1-\zeta,\delta}}dp$$
$$\leq\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\Big[|K|\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu+\zeta-1}+\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu}\Big]\times||g(\cdot,\ u_n(\cdot),\ H(u_n(\cdot)))-g(\cdot,\ u(\cdot),\ H(u(\cdot)))||_{C_{1-\zeta,\delta}}$$
$$\leq\Omega_1||g(\cdot,\ u_n(\cdot),\ H(u_n(\cdot)))-g(\cdot,\ u(\cdot),\ H(u(\cdot)))||_{C_{1-\zeta,\delta}},$$

which shows that $\mathcal{F}$ is completely continuous.

Claim 3: $\mathcal{F}(B_\epsilon)$ is relatively compact. As $\mathcal{F}(B_\epsilon)\subset B_\epsilon$, it implies that $\mathcal{F}$ is uniformly bounded. Also, $\mathcal{F}$ is equicontinuous. For, for any $0<a<t_1<t_2\leq T$, consider

$$|(\mathcal{F}u)(t_1)-(\mathcal{F}u)(t_2)|$$
$$\leq\frac{|K|}{\Gamma(\mu)}\Big[\Big(\frac{t_1^\delta-a^\delta}{\delta}\Big)^{\zeta-1}-\Big(\frac{t_2^\delta-a^\delta}{\delta}\Big)^{\zeta-1}\Big]\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i}p^{\delta-1}\Big(\frac{\xi_i^\delta-p^\delta}{\delta}\Big)^{\mu-1}|g(p,u(p),H(u(p)))|dp$$
$$+\frac{1}{\Gamma(\mu)}\Big[\int_a^{t_1}p^{\delta-1}\Big(\frac{t_1^\delta-p^\delta}{\delta}\Big)^{\mu-1}-\int_a^{t_2}p^{\delta-1}\Big(\frac{t_2^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big]|g(p,u(p),H(u(p)))|dp$$
$$\leq\frac{|K|}{\Gamma(\mu)}\Big[\Big(\frac{t_1^\delta-a^\delta}{\delta}\Big)^{\zeta-1}-\Big(\frac{t_2^\delta-a^\delta}{\delta}\Big)^{\zeta-1}\Big]\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i}p^{\delta-1}\Big(\frac{\xi_i^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{\zeta-1}||g||_{C_{1-\zeta,\delta}}dp$$
$$+\frac{1}{\Gamma(\mu)}\Big[\int_a^{t_1}p^{\delta-1}\Big(\frac{t_1^\delta-p^\delta}{\delta}\Big)^{\mu-1}-\int_a^{t_2}p^{\delta-1}\Big(\frac{t_2^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big]\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{\zeta-1}||g||_{C_{1-\zeta,\delta}}dp$$
$$\leq||g||_{C_{1-\zeta,\delta}}\frac{|K|\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\Big[\Big(\frac{t_1^\delta-a^\delta}{\delta}\Big)^{\zeta-1}-\Big(\frac{t_2^\delta-a^\delta}{\delta}\Big)^{\zeta-1}\Big]\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu+\zeta-1}$$
$$+||g||_{C_{1-\zeta,\delta}}\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\Big[\Big(\frac{t_1^\delta-a^\delta}{\delta}\Big)^{\mu+\zeta-1}-\Big(\frac{t_2^\delta-a^\delta}{\delta}\Big)^{\mu+\zeta-1}\Big]\to 0,$$

as $t_2\to t_1$. Therefore, $\mathcal{F}$ is equicontinuous.

Hence, $\mathcal{F}(B_\epsilon)$ is an euquicontinuous set which implies that $\mathcal{F}(B_\epsilon)$ is relatively compact. Thus, from claims (1)-(3) and Arzela-Ascoli theorem, we can say that $\mathcal{F}:B_\epsilon\to B_\epsilon$ is completely continuous. Hence, by Schauder fixed point theorem, the operator $\mathcal{F}$ has at least one fixed point and hence the problem has at least one solution. □

Now, we prove uniqueness theorem.

Theorem 3.2. *If the hypotheses $(H_1)-(H_2)$ holds, then the problem (1.1) has unique solution.*

Proof. From the operator defined in the previous Theorem (3.1), we have $\mathcal{F}:C_{1-\zeta,\delta}[a,T]\to C_{1-\zeta,\delta}[a,T]$ by

$$(\mathcal{F}u)(t)=\frac{K}{\Gamma(\mu)}\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^{\zeta-1}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i}p^{\delta-1}\Big(\frac{\xi_i^\delta-p^\delta}{\delta}\Big)^{\mu-1}g\Big(p,u(p),H(u(p))\Big)dp$$
$$+\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta-p^\delta}{\delta}\Big)^{\mu-1}g\Big(p,u(p),H(u(p))\Big)dp, \tag{3.7}$$

from the Claim (1) of the previous Theorem 3.1, we can see that for $u \in B_\epsilon$, $||\mathcal{F}u||_{C_{1-\zeta,\delta}} \leq \epsilon$. Next, we prove that $\mathcal{F}$ is a contraction. Consider

$$
\begin{aligned}
&\Big|((\mathcal{F}u)-(\mathcal{F}v))(t)\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\Big|\\
&\leq \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i}p^{\delta-1}\Big(\frac{\xi_i^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big|g\Big(p,u(p),\int_0^t k(p,\tau)u(\tau)d\tau\Big)-g\Big(p,v(p),\int_0^t k(p,\tau)u(\tau)d\tau\Big)\Big|dp\\
&+\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta-p^\delta}{\delta}\Big)^{\mu-1}\\
&\times\Big|g\Big(p,u(p),\int_0^t k(p,\tau)u(\tau)d\tau\Big)-g\Big(p,(v(p),\int_0^t k(p,\tau)u(\tau)d\tau\Big)\Big|dp\\
&\leq \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i}p^{\delta-1}\Big(\frac{\xi_i^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big[M|u(p)-v(p)|+N\Big|\int_0^t k(p,\tau)[u(\tau)-v(\tau)]d\tau\Big|\Big]dp\\
&+\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big[M|u(p)-v(p)|+N\Big|\int_0^t k(p,\tau)[u(\tau)-v(\tau)]d\tau\Big|\Big]dp\\
&\leq \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i}p^{\delta-1}\Big(\frac{\xi_i^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{\zeta-1}\\
&\times\Big[M\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{1-\zeta}|u(p)-v(p)|+Nk_T\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\Big|\int_0^p\Big(\frac{\tau^\delta-a^\delta}{\delta}\Big)^{\zeta-1}||u-v||_{C_{1-\zeta,\delta}}d\tau\Big|\Big]dp\\
&+\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{\zeta-1}\\
&\times\Big[M\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{1-\zeta}|u(p)-v(p)|+Nk_T\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\Big|\int_0^p\Big(\frac{\tau^\delta-a^\delta}{\delta}\Big)^{\zeta-1}||u-v||_{C_{1-\zeta,\delta}}d\tau\Big|\Big]dp\\
&\leq \frac{|K|}{\Gamma(\mu)}\sum_{i=1}^{n}\lambda_i\int_a^{\xi_i}p^{\delta-1}\Big(\frac{\xi_i^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\Big[M+Nk_TT^\zeta\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{\zeta-1}\Big]dp\;||u-v||_{C_{1-\zeta,\delta}}\\
&+\Big(\frac{t^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\frac{1}{\Gamma(\mu)}\int_a^t p^{\delta-1}\Big(\frac{t^\delta-p^\delta}{\delta}\Big)^{\mu-1}\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{\zeta-1}\\
&\times\Big[M+Nk_TT^\zeta\Big(\frac{p^\delta-a^\delta}{\delta}\Big)^{1-\zeta}\Big]dp||u-v||_{C_{1-\zeta,\delta}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq |K|\Big[\frac{M\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu+\zeta-1}+\frac{T^\zeta Nk_T}{\Gamma(\mu+1)}\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu}\Big]||u-v||_{C_{1-\zeta,\delta}}\\
&+\Big[M\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu}+NT^\zeta k_T\frac{1}{\Gamma(\mu+1)}\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu-\zeta+1}\Big]||u-v||_{C_{1-\zeta,\delta}}\\
&\leq\Big\{M\frac{\Gamma(\zeta)}{\Gamma(\mu+\zeta)}\Big[|K|\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu+\zeta-1}+\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu}\Big]\\
&+\frac{T^\zeta k_TN}{\Gamma(\mu+1)}\Big[|K|\sum_{i=1}^{n}\lambda_i\Big(\frac{\xi_i^\delta-a^\delta}{\delta}\Big)^{\mu}+\Big(\frac{T^\delta-a^\delta}{\delta}\Big)^{\mu-\zeta+1}\Big]\Big\}||u-v||_{C_{1-\zeta,\delta}}\\
&\leq(M\Omega_1+\Omega_2)||u-v||_{C_{1-\zeta,\delta}}.
\end{aligned}
$$

$$||\mathcal{F}u-\mathcal{F}v||_{C_{1-\zeta,\delta}}\leq(M\Omega_1+\Omega_2)||u-v||_{C_{1-\zeta,\delta}},\forall u,v\in B_\epsilon. \tag{3.8}$$

Thus, $\mathcal{F}$ is a contraction map and hence by the Banach fixed point theorem, the operator has unique solution in $C_{1-\zeta,\delta}[a,T]$. □

4. Examples

In this section, we apply the result to examples.

Example 4.1. Consider the nonlocal problem

$$\begin{cases} ({}^{\delta}D^{\mu,\eta}_{a+}u)(t) &= g\left(t, u(t), \int_2^t k(t,p)u(p)dp\right), \quad t\in(2,3], \\ ({}^{\delta}I^{1-\zeta}_{a+}u)(3-) &= 4u(\frac{7}{3}), \qquad \zeta=\mu+\eta(1-\mu). \end{cases} \tag{4.1}$$

Solution:
Set $\mu=\frac{1}{2},\ \eta=\frac{1}{4}$ then $\zeta=\frac{11}{16}$. Also, let $\delta=\frac{1}{2}$ and

$$g\Big(t, u(t), \int_1^t k(t,p)u(p)dp\Big) = \frac{2}{11}\sin u + \frac{1}{5}\int_2^t u(p)dp,\ k(t,p)=1, \text{so that } k_T=1,\ T=3,$$

Also, $|g(t,u_1,u_2)-g(t,v_1,v_2)| \le \frac{2}{11}|u_1-v_1| + \frac{1}{5}|u_2-v_2|,\ \forall\ u_1,\ u_2,\ v_1,\ v_2$ so $M=\frac{2}{11},\ N=\frac{1}{5}$.

$$|K| = \left|\left(\Gamma(\frac{11}{16}) - 4\Big(\frac{(7/3)^{\frac{1}{2}}-(2)^{\frac{1}{2}}}{\frac{1}{2}}\Big)^{-5/16}\right)^{-1}\right| \approx 0.198305 < 1.$$

Now, we find Ω_1 and Ω_2.

$$\Omega_1 = \frac{\Gamma(11/16)}{\Gamma(19/16)}\left[0.1676673047(4)\Big(\frac{(7/3)^{\frac{1}{2}}-(2)^{\frac{1}{2}}}{\frac{1}{2}}\Big)^{\frac{3}{16}} + \Big(\frac{(3)^{\frac{1}{2}}-(2)^{\frac{1}{2}}}{\frac{1}{2}}\Big)^{\frac{1}{2}}\right] \approx 1.68066,$$

$$\Omega_2 = \frac{3^{11/16}\frac{1}{5}\cdot 1}{\Gamma(\frac{3}{2})}\left[0.1676673047(4)\Big(\frac{(7/3)^{\frac{1}{2}}-(2)^{\frac{1}{2}}}{\frac{1}{2}}\Big)^{\frac{1}{2}} + \Big(\frac{(3)^{\frac{1}{2}}-(2)^{\frac{1}{2}}}{\frac{1}{2}}\Big)^{\frac{13}{16}}\right] \approx 0.513743.$$

Then $M\Omega_1+\Omega_2 = \frac{2}{11}\times\frac{2}{11}\times 1.68066 + 0.513743 = 0.819317 < 1$ Hence, by the Theorem 3.2, the problem (4.1) has unique solution in $C_{5/16,1/2}[2,3]$.

Example 4.2. Consider the nonlocal problem

$$\begin{cases} ({}^{\delta}D^{\mu,\eta}_{a+}u)(t) &= g\left(t, u(t), \int_2^t k(t,p)u(p)dp\right), \quad t\in(2,3], \\ ({}^{\delta}I^{1-\zeta}_{a+}u)(3-) &= \frac{1}{2}u(2.1)+u(2.2), \qquad \zeta=\mu+\eta(1-\mu). \end{cases} \tag{4.2}$$

Solution:
Set $\mu=\frac{1}{3},\ \eta=\frac{1}{2}$ then $\zeta=\frac{2}{3}$. Also, let $\delta=\frac{1}{2}$ and

$$g\Big(t, u(t), \int_1^t k(t,p)u(p)dp\Big) = \frac{1}{5}\cos u + \frac{1}{4}\int_2^t e^{-(t+p)}u(p)dp,\ k(t,p)=e^{-(t+p)},$$

so that $k_T=e^{-4},\ T=3\ ,\xi_1=2.1,\ \xi_2=2.2,\ \lambda_1=\frac{1}{2},\ \lambda_2=1.$

Also, $|g(t,u_1,u_2)-g(t,v_1,v_2)| \le \frac{1}{3}|u_1-v_1| + \frac{1}{4}|u_2-v_2|,\ \forall\ u_1,\ u_2,\ v_1,\ v_2$ so $M=\frac{1}{3},\ N=\frac{1}{4}$.

$$|K| = \left|\left(\Gamma(\frac{2}{3}) - \left[\frac{1}{2}\Big(\frac{(2.1)^{\frac{1}{2}}-(2)^{\frac{1}{2}}}{\frac{1}{2}}\Big)^{-1/3} + \Big(\frac{(2.2)^{\frac{1}{2}}-(2)^{\frac{1}{2}}}{\frac{1}{2}}\Big)^{-1/3}\right]\right)^{-1}\right| \approx 0.5571505 < 1.$$

Now, we find Ω_1 and Ω_2.

$$\Omega_1 = \frac{\Gamma(2/3)}{\Gamma(1)}\left[0.5571505\left(\frac{1}{2}\times 1+1\right)+\left(\frac{(3)^{\frac{1}{2}}-(2)^{\frac{1}{2}}}{\frac{1}{2}}\right)^{\frac{1}{3}}\right] \approx 2.29597941,$$

$$\Omega_2 = \frac{3^{2/3}\times 0.25\times e^{-4}}{\Gamma(\frac{4}{3})}\left[0.5571505\left(\frac{1}{2}\left(\frac{(2.1)^{\frac{1}{2}}-(2)^{\frac{1}{2}}}{\frac{1}{2}}\right)^{\frac{1}{3}}+\left(\frac{(2.2)^{\frac{1}{2}}-(2)^{\frac{1}{2}}}{\frac{1}{2}}\right)^{\frac{1}{3}}\right)+\left(\frac{(3)^{\frac{1}{2}}-(2)^{\frac{1}{2}}}{\frac{1}{2}}\right)^{\frac{2}{3}}\right]$$
$$\approx 0.0121803.$$

Then $M\Omega_1+\Omega_2 = \frac{1}{3}\times 2.29597941 + 0.0121803 = 0.77750677 < 1$ Hence, by the Theorem 3.2, the problem (4.2) has unique solution in $C_{1/3,1/2}[2,3]$.

5. Conclusion

Here, we have studied the existence and uniqueness results for generalized fractional integrodifferential equations via terminal value condition and minimal hypothesis. Further, the results obtained are justified by examples. In the future, we are planning to extend the results for different fractional derivatives and numerous boundary conditions.

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