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# Existence, uniqueness, and stability analysis of coupled random fractional boundary value problems with nonlocal conditions

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**Abstract**

In this paper, the existence, uniqueness, compactness, and stability of a coupled random differential equations involving the Hilfer fractional derivatives with nonlocal boundary conditions are discussed. Arguments are discussed via some random fixed point theorems in a separable vector Banach spaces and Ulam type stability. Some examples are presented to ensure the abstract results.

**Keywords.** Coupled fractional differential systems, Random variable, Hilfer fractional derivatives, Nonlocal conditions, Fixed point theorem, Ulam-Hyers stability.

**2010 Mathematics Subject Classification.** 26A33, 34A37, 34G20.

## 1. Introduction

Fractional calculus gives an amazing instrument for the portrayal of memory and innate properties of different materials and forms due to the presence of terms which insurrection the history and its affect on the present and future in a model [16, 24]. Fractional differential equations can depict numerous wonders in different areas of science and engineering. In result, the subject of fractional differential equations is picking up much significance and consideration. There are an expansive number of papers related to the existence, uniqueness, and multiple solutions of fractional order boundary value problems, (see [8, 9, 18, 26, 29, 30, 32–35, 38, 42–44, 51] and references therein). On the other hand, Perov's et al. [46] extended the classical Banach contraction principle for contrastive maps on space endowed with a vector-valued metric. Later, they attempted to generalize the Perov's fixed point theorem in several directions which has a number of applications in various fields of nonlinear analysis, semilinear differential equations, and system of ordinary differential equations.

Fractional derivatives in the sense of Riemann-Liouville and Caputo are among the most often used definitions of fractional integrals and derivatives in the literature. The Hadamard fractional derivative, the Erdeyl-Kober fractional derivative, and so forth are other, less well-known definitions. When R. Hilfer [24] examined the fractional time history of physical processes, he provided a generalization of the derivatives of both Riemann-Liouville and Caputo in [24]. It was described by him as a generalized fractional derivative of order $\alpha \in (0, 1)$ and a type $\beta \in [0, 1]$ which can be reduced to the Riemann-Liouville and Caputo fractional derivatives when $\beta = 0$ and $\beta = 1$, respectively. The Hilfer fractional derivative is the term used by several authors. A derivative of this kind in certain ways interpolates between the Riemann-Liouville and Caputo derivatives. According to [24, 25], and the sources listed therein, the Hilfer derivative has several specific characteristics and uses. By adding a degree of freedom to the initial condition, this kind of two-parameter family produces more stationary states. In the models [2, 3, 14, 15, 17, 21, 24, 28, 49, 52] and references therein, this derivative is employed. On the other hand, the precision of our knowledge of the parameters

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that characterize a dynamic system in engineering or the natural sciences determines how that system behaves. A deterministic dynamical system develops when knowledge of a dynamic system is exact. Unfortunately, the information that is currently accessible for the description and assessment of a dynamic system's parameters is frequently incorrect, vague, or misleading. So there are certain uncertainties involved in evaluating parameters in a dynamical system. The prevalent method in mathematical modelling of such systems is the use of random differential equations or stochastic differential equations when our knowledge of the parameters of a dynamic system is of a statistical nature, i.e., the information is probabilistic. As logical expansions of deterministic ones, random differential equations appear in many applications and have attracted the attention of several mathematicians. The reader is referred to references [13, 37].

In this paper, we establish the existence, uniqueness, compactness, and stability of the following coupled system of fractional order boundary value problems with random effects,

$$\begin{cases} \mathtt{D}_{0^+}^{\ell_1,\mathfrak{n}_1}\mathtt{z}_1(\mathtt{u},\varpi) = f(\mathtt{u},\mathtt{z}_1(\mathtt{u},\varpi),\mathtt{z}_2(\mathtt{u},\varpi),\varpi),\ 0<\ell_1<1,\ 0\le\mathfrak{n}_1\le 1,\\ \mathtt{D}_{0^+}^{\ell_2,\mathfrak{n}_2}\mathtt{z}_2(\mathtt{u},\varpi) = g(\mathtt{u},\mathtt{z}_1(\mathtt{u},\varpi),\mathtt{z}_2(\mathtt{u},\varpi),\varpi),\ 0<\ell_2<1,\ 0\le\mathfrak{n}_2\le 1, \end{cases} \tag{1.1}$$

with nonlocal boundary conditions,

$$\begin{cases} \left(I_{0^+}^{1-\mathfrak{r}_1}\mathtt{z}_1\right)(\varpi,0) = \sum_{i=1}^{p} c_i(\mathfrak{z}_i,\varpi),\\ \left(I_{0^+}^{1-\mathfrak{r}_2}\mathtt{z}_2\right)(\varpi,0) = \sum_{i=1}^{p} c_i'(\mathfrak{z}_i,\varpi), \end{cases} \tag{1.2}$$

where $\mathtt{u},\mathfrak{z}_i \in J' = (0,\mathfrak{d}]$, $\mathtt{D}_{0^+}^{\ell_i,\mathfrak{n}_i}$ are the left-sided Hilfer fractional derivatives of order $\ell_i$ and type $\mathfrak{n}_i$, $\ell_i \le \mathfrak{r}_i = \ell_i + \mathfrak{n}_i - \ell_i\mathfrak{n}_i < 1$, $i=1,2$, which is an interpolator between Riemann-Liouville and Caputo fractional derivatives. $I_{0^+}^{1-\mathfrak{r}_i}$ is the left-side Riemann-Liouville integral of order $1-\mathfrak{r}_i$, $i=1,2$. Also $f,g : J\times X\times X\times\Omega \to X$ are given functions and $X$ is a separable Banach space. $c_i, c_i'$ $(i=1,2,\cdots,m)$ are real numbers, $\mathfrak{z}_i$ are prefixed points satisfying $0<\mathfrak{z}_1\le\mathfrak{z}_2\le\cdots\le\mathfrak{z}_{p+1}=b$ with $\Gamma(\mathfrak{r}_1)\neq\sum_{i=1}^{p} c_i\mathfrak{z}_i^{\mathfrak{r}_1-1}$, $\quad\mathfrak{r}_1>0$ and $\Gamma(\mathfrak{r}_2)\neq\sum_{i=1}^{p} c_i'\mathfrak{z}_i^{\mathfrak{r}_2-1}$, $\quad\mathfrak{r}_2>0$. The behavior of systems with randomly varying transfer characteristics or with random parametric excitations is described by (1.1) with stochastic processes as coefficients and conditions (1.2) says that some initial measurements were made at the times 0 and $\mathfrak{z}_i$, and the observer uses this previous information in their model. This type of situation can lead us to a better description of the phenomenon. Considers the phenomenon of diffusion of a small amount of gas in a tube and assumes that the diffusion is observed via the surface of the tube. The nonlocal conditions allow additional measurement which is more precise than the measurement just at $\mathtt{u}=0$.

## 2. Preliminaries

In this section, we provide definitions, and auxiliary results which will be useful in the sequel.

**Definition 2.1.** *A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\varrho(\mathtt{Q})$ is strictly less than 1. In other words, this means that all the eigenvalues of $\mathtt{Q}$ are in the open unit disc, i.e., $|\lambda|<1$, for every $\lambda\in\mathbb{C}$ with $det(\mathtt{Q}-\lambda\mathtt{I})=0$, where $\mathtt{I}$ denotes the unit matrix.*

**Lemma 2.2.** *[50] Let $\mathtt{Q}$ be a real square matrix. Then the following statements are equivalent.*

(a) $\mathtt{Q}$ *converges to zero,*
(b) $\mathtt{Q}^{\mathtt{j}}\to 0$ *as* $\mathtt{j}\to\infty$,
(c) $\det(\mathtt{I}-\mathtt{Q})\neq 0$ *and* $(\mathtt{I}-\mathtt{Q})^{-1} = \mathtt{I}+\mathtt{Q}+\mathtt{Q}^2+\ldots+\mathtt{Q}^{\mathtt{j}}+\cdots$,
(d) $\det(\mathtt{I}-\mathtt{Q})\neq 0$ *and* $(\mathtt{I}-\mathtt{Q})^{-1}$ *has nonnegative elements.*

Let $J:=[0,\mathfrak{d}]$, $\mathfrak{d}>0$ and $(X,|.|)$ be a Banach space, $C(J,X)$ be the space of $X$-valued continuous functions on $J$ endowed with the uniform norm topology

$$\|\mathtt{z}\|_\infty = \sup_{\mathtt{u}\in J} |\mathtt{z}(\mathtt{u})| . \tag{2.1}$$

Let $L^1(J, X)$ the space of $X$-valued Bochner integrable functions on $J$ with norm

$$\|f\|_\infty = \int_0^{\mathfrak{d}} |f(\mathtt{u})|\, d\mathtt{u}. \tag{2.2}$$

We consider the Banach space of continuous functions

$$C_{1-\mathfrak{r}}(J, X) = \{\mathtt{z} \in C(J', X) : \lim_{\mathtt{u}\to 0^+} \mathtt{u}^{1-\mathfrak{r}}\mathtt{z}(\mathtt{u}) \text{ exists}\}. \tag{2.3}$$

A norm in this space is given by

$$\|\mathtt{z}\|_\ell = \sup_{\mathtt{u}\in J} \mathtt{u}^{1-\mathfrak{r}} |\mathtt{z}(\mathtt{u})| . \tag{2.4}$$

Let the product weighted space be $C_{\mathfrak{r}_1,\mathfrak{r}_2} := C_{1-\mathfrak{r}_1}(J, X) \times C_{1-\mathfrak{r}_2}(J, X)$, with norm,

$$\|\mathtt{u}\|_{C_{\mathfrak{r}_1,\mathfrak{r}_2}} = \|\mathtt{u}\|_{\mathfrak{r}_1} + \|\mathtt{u}\|_{\mathfrak{r}_2} .$$

For $\Omega$ a subset of the space $C_{1-\mathfrak{r}}(J, X)$, define $\Omega_\mathfrak{r}$ by

$$\Omega_\mathfrak{r} = \{\mathtt{z}_\mathfrak{r},\ \mathtt{z} \in \Omega\},$$

where

$$\mathtt{z}_\mathfrak{r}(\mathtt{u}) = \begin{cases} \mathtt{u}^{1-\mathfrak{r}}\mathtt{z}(\mathtt{u}), & \text{if } \mathtt{u} \in (0, \mathfrak{d}), \\ \lim_{\mathtt{u}\to 0^+} \mathtt{u}^{1-\mathfrak{r}}\mathtt{z}(\mathtt{u}) & \text{if } \mathtt{u} = 0. \end{cases}$$

It is clear that $\mathtt{z}_\mathfrak{r} \in C(J, X)$.

**Lemma 2.3.** *[9] A set $\varpi \subset C_{1-\mathfrak{r}}(J, X)$ is relatively compact if and only if $\varpi_\mathfrak{r}$ is relatively compact in $C(J, X)$.*

**Definition 2.4.** *[16, 36] The Riemann-Liouville fractional integral of order $\ell > 0$ of a function $f \in L^1(J, X)$ is the function $I^\ell_{0+} f$ of the following form :*

$$I^\ell_{0+} f(\mathtt{u}) = \frac{1}{\Gamma(\ell)} \int_0^{\mathtt{u}} \frac{f(s)}{(\mathtt{u}-s)^{1-\ell}} ds, \quad t > 0, \quad \ell > 0.$$

**Definition 2.5.** *[16, 36] The Riemann-Liouville derivative of order $\ell$ with the lower limit zero for a function $f : [0, \infty) \to X$ can be written as*

$${}^{L}\mathtt{D}^\ell_{0+} f(\mathtt{u}) = \frac{1}{\Gamma(n-\ell)} \left(\frac{d}{dt}\right)^n \int_0^{\mathtt{u}} \frac{f(s)}{(\mathtt{u}-s)^{\ell-n+1}} ds, \quad t > 0, \quad n-1 < \ell < n.$$

**Definition 2.6.** *[24] The Hilfer fractional derivative of type $0 \le \mathfrak{n} \le 1$ and of order $0 < \ell < 1$ for a function $f(\mathtt{u})$ is defined by*

$$\mathtt{D}^{\ell,\mathfrak{n}}_{0+} f(\mathtt{u}) = \left(I^{\mathfrak{n}(1-\ell)}_{0+} \frac{d}{dt}(I^{(1-\mathfrak{n})(1-\ell)}_{0+} f)\right) = \left(I^{\mathfrak{n}(1-\ell)}_{0+} (\mathtt{D}^{\mathfrak{r}}_{0+} f)\right)(\mathtt{u}).$$

**Remark 2.7.** *If $\mathfrak{n} = 0$ and $0 < \ell < 1$, then Hilfer fractional derivative becomes Riemann-Liouville fractional derivative of order $\ell$ and if $\mathfrak{n} = 1$ and $0 < \ell < 1$, then Hilfer fractional derivative becomes Caputo fractional derivative of order $\ell$.*

**Lemma 2.8.** *Suppose $\Gamma(\mathfrak{r}) \neq \sum_{i=1}^{p} c_i \mathfrak{z}_i^{\mathfrak{r}-1}$ and $h \in C_\mathfrak{r}(J)$. Then the Cauchy problem with nonlocal conditions*

$$\begin{cases} \mathtt{D}^{\ell,\mathfrak{n}}_{0+}\mathtt{z}(\mathtt{u}) = h(\mathtt{u}), \quad 0 < \ell < 1,\ 0 \le \mathfrak{n} \le 1, \\ \left(I^{1-\mathfrak{r}}_{0+}\mathtt{z}\right)(0) = \sum_{i=1}^{p} c_i \mathtt{z}(\mathfrak{z}_i), \quad \ell \le \mathfrak{r} = \ell + \mathfrak{n} - \ell\mathfrak{n},\ \mathtt{u}, \mathfrak{z}_i \in J'. \end{cases}$$

*has a unique solution given by*

$$\mathtt{z}(\mathtt{u}) = \frac{\mathfrak{T}}{\Gamma(\ell)}\mathtt{u}^{\mathfrak{r}-1}\sum_{i=1}^{p} c_i \int_0^{\mathfrak{z}_i} (\mathfrak{z}_i - s)^{1-\ell} h(s)ds + \frac{1}{\Gamma(\ell)}\int_0^{\mathtt{u}} (\mathtt{u}-s)^{\ell-1}h(s)ds, \tag{2.5}$$

*where* $\mathfrak{T} = \dfrac{1}{\Gamma(\mathfrak{r}) - \sum\limits_{i=1}^{p} c_i \mathfrak{z}_i^{\mathfrak{r}-1}}$.

We adopt the following definitions from [27, 48].

**Definition 2.9.** *The system* (1.1) *is said to be Ulam-Hyers stable, if there exists* $L = (L_1, L_2) > 0$ *such that for some* $\epsilon = (\epsilon_1, \epsilon_2) > 0$ *and for every solution* $(\mathtt{z}_1, \mathtt{z}_2) \in C_{\mathfrak{r}_1,\mathfrak{r}_2}$ *of the inequality*

$$\begin{cases} \left|\mathtt{D}_{0^+}^{\ell_1,\mathfrak{n}_1}\mathtt{z}_1(\mathtt{u},\varpi) - f(\mathtt{u},\mathtt{z}_1(\mathtt{u},\varpi),\mathtt{z}_2(\mathtt{u},\varpi),\varpi)\right| < \epsilon_1, \\ \left|\mathtt{D}_{0^+}^{\ell_2,\mathfrak{n}_2}\mathtt{z}_2(\mathtt{u},\varpi) - g(\mathtt{u},\mathtt{z}_1(\mathtt{u},\varpi),\mathtt{z}_2(\mathtt{u},\varpi),\varpi)\right| < \epsilon_2, \end{cases} \tag{2.6}$$

*for* $\mathtt{u} \in J'$. *Then, there exists a unique solution* $(\hat{\mathtt{z}}_1, \hat{\mathtt{z}}_2) \in C_{\mathfrak{r}_1,\mathfrak{r}_2}$ *with*

$$|(\mathtt{z}_1,\mathtt{z}_2) - (\hat{\mathtt{z}}_1,\hat{\mathtt{z}}_2)| \le L\epsilon.$$

**Definition 2.10.** *The system* (1.1) *is said to be generalized Ulam-Hyers stable, if there exists* $\psi \in C(\mathbb{R}^+,\mathbb{R}^+)$ *with* $\psi(0) = 0$, *such that for every solution* $(\mathtt{z}_1,\mathtt{z}_2) \in C_{\mathfrak{r}_1,\mathfrak{r}_2}$ *of the inequality* (2.6), *there exists a unique solution* $(\hat{\mathtt{z}}_1,\hat{\mathtt{z}}_2) \in C_{\mathfrak{r}_1,\mathfrak{r}_2}$ *of* (1.1) *which satisfies*

$$|(\mathtt{z}_1,\mathtt{z}_2) - (\hat{\mathtt{z}}_1,\hat{\mathtt{z}}_2)| \le \psi(\epsilon).$$

**Remark 2.11.** *We say that* $(\mathtt{z}_1,\mathtt{z}_2)C_{\mathfrak{r}_1,\mathfrak{r}_2}$ *is a solution of the system of inequalities* (2.6), *if there exist functons* $\phi,\psi \in C(J',\mathbb{R})$ *which depend upon* $x, y$, *respectively, such that*

(i) $\phi(\mathtt{u}) \le \epsilon_1,\ \psi(\mathtt{u}) \le \epsilon_2,\ \mathtt{u} \in J'$ *and*

(ii) $\mathtt{D}_{0^+}^{\ell_1,\mathfrak{n}_1}\mathtt{z}_1(\mathtt{u},\varpi) = f(\mathtt{u},\mathtt{z}_1(\mathtt{u},\varpi),\mathtt{z}_2(\mathtt{u},\varpi),\varpi) + \phi(\mathtt{u}), \quad 0 < \ell_1 < 1,\ 0 \le \mathfrak{n}_1 \le 1,$

$\mathtt{D}_{0^+}^{\ell_2,\mathfrak{n}_2}\mathtt{z}_2(\mathtt{u},\varpi) = g(\mathtt{u},\mathtt{z}_1(\mathtt{u},\varpi),\mathtt{z}_2(\mathtt{u},\varpi),\varpi) + \psi(\mathtt{u}), \quad 0 < \ell_2 < 1,\ 0 \le \mathfrak{n}_2 \le 1.$

2.1. **Random Variable and Multivalued Maps.** Now, we also introduce some basic definitions on multivalued maps. For more details, see [9, 31]. Let $X$ be a Banach space. We denote by $\mathcal{P}(X) = \{A \subseteq X : A \neq \emptyset\}$ the family of all nonempty subsets of $X$, $\mathcal{P}_{cp}(X) = \{A \in \mathcal{P}(X) : A \text{ is compact}\}, \mathcal{P}_b(X) = \{A \in \mathcal{P}(X) : A \text{ is bounded}\}$, $\mathcal{P}_{cl}(X) = \{C \in \mathcal{P}(E) : A \text{ is closed}\}$, $,\mathcal{P}_{cv}(X) = \{A \in \mathcal{P}(X) : A \text{ is convex}\}$, and the collection of all non-empty compact and convex subsets of $X$ is denoted by $\mathcal{P}_{cp,cv}(X) = \mathcal{P}_{cp}(X) \cap \mathcal{P}_{cv}(X)$.

Let $(\Omega,\Sigma)$ be a measurable space and $F : \varpi \to \mathcal{P}(X)$ be a multivalued mapping, $F$ is called measurable if $F^+(Q) = \{\varpi \in \Omega : F(\varpi) \subset Q\}$ for every $Q \in \mathcal{P}_{cl}(X)$ equivalently, for every $\mathcal{U}$ open set of $X$, the set $F^-(Q) = \{\varpi \in \Omega : F(\varpi) \cap \mathcal{U} \neq \emptyset\}$ is measurable. If $X$ is a metric space, we shall use $\mathcal{B}(X)$ to denote the Borel $\sigma$-algebra on $X$. The $\Sigma \bigotimes \mathcal{B}$ denotes the smallest $\sigma$-algebra on $\varpi \times X$ which contains all the sets $A \times S$, where $Q \in \Sigma$ and $S \in \mathcal{B}(X)$. Let $F : X \to \mathcal{P}(Y)$ be a multivalued map. A single-valued map $f : X \to Y$ is said to be a selection of $G$, and we write $(f \subset F)$ whenever $f(\mathtt{z}) \in F(\mathtt{z})$ for every $\mathtt{z} \in X$.

**Definition 2.12.** *[12] A mapping* $F : \Omega \times X \to X$ *is said to be a random operator if, for any* $\mathtt{z} \in X$, $f(\cdot,\mathtt{z})$ *is measurable.*

**Definition 2.13.** *[12] A random fixed point of* $f$ *is measurable function* $\mathtt{z} : \Omega \to X$ *such that* $\mathtt{z}(\varpi) = f(\varpi,\mathtt{z}(\varpi))$ *for all* $\varpi \in \Omega$. *Equivalently, a measurable selection for the multivalued map* $\mathsf{F}ixF_\varpi : \Omega \to \mathcal{P}(X)$ *is defined by*

$$\mathsf{F}ixF_\varpi(\mathtt{z}) = \{\mathtt{z} \in X : \mathtt{z} = f(\varpi,\mathtt{z})\}.$$

**Theorem 2.14.** *[12] Let* $(\Omega,\Sigma)$, $Y$ *be a separable metric space and* $F : \Omega \to \mathcal{P}_{cl}(Y)$ *be measurable multivalued. Then* $F$ *has a measurable selection.*

**Theorem 2.15.** *[12] Let $X$ be a separable metric space, $Y$ be a metric space, $f : \Omega \times X \to X$ be a Carathéodory function, and $\mathcal{U}$ be an open subset of $Y$. Then the multivalued map $F : \Omega \to \mathcal{P}(X)$ defined by $F(\varpi) = \{\varpi \in \Omega : f(\varpi, \mathtt{z}) \in \mathcal{U}\}$ is measurable.*

Next, we present some random fixed theorem in a separable vector Banach space.

**Theorem 2.16.** *[12] Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, $X$ be a real separable generalized Banach space and $F : \Omega \times X \to X$ be a continuous random operator, and let $M(\varpi) \in \mathcal{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matrix such that $M(\varpi)$ converges to $0$ a.e. and*

$$d(F(\varpi, \mathtt{z}_1), F(\varpi, \mathtt{z}_2)) \leq M(\varpi) d(\mathtt{z}_1, \mathtt{z}_2), \quad \text{for each } \mathtt{z}_1, \mathtt{z}_2 \in X,\ \varpi \in \Omega.$$

*Then there exists any random variable $\mathtt{z} : \Omega \to X$ which is a unique random fixed point of $F$.*

**Theorem 2.17.** *[12] Let $X$ be a separable generalized Banach space, and $F : \Omega \times X \to X$ a completely continuous random operator. Then either of the following holds:*

- *(i) The random equation $F(\varpi, \mathtt{z}) = \mathtt{z}$ has a random solution, i.e., there is a measurable function $\mathtt{z} : \Omega \to X$ such that $F(\varpi, \mathtt{z}(\varpi)) = \mathtt{z}(\varpi)$ for all $\varpi \in \Omega$, or*
- *(ii) The set $\mathcal{M} = \{\mathtt{z} : \Omega \to X$ is measurable $\lambda(\varpi)F(\varpi, \mathtt{z}) = \mathtt{z}\}$ is unbounded for some measurable function $\lambda : \Omega \to X$ with $0 < \lambda(\varpi) < 1$ on $\Omega$.*

## 3. Existence and Uniqueness of Solutions

Now we give our main existence results. The following hypotheses will be used in the sequel.

$(H_1)$ There exists random variables $K_i, \widetilde{K}_i : \Omega \to \mathbb{R}_+$ such that

$$\|f_i(\mathtt{u}, \mathtt{z}_1, \mathtt{z}_2, \varpi) - f_i(\mathtt{u}, \widetilde{\mathtt{z}}_1, \widetilde{\mathtt{z}}_2, \varpi)\| \leq K_i(\varpi)\|\mathtt{z}_1 - \widetilde{\mathtt{z}}_1\| + \widetilde{K}_i(\varpi)\|\mathtt{z}_2 - \widetilde{\mathtt{z}}_2\|,$$

for $\mathtt{z}_1, \mathtt{z}_2, \widetilde{\mathtt{z}}_1, \widetilde{\mathtt{z}}_2 \in X, \mathtt{u} \in J', \varpi \in \Omega, i = 1, 2$.

**Theorem 3.1.** *Assume that the hypothesis $(H_1)$ holds. If the matrix $M(\varpi)$ converges to $0$, then problem (1.1)-(1.2) has a unique random solution.*

*Proof.* Define the random operators $\aleph : C_{\mathfrak{r}_1,\mathfrak{r}_2} \times \Omega \longrightarrow C_{\mathfrak{r}_1,\mathfrak{r}_2}$ by

$$\aleph(\mathtt{z}_1, \mathtt{z}_2) = (\aleph_1(\varpi, \mathtt{z}_1, \mathtt{z}_2); \aleph_2(\varpi, \mathtt{z}_1, \mathtt{z}_2)),$$

where

$$\begin{aligned}\aleph_1(\mathtt{z}_1(\mathtt{u}, \varpi), \mathtt{z}_2(\mathtt{u}, \varpi), \varpi) =& \frac{\mathfrak{T}_1 \mathtt{u}^{\mathfrak{r}_1 - 1}}{\Gamma(\ell_1)} \sum_{i=1}^{p} c_i \int_0^{\mathfrak{z}_i} (\mathfrak{z}_i - s)^{\ell_1 - 1} f_1(s, \mathtt{z}_1(s, \varpi), \mathtt{z}_2(s, \varpi), \varpi) ds \\ &+ \frac{1}{\Gamma(\ell_1)} \int_0^{\mathtt{u}} (\mathtt{u} - s)^{\ell_1 - 1} f_1(s, \mathtt{z}_1(s, \varpi), \mathtt{z}_2(s, \varpi), \varpi) ds,\end{aligned}$$

and

$$\begin{aligned}\aleph_2(\mathtt{z}_1(\mathtt{u}, \varpi), \mathtt{z}_2(\mathtt{u}, \varpi), \varpi) =& \frac{\mathfrak{T}_2 \mathtt{u}^{\mathfrak{r}_2 - 1}}{\Gamma(\ell_2)} \sum_{i=1}^{p} c_i' \int_0^{\mathfrak{z}_i} (\mathfrak{z}_i - s)^{\ell_2 - 1} f_2(s, \mathtt{z}_1(s, \varpi), \mathtt{z}_2(s, \varpi), \varpi) ds \\ &+ \frac{1}{\Gamma(\ell_2)} \int_0^{\mathtt{u}} (\mathtt{u} - s)^{\ell_2 - 1} f_2(s, \mathtt{z}_1(s, \varpi), \mathtt{z}_2(s, \varpi), \varpi) ds.\end{aligned}$$

Next, we prove $\aleph_1$ meets all conditions of Theorem 3.1 on $C_{\mathfrak{r}_1,\mathfrak{r}_2}$. Let $(\mathtt{z}_1,\mathtt{z}_2),(\widetilde{\mathtt{z}}_1,\widetilde{\mathtt{z}}_2)\in C_{\mathfrak{r}_1,\mathfrak{r}_2}$, then

$$\begin{aligned}
&\mathtt{u}^{1-\mathfrak{r}_1}\Big\|\aleph_1(\mathtt{z}_1(\mathtt{u},\varpi),\mathtt{z}_2(\mathtt{u},\varpi),\varpi)-\aleph_1(\widetilde{\mathtt{z}}_1,\widetilde{\mathtt{z}}_2(\mathtt{u},\varpi),\varpi)\Big\|\\
&\le \frac{|\mathfrak{T}_1|}{\Gamma(\ell_1)}\sum_{i=1}^{p}c_i\int_0^{\mathfrak{z}_i}(\mathfrak{z}_i-s)^{\ell_1-1}\Big\|f_1(s,\mathtt{z}_1(s,\varpi),\mathtt{z}_2(s,\varpi),\varpi)-f_1(s,\widetilde{\mathtt{z}}_1(s,\varpi),\widetilde{\mathtt{z}}_{\mathtt{z}}(s,\varpi),\varpi)\Big\|ds\\
&+\frac{\mathtt{u}^{1-\mathfrak{r}_1}}{\Gamma(\ell_1)}\int_0^{\mathtt{u}}(\mathtt{u}-s)^{\ell_1-1}\|f_1(s,\mathtt{z}_1(s,\varpi),\mathtt{z}_2(s,\varpi),\varpi)-f_1(s,\widetilde{\mathtt{z}}_1(s,\varpi),\widetilde{\mathtt{z}}_2(s,\varpi),\varpi)\|ds\\
&\le \frac{|\mathfrak{T}_1|}{\Gamma(\ell_1)}\sum_{i=1}^{p}c_i\int_0^{\mathfrak{z}_i}(\mathfrak{z}_i-s)^{\ell_1-1}(\text{ß}_1(\varpi)\|\mathtt{z}_1-\widetilde{\mathtt{z}}_1\|+\hat{\text{ß}}_1(\varpi)\|\mathtt{z}_2-\widetilde{\mathtt{z}}_2\|)ds\\
&+\frac{\mathtt{u}^{1-\mathfrak{r}_1}}{\Gamma(\ell_1)}\int_0^{\mathtt{u}}(\mathtt{u}-s)^{\ell_1-1}(\text{ß}_1(\varpi)\|\mathtt{z}_1-\widetilde{\mathtt{z}}_1\|+\hat{\text{ß}}_1(\varpi)\|\mathtt{z}_2-\widetilde{\mathtt{z}}_2\|)ds\\
&\le \text{ß}_1(\varpi)|\mathfrak{T}_1|\frac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_1+\mathfrak{r}_1)}\sum_{i=1}^{p}c_i\mathfrak{z}_i^{\ell_1+\mathfrak{r}_1-1}\|\mathtt{z}_1-\widetilde{\mathtt{z}}_2\|_{\mathfrak{r}_1}\\
&+\hat{\text{ß}}_1(\varpi)|\mathfrak{T}_1|\frac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_1+\mathfrak{r}_2)}\sum_{i=1}^{p}c_i\mathfrak{z}_i^{\ell_1+\mathfrak{r}_2-1}\|\mathtt{z}_2-\widetilde{\mathtt{z}}_2\|_{\mathfrak{r}_2}\\
&+\text{ß}_1(\varpi)|\mathfrak{T}_1|\frac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_1+\mathfrak{r}_1)}\mathfrak{d}^{\ell_1}\|\mathtt{z}_1-\widetilde{\mathtt{z}}_2\|_{\mathfrak{r}_1}\\
&+\hat{\text{ß}}_1(\varpi)|\mathfrak{T}_1|\frac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_1+\mathfrak{r}_2)}\mathfrak{d}^{\ell_1-\mathfrak{r}_1+\mathfrak{r}_2}\|\mathtt{z}_2-\widetilde{\mathtt{z}}_2\|_{\mathfrak{r}_2}.
\end{aligned}$$

Then

$$\begin{aligned}
&\|\aleph_1(\mathtt{z}_1(\mathtt{u},\varpi),\mathtt{z}_2(\mathtt{u},\varpi),\varpi)-\aleph_1(\widetilde{\mathtt{z}}_1,\widetilde{\mathtt{z}}_2(\mathtt{u},\varpi),\varpi)\|_{\mathfrak{r}_1}\\
&\le \frac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_1+\mathfrak{r}_1)}\left(|\mathfrak{T}_1|\textstyle\sum_{i=1}^{p}c_i\mathfrak{z}_i^{\ell_1+\mathfrak{r}_1-1}+\mathfrak{d}^{\ell_1}\right)\text{ß}_1(\varpi)\|\mathtt{z}_1-\widetilde{\mathtt{z}}_2\|_{\mathfrak{r}_1}\\
&+\frac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_1+\mathfrak{r}_2}\left(|\mathfrak{T}_1|\sum_{i=1}^{p}c_i\mathfrak{z}_i^{\ell_1+\mathfrak{r}_2-1}+\mathfrak{d}^{\ell_1-\mathfrak{r}_1+\mathfrak{r}_2}\right)\hat{\text{ß}}_1(\varpi)\|\mathtt{z}_2-\widetilde{\mathtt{z}}_2\|_{\mathfrak{r}_2}.
\end{aligned}\tag{3.1}$$

Similarly, for any $\varpi\in\Omega$ and each $(\mathtt{z}_1,\mathtt{z}_2),(\widetilde{\mathtt{z}}_1,\widetilde{\mathtt{z}}_2)\in C_{\mathfrak{r}_1,\mathfrak{r}_2}$, and $\mathtt{u}\in J$, we get

$$\begin{aligned}
&\|\aleph_2(\mathtt{z}_1(\mathtt{u},\varpi),\mathtt{z}_2(\mathtt{u},\varpi),\varpi)-\aleph_2(\widetilde{\mathtt{z}}_1,\widetilde{\mathtt{z}}_2(\mathtt{u},\varpi),\varpi)\|_{\mathfrak{r}_1}\\
&\le \frac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_2+\mathfrak{r}_1)}\left(|\mathfrak{T}_1|\textstyle\sum_{i=1}^{p}c_i'\mathfrak{z}_i^{\ell_2+\mathfrak{r}_1-1}+\mathfrak{d}^{\ell_2+\mathfrak{r}_1-\mathfrak{r}_2}\right)\text{ß}_2(\varpi)\|\mathtt{z}_1-\widetilde{\mathtt{z}}_1\|_{\mathfrak{r}_1}\\
&+\frac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_1+\mathfrak{r}_2}\left(|\mathfrak{T}_1|\sum_{i=1}^{p}c_i'\mathfrak{z}_i^{\ell_1+\mathfrak{r}_2-1}+\mathfrak{d}^{\ell_2}\right)\hat{\text{ß}}_2(\varpi)\|\mathtt{z}_2-\widetilde{\mathtt{z}}_2\|_{\mathfrak{r}_2}.
\end{aligned}$$

Therefore, we have

$$d(\aleph(\mathtt{z}_1,\mathtt{z}_2,\varpi)-\aleph(\widetilde{\mathtt{z}}_1,\widetilde{\mathtt{z}}_2,\varpi))\le M(\varpi)d((\mathtt{z}_1,\mathtt{z}_2),(\widetilde{\mathtt{z}}_1,\widetilde{\mathtt{z}}_2)),$$

where

$$d((\mathtt{z}_1,\mathtt{z}_2),(\widetilde{\mathtt{z}}_1,\widetilde{\mathtt{z}}_2))=\begin{pmatrix}\|\mathtt{z}_1-\widetilde{\mathtt{z}}_1\|_{\mathfrak{r}_1}\\ \|\mathtt{z}_2-\widetilde{\mathtt{z}}_2\|_{\mathfrak{r}_2}\end{pmatrix},$$

and

$$M(\varpi):=\begin{pmatrix}(A_1+B_1)\text{ß}_1(\varpi) & (\widetilde{A}_1+\widetilde{B}_1)\hat{\text{ß}}_1(\varpi)\\ (A_2+B_2)\text{ß}_2(\varpi) & (\widetilde{A}_2+\widetilde{B}_2)\hat{\text{ß}}_2(\varpi)\end{pmatrix},$$

where $A_1=\dfrac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_1+\mathfrak{r}_1)}|\mathfrak{T}_1|\sum\limits_{i=1}^{p}c_i\mathfrak{z}_i^{\ell_1+\mathfrak{r}_1-1}$, $B_1=\dfrac{\Gamma(\mathfrak{r}_1)\mathfrak{d}^{\ell_1}}{\Gamma(\ell_1+\mathfrak{r}_1)}$,
$\widetilde{A}_1=\dfrac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_1+\mathfrak{r}_2)}|\mathfrak{T}_1|\sum\limits_{i=1}^{p}c_i\mathfrak{z}_i^{\ell_1+\mathfrak{r}_2-1}$, $\widetilde{B}_1=\dfrac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_1+\mathfrak{r}_2)}\mathfrak{d}^{\ell_1-\mathfrak{r}_1+\mathfrak{r}_2}$,
$A_2=\dfrac{\Gamma(\mathfrak{r}_1)}{(\ell_2+\mathfrak{r}_1)}|\mathfrak{T}_2|\sum\limits_{i=1}^{p}c_i'\mathfrak{z}_i^{\ell_2+\mathfrak{r}_1-1}$, $B_2=\dfrac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_2+\mathfrak{r}_1)}\mathfrak{d}^{\ell_2-\mathfrak{r}_2+\mathfrak{r}_1}$,
$\widetilde{A}_2=\dfrac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_2+\mathfrak{r}_2)}|\mathfrak{T}_2|\sum\limits_{i=1}^{p}c_i'\mathfrak{z}_i^{\ell_2+\mathfrak{r}_2-1}$, $\widetilde{B}_2=\dfrac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_2+\mathfrak{r}_2)}\mathfrak{d}^{\ell_2}$.
Hence, by Theorem 2.16, the operator $\aleph$ has a unique fixed point which is a random solution of (1.1)-(1.2). ☐

## 4. Existence and Compactness of Solution Sets

For the existence and compactness of solutions for the system (1.1)- (1.2), we need to make the following assumptions

($H_2$) The functions $f_i$, $i=1,2$ are random Carathéodory on $J\times X\times X\times\Omega$,

($H_3$) There exists measurable functions $a_i,b_i:\Omega\to\mathbb{R}_+$, $i=1,2$ such that
$$\|f_i(\mathtt{u},\mathtt{z}_1,\mathtt{z}_2,\varpi)\|\le a_i(\varpi)\|\mathtt{z}_1\|+b_i(\varpi)\|\mathtt{z}_2\|,\ \text{for a.e., } \mathtt{u}\in J,\ \text{and each } \mathtt{z}_1,\mathtt{z}_2\in X.$$

**Theorem 4.1.** *Assume that the hypotheses* ($H_2$)-($H_3$) *hold. Then, the coupled system* (1.1) *and* (1.2) *has at least one random solution.*

*Proof.* We prove it in the following four steps.
**Step 1.** $N(\cdot,\cdot,\varpi)$ is continuous.
Let $(\mathtt{z}_{1n},\mathtt{z}_{2n})_n$ be a sequence such that $(\mathtt{z}_{1n},\mathtt{z}_{2n})\to(\mathtt{z}_1,\mathtt{z}_2)\in C_{\mathfrak{r}_1,\mathfrak{r}_2}$ as $n\to\infty$. For any $\varpi\in\Omega$ and each $\mathtt{u}\in J$, we have
$$\begin{aligned}
&\mathtt{u}^{1-\mathfrak{r}_1}\Big\|\aleph_1(\mathtt{u},\mathtt{z}_{1n}(\mathtt{u},\varpi),\mathtt{z}_{2n}(\mathtt{u},\varpi),\varpi)-\aleph_1(\mathtt{u},\mathtt{z}_1(\mathtt{u},\varpi),\mathtt{z}_2(\mathtt{u},\varpi),\varpi)\Big\|\\
&\le\frac{|\mathfrak{T}_1|}{\Gamma(\ell_1)}\sum_{i=1}^{p}c_i\int_0^{\mathfrak{z}_i}(\mathfrak{z}_i-s)^{\ell_1-1}\Big\|f_1(s,\mathtt{z}_{1n}(s,\varpi),\mathtt{z}_{2n}(s,\varpi),\varpi)-f_1(s,\mathtt{z}_1(s,\varpi),\mathtt{z}_2(s,\varpi),\varpi)\Big\|ds\\
&+\frac{\mathfrak{d}^{1-\mathfrak{r}_1}}{\Gamma(\ell_1)}\int_0^{\mathtt{u}}(\mathtt{u}-s)^{\ell_1-1}\Big\|f_1(s,\mathtt{z}_{1n}(s,\varpi),\mathtt{z}_{2n}(s,\varpi),\varpi)-f_1(s,\mathtt{z}_1(s,\varpi),\mathtt{z}_2(s,\varpi),\varpi)\Big\|ds
\end{aligned}$$
Then
$$\begin{aligned}
\|\aleph_1(t,\mathtt{z}_{1n}(\mathtt{u},\varpi),\mathtt{z}_{2n}(\mathtt{u},\varpi),\varpi)-\aleph_1(\mathtt{u},\mathtt{z}_1(\mathtt{u},\varpi),\mathtt{z}_2(\mathtt{u},\varpi),\varpi)\|_{\mathfrak{r}_1}&\le\left(\frac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_1+\mathfrak{r}_1)}|\mathfrak{T}_1|\sum_{i=1}^{p}c_i\mathfrak{z}_i^{\ell_1+\mathfrak{r}_1-1}+\frac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_1+\mathfrak{r}_1)}\mathfrak{d}^{\ell_1}\right)\\
&\times\Big\|f_1(\cdot,\mathtt{z}_{1n}(\cdot,\varpi),\mathtt{z}_{2n}(\cdot,\varpi),\varpi)-f_1(\cdot,\mathtt{z}_1(\cdot,\varpi),\mathtt{z}_2(\cdot,\varpi),\varpi)\Big\|_{\mathfrak{r}_1}.
\end{aligned}$$
Since $f_1$ is Carathéodory, we have
$$\|\aleph_1(\cdot,\mathtt{z}_{1n}(\cdot,\varpi),\mathtt{z}_{2n}(\cdot,\varpi),\varpi)-\aleph_1(\cdot,\mathtt{z}_1(\cdot,\varpi),\mathtt{z}_2(\cdot,\varpi),\varpi)\|_{\mathfrak{r}_1}\to0\text{ as }n\to\infty.$$
On the other hand, for any $\varpi\in\Omega$ and each $\mathtt{u}\in J$ we obtain
$$\begin{aligned}
&\|\aleph_2(\mathtt{u},\mathtt{z}_{1n}(\mathtt{u},\varpi),\mathtt{z}_{2n}(\mathtt{u},\varpi),\varpi)-\aleph_2(\mathtt{u},\mathtt{z}_1(\mathtt{u},\varpi),\mathtt{z}_2(\mathtt{u},\varpi),\varpi)\|_{\mathfrak{r}_2}\\
&\le\left(\frac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_2+\mathfrak{r}_2)}|\mathtt{u}_2|\sum_{i=1}^{p}c_i'\mathfrak{z}_i^{\ell_2+\mathfrak{r}_2-1}+\frac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_2+\mathfrak{r}_2)}\mathfrak{d}^{\ell_2}\right)\\
&\qquad\times\|f_2(\cdot,\mathtt{z}_{1n}(\cdot,\varpi),\mathtt{z}_{2n}(\cdot,\varpi),\varpi)-f_2(\cdot,\mathtt{z}_1(\cdot,\varpi),\mathtt{z}_2(\cdot,\varpi),\varpi)\|_{\mathfrak{r}_2}.
\end{aligned}$$
Furthermore, from the fact that $f_2$ is Carathéodory, we have
$$\|\aleph_2(\cdot,\mathtt{z}_{1n}(\cdot,\varpi),\mathtt{z}_{2n}(\cdot,\varpi),\varpi)-\aleph_2(\cdot,\mathtt{z}_1(\cdot,\varpi),\mathtt{z}_2(\cdot,\varpi),\varpi)\|_{\mathfrak{r}_2}\to0\text{ as }n\to\infty.$$
Hence, $N(\cdot,\cdot,\varpi)$ is continuous.
**Step2.** $N(\cdot,\cdot,\varpi)$ maps bounded sets into bounded sets in $C_{\mathfrak{r}_1,\mathfrak{r}_2}$.

For $R>0$, denote $B_R := \{(\mathrm{z}_1,\mathrm{z}_2)\in C_{\mathfrak{r}_1,\mathfrak{r}_2} :\ \|\mathrm{z}_1\|_{\mathfrak{r}_1}\le R,\ \|\mathrm{z}_2\|_{\mathfrak{r}_2}\le R\}$. For any $\varpi\in\Omega$ and each $(\mathrm{z}_1,\mathrm{z}_2)\in B_R$ and $\mathrm{u}\in J$, we have

$$\begin{aligned}
\mathrm{u}^{1-\mathfrak{r}_1}\|\aleph_1(\mathrm{u},\mathrm{z}_1(\mathrm{u},\varpi),\mathrm{z}_2(\mathrm{u},\varpi),\varpi)\| &\le \frac{|\mathfrak{T}_1|}{\Gamma(\ell_1)}\sum_{i=1}^{p} c_i\int_0^{\mathfrak{z}_i}(\mathfrak{z}_i-s)^{\ell_1-1}\|f_1(s,\mathrm{z}_1(s,\varpi),\mathrm{z}_2(s,\varpi),\varpi)\|ds\\
&\quad+\frac{\mathfrak{d}^{1-\mathfrak{r}_1}}{\Gamma(\ell_1)}\int_0^{\mathrm{u}}(\mathrm{u}-s)^{\ell_1-1}\|f_1(s,\mathrm{z}_1(s,\varpi),\mathrm{z}_2(s,\varpi),\varpi)\|ds\\
&\le \frac{|\mathfrak{T}_1|}{\Gamma(\ell_1)}\sum_{i=1}^{p} c_i\int_0^{\mathfrak{z}_i}(\mathfrak{z}_i-s)^{\ell_1-1}s^{1-\mathfrak{r}_1}(a_1(\varpi)\|x\|+b_1(\varpi)\|y\|)ds\\
&\quad+\frac{\mathfrak{d}^{1-\mathfrak{r}_1}}{\Gamma(\ell_1)}\int_0^{\mathrm{u}}(\mathrm{u}-s)^{\ell_1-1}s^{1-\mathfrak{r}_1}(a_1(\varpi)\|x\|+b_1(\varpi)\|y\|)ds\\
&\le \frac{R}{\Gamma(\ell_1)}|\mathfrak{T}_1|\sum_{i=1}^{p} c_i\int_0^{\mathfrak{z}_i}(\mathfrak{z}_i-s)^{\ell_1-1}s^{1-\mathfrak{r}_1}(a_1(\varpi)+b_1(\varpi))ds\\
&\quad+\frac{R}{\Gamma(\ell_1)}\mathfrak{d}^{1-\mathfrak{r}_1}\int_0^{\mathrm{u}}(\mathrm{u}-s)^{\ell_1-1}s^{1-\mathfrak{r}_1}(a_1(\varpi)+b_1(\varpi))ds\\
&\le R\left(\frac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_1+\mathfrak{r}_1)}|\mathfrak{T}_1|\sum_{i=1}^{p}c_i\mathfrak{z}_i^{\ell_1+\mathfrak{r}_1-1}+\frac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_1+\mathfrak{r}_1)}\mathfrak{d}^{\ell_1}\right)\Big(a_1(\varpi)+b_1(\varpi)\Big)\\
&:= l_1.
\end{aligned}$$

Thus, $\|\aleph_1(\mathrm{u},\mathrm{z}_1(\mathrm{u},\varpi),\mathrm{z}_2(\mathrm{u},\varpi),\varpi)\|_{C_{\mathfrak{r}_1}}\le l_1$. Similarly, we have $N_2(\mathrm{u},\mathrm{z}_1(\mathrm{u},\varpi),\mathrm{z}_2(\mathrm{u},\varpi),\varpi)\|_{C_{\mathfrak{r}_2}}\le l_2$, where

$$l_2 = R\left(\frac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_2+\mathfrak{r}_2)}|\mathrm{u}_2|\sum_{i=1}^{p}c_i'\mathfrak{z}_i^{\ell_2+\mathfrak{r}_2-1}+\frac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_2+\mathfrak{r}_2)}\mathfrak{d}^{\ell_2}\right)\Big(a_2(\varpi)+b_2(\varpi)\Big).$$

Hence $\|\aleph(\mathrm{u},\mathrm{z}_1(\mathrm{u},\varpi),\mathrm{z}_2(\mathrm{u},\varpi),\varpi)\|_{C_{\mathfrak{r}_1,\mathfrak{r}_2}}\le (l_1,l_2) := l.$

**Step 3.** $\aleph(\cdot,\cdot,\varpi)$ maps bounded sets into equicontinuous sets in $C_{\mathfrak{r}_1,\mathfrak{r}_2}$.

Let $\mathrm{u}_1,\mathrm{u}_2\in(0,\mathfrak{d}]$, $\mathrm{u}_1\le\mathrm{u}_2$, $(\mathrm{z}_1,\mathrm{z}_2)\in B_R$ and $\varpi\in\Omega$, we have:

$$\begin{aligned}
&\|\mathrm{u}_2^{1-\mathfrak{r}_1}\aleph_1(\mathrm{u}_2,\mathrm{z}_1(\mathrm{u}_2,\varpi),\mathrm{z}_2(\mathrm{u}_2,\varpi),\varpi)-\mathrm{u}_1^{1-\mathfrak{r}_1}\aleph_1(\mathrm{u}_1,\mathrm{z}_1(\mathrm{u}_1,\varpi),\mathrm{z}_2(\mathrm{u}_1,\varpi),\varpi)\|\\
&=\Bigg\|\frac{\mathfrak{T}_1^{1-\mathfrak{r}_1}}{\Gamma(\ell_1}\int_0^{\mathfrak{T}_1}(\mathrm{u}_1-s)^{\ell_1-1}f_1(s,\mathrm{z}_1(s,\varpi),\mathrm{z}_2(s,\varpi),\varpi)ds\\
&-\frac{\mathrm{u}_2^{1-\mathfrak{r}_1}}{\Gamma(\ell_1}\int_0^{\mathrm{u}_2}(\mathrm{u}_2-s)^{\ell_1-1}f_1(s,\mathrm{z}_1(s,\varpi),\mathrm{z}_2(s,\varpi),\varpi)ds\Bigg\|\\
&\le\frac{1}{\Gamma(\ell_1)}\int_0^{\mathfrak{T}_1}\left|\mathrm{u}_2^{1-\mathfrak{r}_1}(\mathrm{u}_2-s)^{\ell_1-1}-\mathrm{u}_1^{1-\mathfrak{r}_1}(\mathrm{u}_1-s)^{\ell_1-1}\right|s^{\mathfrak{r}_1-1}s^{1-\mathfrak{r}_1}\|f(s,\mathrm{z}_1(s,\varpi),\mathrm{z}_2(s,\varpi),\varpi)\|ds\\
&+\frac{\mathrm{u}_2^{1-\mathfrak{r}_1}}{\Gamma(\ell_1)}\int_{\mathfrak{T}_1}^{\mathrm{u}_2}\left|(\mathrm{u}_2-s)^{\ell_1-1}\right|s^{\mathfrak{r}_1-1}s^{1-\mathfrak{r}_1}\|f_1(s,\mathrm{z}_1(s,\varpi),\mathrm{z}_2(s,\varpi),\varpi)\|ds\\
&\le\frac{R}{\Gamma(\ell_1)}\int_0^{\mathfrak{T}_1}\left|\mathrm{u}_2^{1-\mathfrak{r}_1}(\mathrm{u}_2-s)^{\ell_1-1}-\mathrm{u}_1^{1-\mathfrak{r}_1}(\mathrm{u}_1-s)^{\ell_1-1}\right|s^{\mathfrak{r}_1-1}ds\\
&+\frac{R}{\Gamma(\ell_1)}\int_{\mathfrak{T}_1}^{\mathrm{u}_2}\left|\mathrm{u}_2^{1-\mathfrak{r}_1}(\mathrm{u}_2-s)^{\ell_1-1}\right|s^{\mathfrak{r}_1-1}ds\\
&\le\frac{\Gamma(\mathfrak{r}_1)R}{\Gamma(\ell_1+\mathfrak{r}_1)}\left[\mathrm{u}_2^{\ell_1}-\mathfrak{T}_1^{\ell_1}+\mathrm{u}_2^{1-\mathfrak{r}_1}(\mathrm{u}_2-\mathfrak{T}_1)^{\ell_1+\mathfrak{r}_1+1}\right]
\end{aligned}$$

Similarly, we get

$$\begin{aligned}&\|\mathtt{u}_1^{1-\mathfrak{r}_2}\aleph_1(\mathtt{u}_1,\mathtt{z}_1(\mathtt{u}_1,\varpi),\mathtt{z}_2(\mathtt{u}_1,\varpi),\varpi)-\mathtt{u}_2^{1-\mathfrak{r}_2}\aleph_2(\mathtt{u}_2,\mathtt{z}_1(\mathtt{u}_2,\varpi),\mathtt{z}_2(\mathtt{u}_1,\varpi),\varpi)\|\\&\leq\frac{\Gamma(\mathfrak{r}_2)R}{\Gamma(\ell_2+\mathfrak{r}_2)}\left[\mathtt{u}_2^{\ell_2}-\mathtt{u}_1^{\ell_2}+\mathtt{u}_2^{1-\mathfrak{r}_2}(\mathtt{u}_2-\mathtt{u}_1)^{\ell_2+\mathfrak{r}_2+1}\right]\end{aligned}$$

Thus,

$$\|\aleph(\mathtt{u}_1,\mathtt{z}_1(\mathtt{u}_1,\varpi),\mathtt{z}_2(\mathtt{u}_1,\varpi),\varpi)-\aleph(\mathtt{u}_2,\mathtt{z}_1(\mathtt{u}_2,\varpi),\mathtt{z}_2(\mathtt{u}_2,\varpi),\varpi)\|_{C_{\mathfrak{r}_1,\mathfrak{r}_2}}\to 0\quad\text{as } \mathtt{u}_1\to\mathtt{u}_2.$$

As a consequence of Steps 1-3, with the Arzela-Ascoli theorem, we conclude that $\aleph(\cdot,\cdot,\varpi)$ maps $B_R$ into a precompact set in $C_{\mathfrak{r}_1,\mathfrak{r}_2}$.

**Step 4.** The set $E(\varpi)$ consisting of $(\mathtt{z}_1(\cdot,\varpi),\mathtt{z}_2(\cdot,\varpi))$ such that

$$(\mathtt{z}_1(\cdot,\varpi),\mathtt{z}_2(\cdot,\varpi))=\lambda(\varpi)\aleph(\cdot,\mathtt{z}_1(\cdot,\varpi),\mathtt{z}_2(\cdot,\varpi),\varpi),$$

for some measurable function $\lambda:\varpi\to(0,1)$ is bounded in $C_{\mathfrak{r}_1,\mathfrak{r}_2}$. Let $(\mathtt{z}_1(\cdot,\varpi),\mathtt{z}_2(\cdot,\varpi))\in C_{\mathfrak{r}_1,\mathfrak{r}_2}$ such that

$$(\mathtt{z}_1(\cdot,\varpi),\mathtt{z}_2(\cdot,\varpi))=\lambda(\varpi)\aleph(\cdot,\mathtt{z}_1(\cdot,\varpi),\mathtt{z}_2(\cdot,\varpi),\varpi).$$

Then,

$$\mathtt{z}_1(\cdot,\varpi)=\lambda(\varpi)\aleph_1(\cdot,\mathtt{z}_1(\cdot,\varpi),\mathtt{z}_2(\cdot,\varpi),\varpi),$$

and

$$\mathtt{z}_2(\cdot,\varpi)=\lambda(\varpi)\aleph_2(\cdot,\mathtt{z}_1(\cdot,\varpi),\mathtt{z}_2(\cdot,\varpi),\varpi).$$

Thus, for any $\varpi\in\varpi$ and each $\mathtt{u}\in J$, we have:

$$\begin{aligned}\|\mathtt{z}_1(\mathtt{u},\varpi)\|_{C_{\mathfrak{r}_1}}\leq&\frac{|\mathfrak{T}_1|}{\Gamma(\ell_1)}\sum_{i=1}^{p}c_i\int_0^{\mathfrak{z}_i}(\mathfrak{z}_i-s)^{\ell_1-1}(a_1(\varpi)\|\mathtt{z}_1\|+b_1(\varpi)\|\mathtt{z}_2\|)ds\\&+\frac{\mathfrak{d}^{1-\mathfrak{r}_1}}{\Gamma(\ell_1)}\int_0^{\mathtt{u}}(\mathtt{u}-s)^{\ell_1-1}(a_1(\varpi)\|\mathtt{z}_1\|+b_1(\varpi)\|\mathtt{z}_2\|)ds.\end{aligned}$$

Furthermore, we get:

$$\begin{aligned}\|\mathtt{z}_2(\mathtt{u},\varpi)\|_{C_{\mathfrak{r}_2}}\leq&\frac{|\mathfrak{T}_2|}{\Gamma(\ell_2)}\sum_{i=1}^{p}c_i'\int_0^{\mathfrak{z}_i}(\mathfrak{z}_i-s)^{\ell_2-1}(a_2(\varpi)\|\mathtt{z}_1\|+b_2(\varpi)\|\mathtt{z}_2\|)ds\\&+\frac{\mathfrak{d}^{1-\mathfrak{r}_2}}{\Gamma(\ell_2)}\int_0^{\mathtt{u}}(\mathtt{u}-s)^{\ell_2-1}(a_2(\varpi)\|\mathtt{z}_1\|+b_2(\varpi)\|\mathtt{z}_2\|)ds.\end{aligned}$$

Hence, we obtain:

$$\|\mathtt{z}_1(\mathtt{u},\varpi)\|_{C_{\mathfrak{r}_1}}+\|\mathtt{z}_2(\mathtt{u},\varpi)\|_{C_{\mathfrak{r}_2}}\leq\frac{1}{1-C_1}:=C,$$

where

$$C_1=\max\left\{a_1(\varpi)\mathfrak{d}^{\mathfrak{r}_1}\frac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_1+\mathfrak{r}_1)}+a_2(\varpi)\mathfrak{d}^{\ell_2+\mathfrak{r}_1-\mathfrak{r}_2}\frac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_2+\mathfrak{r}_1)},b_1(\varpi)\mathfrak{d}^{\mathfrak{r}_1}\frac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_1+\mathfrak{r}_1)}+b_2(\varpi)\mathfrak{d}^{\ell_1+\mathfrak{r}_2-\mathfrak{r}_1}\frac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_1+\mathfrak{r}_2)}\right\}.$$

Hence

$$\|\mathtt{z}_1(\cdot,\varpi),\mathtt{z}_2(\cdot,\varpi)\|_{\mathfrak{r}_1,\mathfrak{r}_2}\leq C.\tag{4.1}$$

This shows that the set $E(\varpi)$ is bounded. As a consequence of Steps 1-4 together with Theorem 2.17, we can conclude that $N$ has at least one fixed point in $B_R$, which is the solution for the system (1.1) and (1.2). Now, we show that the set

$$S(x_0,y_0)=\{(\mathtt{z}_1,\mathtt{z}_2)\in C_{\mathfrak{r}_1,\mathfrak{r}_2}:(\mathtt{z}_1,\mathtt{z}_2)\text{ is the solution of (1.1) and (1.2)}\}$$

such that $x_0=\sum_{i=1}^{p}c_i(\mathfrak{z}_i,\varpi)$ and $y_0=\sum_0^p c_i'(\mathfrak{z}_i,\varpi)$ for $\varpi\in\Omega$ and $\mathfrak{z}_i\in J$ is compact. It is clear that $S(x_0,y_0)\subset N(S(x_0,y_0))$. From (4.1) we deduced that $S(x_0,y_0)$ is bounded sets in $C_{\mathfrak{r}_1,\mathfrak{r}_2}$. Since $N$ is compact, then $S(x_0,y_0)$ is

compact if and only if $S(x_0, y_0)$ is closed.
Let $\{\mathtt{z}_{1n}, \mathtt{z}_{2n}\}_{n\geq 1} \subset S(x_0, y_0)$ be a sequence converge to $(\mathtt{z}_1, \mathtt{z}_2)$. Thus,

$$\begin{aligned}\mathtt{z}_{1n}(\mathtt{u},\varpi) = & \frac{\mathfrak{T}_1}{\Gamma(\ell_1)}\mathtt{u}^{\mathfrak{r}_1-1}\sum_{i=1}^{p} c_i \int_0^{\mathfrak{z}_i} (\mathfrak{z}_i - s)^{\ell_1-1} f_1(s, \mathtt{z}_{1n}(s,\varpi), \mathtt{z}_{2n}(s,\varpi), \varpi)ds \\ & + \frac{1}{\Gamma(\ell_1)}\int_0^{\mathtt{u}} (\mathtt{u}-s)^{\ell_1-1} f_1(s, \mathtt{z}_{1n}(s,\varpi), \mathtt{z}_{2n}(s,\varpi), \varpi)ds, \quad \mathtt{u} \in J'\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\mathtt{z}_{2n}(\mathtt{u},\varpi) = & \frac{\mathtt{u}_2}{\Gamma(\ell_2)}\mathtt{u}^{\mathfrak{r}_2-1}\sum_{i=1}^{p} c'_i \int_0^{\mathfrak{z}_i} (\mathfrak{z}_i - s)^{\ell_2-1} f_2(s, \mathtt{z}_{1n}(s,\varpi), \mathtt{z}_{2n}(s,\varpi), \varpi)ds \\ & + \frac{1}{\Gamma(\ell_2)}\int_0^{\mathtt{u}} (\mathtt{u}-s)^{\ell_2-1} f_2(s, \mathtt{z}_{1n}(s,\varpi), \mathtt{z}_{2n}(s,\varpi), \varpi)ds \quad \mathtt{u} \in J'.\end{aligned} \tag{4.3}$$

Similarly to step 2, we can prove that

$$\begin{aligned}\mathtt{z}_{1}(\mathtt{u},\varpi) = & \frac{\mathfrak{T}_1}{\Gamma(\ell_1)}\mathtt{u}^{\mathfrak{r}_1-1}\sum_{i=1}^{p} c_i \int_0^{\mathfrak{z}_i} (\mathfrak{z}_i - s)^{\ell_1-1} f_1(s, \mathtt{z}_{1}(s,\varpi), \mathtt{z}_{2}(s,\varpi), \varpi)ds \\ & + \frac{1}{\Gamma(\ell_1)}\int_0^{\mathtt{u}} (\mathtt{u}-s)^{\ell_1-1} f_1(s, \mathtt{z}_{1}(s,\varpi), \mathtt{z}_{2}(s,\varpi), \varpi)ds \quad \mathtt{u} \in J',\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}\mathtt{z}_{2}(\mathtt{u},\varpi) = & \frac{\mathtt{u}_2}{\Gamma(\ell_2)}\mathtt{u}^{\mathfrak{r}_2-1}\sum_{i=1}^{p} c'_i \int_0^{\mathfrak{z}_i} (\mathfrak{z}_i - s)^{\ell_2-1} f_2(s, \mathtt{z}_{1}(s,\varpi), \mathtt{z}_{2}(s,\varpi), \varpi)ds \\ & + \frac{1}{\Gamma(\ell_2)}\int_0^{\mathtt{u}} (\mathtt{u}-s)^{\ell_2-1} f_2(s, \mathtt{z}_{1}(s,\varpi), \mathtt{z}_{2}(s,\varpi), \varpi)ds \quad \mathtt{u} \in J'.\end{aligned} \tag{4.5}$$

This implies that $S(x_0, y_0)$ is compact. □

## 5. Ulam Stability Analysis

In this section, we study the Ulam stability of the problem (1.1) and (1.2).

**Lemma 5.1.** *Suppose $(\mathtt{z}_1, \mathtt{z}_2) \in C_{\mathfrak{r}_1,\mathfrak{r}_2}$ be the solution of the inequality , then we have the system of inequalities given as*

$$\begin{aligned}\mathtt{u}^{1-\mathfrak{r}_1}\Bigg| & \mathtt{z}_1(\mathtt{u},\varpi) - \frac{\mathfrak{T}_1}{\Gamma(\ell_1)}\mathtt{u}^{\mathfrak{r}_1-1}\sum_{i=1}^{p} c_i \int_0^{\mathfrak{z}_i} (\mathfrak{z}_i - s)^{\ell_1-1} f_1(s, \mathtt{z}_1(s,\varpi), \mathtt{z}_2(s,\varpi), \varpi)ds \\ & - \frac{1}{\Gamma(\ell_1)}\int_0^{\mathtt{u}} (\mathtt{u}-s)^{\ell_1-1} f_1(s, \mathtt{z}_1(s,\varpi), \mathtt{z}_2(s,\varpi), \varpi)ds\Bigg| \leq L_1\varepsilon_1, \ \mathtt{u} \in J, \\ \mathtt{u}^{1-\mathfrak{r}_2}\Bigg| & \mathtt{z}_2(\mathtt{u},\varpi) - \frac{\mathfrak{T}_2}{\Gamma(\ell_2)}\mathtt{u}^{\mathfrak{r}_2-1}\sum_{i=1}^{p} c'_i \int_0^{\mathfrak{z}_i} (\mathfrak{z}_i - s)^{\ell_2-1} f_2(s, \mathtt{z}_1(s,\varpi), \mathtt{z}_2(s,\varpi), \varpi)ds \\ & - \frac{1}{\Gamma(\ell_2)}\int_0^{\mathtt{u}} (\mathtt{u}-s)^{\ell_2-1} f_2(s, \mathtt{z}_1(s,\varpi), \mathtt{z}_2(s,\varpi), \varpi)ds\Bigg| \leq L_2\varepsilon_2, \ \mathtt{u} \in J.\end{aligned}$$

*Proof.* We have

$$\begin{cases} \mathtt{D}_{0^+}^{\ell_1,\mathtt{n}_1}\mathtt{z}_1(\mathtt{u},\varpi) = f_1(\mathtt{u}, \mathtt{z}_1(\mathtt{u},\varpi), \mathtt{z}_2(\mathtt{u},\varpi), \varpi) + \phi(\mathtt{u}), \ \mathtt{u} \in J', \\ \mathtt{D}_{0^+}^{\ell_2,\mathtt{n}_2}\mathtt{z}_2(\mathtt{u},\varpi) = f_2(\mathtt{u}, \mathtt{z}_1(\mathtt{u},\varpi), \mathtt{z}_2(\mathtt{u},\varpi), \varpi) + \psi(\mathtt{u}), \ \mathtt{u} \in J', \end{cases} \tag{5.1}$$

with nonlocal boundary conditions (1.2) has a solution

$$
\begin{aligned}
\mathrm{z}_1(\mathrm{u},\varpi) = {} & \frac{\mathfrak{T}_1}{\Gamma(\ell_1)}\mathrm{u}^{\mathfrak{r}_1-1}\sum_{i=1}^{p} c_i \int_0^{\mathfrak{z}_i} (\mathfrak{z}_i - s)^{\ell_1-1} f_1(s,\mathrm{z}_1(s,\varpi),\mathrm{z}_2(s,\varpi),\varpi)ds \\
& + \frac{1}{\Gamma(\ell_1)}\int_0^{\mathrm{u}} (\mathrm{u}-s)^{\ell_1-1} f_1(s,\mathrm{z}_1(s,\varpi),\mathrm{z}_2(s,\varpi),\varpi)ds \\
& + \frac{\mathfrak{T}_1}{\Gamma(\ell_1)}\mathrm{u}^{\mathfrak{r}_1-1}\sum_{i=1}^{p} c_i \int_0^{\mathfrak{z}_i} (\mathfrak{z}_i - s)^{\ell_1-1} \phi(s)ds \\
& + \frac{1}{\Gamma(\ell_1)}\int_0^{\mathrm{u}} (\mathrm{u}-s)^{\ell_1-1} \phi(s)ds, \ \mathrm{u}\in J, \\
\mathrm{z}_2(\mathrm{u},\varpi) = {} & \frac{\mathfrak{T}_2}{\Gamma(\ell_2)}\mathrm{u}^{\mathfrak{r}_2-1}\sum_{i=1}^{p} c_i' \int_0^{\mathfrak{z}_i} (\mathfrak{z}_i - s)^{\ell_2-1} f_2(s,\mathrm{z}_1(s,\varpi),\mathrm{z}_2(s,\varpi),\varpi)ds \\
& + \frac{1}{\Gamma(\ell_2)}\int_0^{\mathrm{u}} (\mathrm{u}-s)^{\ell_2-1} f_2(s,\mathrm{z}_1(s,\varpi),\mathrm{z}_2(s,\varpi),\varpi)ds \\
& + \frac{\mathfrak{T}_2}{\Gamma(\ell_2)}\mathrm{u}^{\mathfrak{r}_2-1}\sum_{i=1}^{p} c_i' \int_0^{\mathfrak{z}_i} (\mathfrak{z}_i - s)^{\ell_1-1} \psi(s)ds \\
& + \frac{1}{\Gamma(\ell_2)}\int_0^{\mathrm{u}} (\mathrm{u}-s)^{\ell_2-1} \psi(s)ds, \ \mathrm{u}\in J.
\end{aligned}
$$

Then from first equation, we have

$$
\begin{aligned}
& \mathrm{u}^{1-\mathfrak{r}_1}\Bigg| \mathrm{z}_1(\mathrm{u},\varpi) - \frac{\mathfrak{T}_1}{\Gamma(\ell_1)}\mathrm{u}^{\mathfrak{r}_1-1}\sum_{i=1}^{p} c_i \int_0^{\mathfrak{z}_i} (\mathfrak{z}_i - s)^{\ell_1-1} f_1(s,\mathrm{z}_1(s,\varpi),\mathrm{z}_2(s,\varpi),\varpi)ds \\
& \qquad - \frac{1}{\Gamma(\ell_1)}\int_0^{\mathrm{u}} (\mathrm{u}-s)^{\ell_1-1} f_1(s,\mathrm{z}_1(s,\varpi),\mathrm{z}_2(s,\varpi),\varpi)ds \Bigg| \\
& = \mathrm{u}^{1-\mathfrak{r}_1}\left| \frac{\mathfrak{T}_1}{\Gamma(\ell_1)}\mathrm{u}^{\mathfrak{r}_1-1}\sum_{i=1}^{p} c_i \int_0^{\mathfrak{z}_i} (\mathfrak{z}_i - s)^{\ell_1-1} \phi(s)ds + \frac{1}{\Gamma(\ell_1)}\int_0^{\mathrm{u}} (\mathrm{u}-s)^{\ell_1-1}\phi(s)ds \right| \\
& \le \frac{1}{\Gamma(\ell_1+1)}\left[ \mathfrak{T}_1 \sum_{i=1}^{p} c_i \mathfrak{z}_i^{\ell_1} + \mathfrak{d}^{1-\mathfrak{r}+\ell_1} \right] \epsilon_1 := L_1\epsilon_1.
\end{aligned}
$$

In similar manner we can obtain other inequality. □

**Theorem 5.2.** *Assume that* $(H_1)$ *holds and let* $\wp^{-1} = \sigma_2\eta_1 + \sigma_1(\eta_2 - 1) - \eta_2 + 1 \neq 0$, *where*

$$
\begin{cases}
\sigma_1 = \dfrac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_1+\mathfrak{r}_1)}\left( |\mathfrak{T}_1| \displaystyle\sum_{i=1}^{p} c_i \mathfrak{z}_i^{\ell_1+\mathfrak{r}_1-1} + \mathfrak{d}^{\ell_1} \right) \text{ß}_1(\varpi), \\[2ex]
\sigma_2 = \dfrac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_2+\mathfrak{r}_1)}\left( |\mathfrak{T}_1| \displaystyle\sum_{i=1}^{p} c_i' \mathfrak{z}_i^{\ell_2+\mathfrak{r}_1-1} + \mathfrak{d}^{\ell_2+\mathfrak{r}_1-\mathfrak{r}_2} \right) \text{ß}_2(\varpi), \\[2ex]
\eta_1 = \dfrac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_1+\mathfrak{r}_2)}\left( |\mathfrak{T}_1| \displaystyle\sum_{i=1}^{p} c_i \mathfrak{z}_i^{\ell_1+\mathfrak{r}_2-1} + \mathfrak{d}^{\ell_1-\mathfrak{r}_1+\mathfrak{r}_2} \right) \hat{\text{ß}}_1(\varpi), \\[2ex]
\eta_2 = \dfrac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_1+\mathfrak{r}_2)}\left( |\mathfrak{T}_1| \displaystyle\sum_{i=1}^{p} c_i' \mathfrak{z}_i^{\ell_1+\mathfrak{r}_2-1} + \mathfrak{d}^{\ell_2} \right) \hat{\text{ß}}_2(\varpi).
\end{cases}
\tag{5.2}
$$

*Then the boundary value problem* (1.1)-(1.2) *is Ulam-Hyers stable and consequently generalized Ulam-Hyers stable.*

*Proof.* Let $(\mathtt{z}_1,\mathtt{z}_2)\in C_{\mathfrak{r}_1,\mathfrak{r}_2}$ be the solution of the inequalities given in and $(\hat{\mathtt{z}}_1,\hat{\mathtt{z}}_2)\in C_{\mathfrak{r}_1,\mathfrak{r}_2}$ be the unique solution to the following system

$$\begin{cases} \mathtt{D}_{0^+}^{\ell_1,\mathtt{n}_1}\hat{\mathtt{z}}_1(\mathtt{u},\varpi)=f_1(\mathtt{u},\hat{\mathtt{z}}_1(\mathtt{u},\varpi),\hat{\mathtt{z}}_2(\mathtt{u},\varpi),\varpi),\ \mathtt{u}\in J', \\ \mathtt{D}_{0^+}^{\ell_2,\mathtt{n}_2}\hat{\mathtt{z}}_2(\mathtt{u},\varpi)=f_2(\mathtt{u},\hat{\mathtt{z}}_1(\mathtt{u},\varpi),\hat{\mathtt{z}}_2(\mathtt{u},\varpi),\varpi),\ \mathtt{u}\in J', \end{cases} \tag{5.3}$$

with nonlocal boundary conditions,

$$\begin{cases} \left(I_{0^+}^{1-\mathfrak{r}_1}\hat{\mathtt{z}}_1\right)(\varpi,0)=\sum\limits_{i=1}^{p}c_i(\mathfrak{z}_i,\varpi),\ \mathfrak{z}_i\in J', \\ \left(I_{0^+}^{1-\mathfrak{r}_2}\hat{\mathtt{z}}_2\right)(\varpi,0)=\sum\limits_{i=1}^{p}c_i'(\mathfrak{z}_i,\varpi),\ \mathfrak{z}_i\in J'. \end{cases} \tag{5.4}$$

Then, in view of Lemma 2.8, the solution of (5.3) and (5.4) is given by

$$\begin{aligned} \hat{\mathtt{z}}_1(\mathtt{u},\varpi)=&\frac{\mathfrak{T}_1}{\Gamma(\ell_1)}\mathtt{u}^{\mathfrak{r}_1-1}\sum_{i=1}^{p}c_i\int_0^{\mathfrak{z}_i}(\mathfrak{z}_i-s)^{\ell_1-1}f_1(s,\hat{\mathtt{z}}_1(s,\varpi),\hat{\mathtt{z}}_2(s,\varpi),\varpi)ds \\ &+\frac{1}{\Gamma(\ell_1)}\int_0^{\mathtt{u}}(\mathtt{u}-s)^{\ell_1-1}f_1(s,\hat{\mathtt{z}}_1(s,\varpi),\hat{\mathtt{z}}_2(s,\varpi),\varpi)ds \\ \hat{\mathtt{z}}_2(\mathtt{u},\varpi)=&\frac{\mathfrak{T}_2}{\Gamma(\ell_2)}\mathtt{u}^{\mathfrak{r}_2-1}\sum_{i=1}^{p}c_i'\int_0^{\mathfrak{z}_i}(\mathfrak{z}_i-s)^{\ell_2-1}f_2(s,\hat{\mathtt{z}}_1(s,\varpi),\hat{\mathtt{z}}_2(s,\varpi),\varpi)ds \\ &+\frac{1}{\Gamma(\ell_2)}\int_0^{\mathtt{u}}(\mathtt{u}-s)^{\ell_2-1}f_2(s,\hat{\mathtt{z}}_1(s,\varpi),\hat{\mathtt{z}}_2(s,\varpi),\varpi)ds \end{aligned}$$

From Lemma 5.1 and (3.1), we get

$$\begin{aligned} \mathtt{u}^{1-\mathfrak{r}_1}\big|\mathtt{z}_1(\mathtt{u},\varpi)-\hat{\mathtt{z}}_1(\mathtt{u},\varpi)\big|=\mathtt{u}^{1-\mathfrak{r}_1}&\Bigg|\mathtt{z}_1(\mathtt{u},\varpi)-\frac{\mathfrak{T}_1}{\Gamma(\ell_1)}\mathtt{u}^{\mathfrak{r}_1-1}\sum_{i=1}^{p}c_i\int_0^{\mathfrak{z}_i}(\mathfrak{z}_i-s)^{\ell_1-1}f_1(s,\mathtt{z}_1(s,\varpi),\mathtt{z}_2(s,\varpi),\varpi)ds \\ &-\frac{1}{\Gamma(\ell_1)}\int_0^{\mathtt{u}}(\mathtt{u}-s)^{\ell_1-1}f_1(s,\mathtt{z}_1(s,\varpi),\mathtt{z}_2(s,\varpi),\varpi)ds\Bigg| \\ &+\mathtt{u}^{1-\mathfrak{r}_1}\Bigg|\frac{\mathfrak{T}_1}{\Gamma(\ell_1)}\mathtt{u}^{\mathfrak{r}_1-1}\sum_{i=1}^{p}c_i\int_0^{\mathfrak{z}_i}(\mathfrak{z}_i-s)^{\ell_1-1}\big[f_1(s,\mathtt{z}_1(s,\varpi),\mathtt{z}_2(s,\varpi),\varpi) \\ &-f_1(s,\hat{\mathtt{z}}_1(s,\varpi),\hat{\mathtt{z}}_2(s,\varpi),\varpi)\big]ds \\ &+\frac{1}{\Gamma(\ell_1)}\int_0^{\mathtt{u}}(\mathtt{u}-s)^{\ell_1-1}\big[f_1(s,\mathtt{z}_1(s,\varpi),\mathtt{z}_2(s,\varpi),\varpi)-f_1(s,\hat{\mathtt{z}}_1(s,\varpi),\hat{\mathtt{z}}_2(s,\varpi),\varpi)\big]ds\Bigg| \\ &\le L_1\epsilon_1+\frac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_1+\mathfrak{r}_1)}\left(|\mathfrak{T}_1|\sum_{i=1}^{p}c_i\mathfrak{z}_i^{\ell_1+\mathfrak{r}_1-1}+\mathfrak{d}^{\ell_1}\right)\text{ß}_1(\varpi)\|\mathtt{z}_1-\widetilde{\mathtt{z}}_1\|_{\mathfrak{r}_1} \\ &+\frac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_1+\mathfrak{r}_2)}\left(|\mathfrak{T}_1|\sum_{i=1}^{p}c_i\mathfrak{z}_i^{\ell_1+\mathfrak{r}_2-1}+\mathfrak{d}^{\ell_1-\mathfrak{r}_1+\mathfrak{r}_2}\right)\hat{\text{ß}}_1(\varpi)\|\mathtt{z}_2-\widetilde{\mathtt{z}}_2\|_{\mathfrak{r}_2} \end{aligned}$$

So,

$$\begin{aligned} \big\|\mathtt{z}_1(\mathtt{u},\varpi)-\hat{\mathtt{z}}_1(\mathtt{u},\varpi)\big\|_{\mathfrak{r}_1}\le L_1\epsilon_1&+\frac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_1+\mathfrak{r}_1)}\left(|\mathfrak{T}_1|\sum_{i=1}^{p}c_i\mathfrak{z}_i^{\ell_1+\mathfrak{r}_1-1}+\mathfrak{d}^{\ell_1}\right)\text{ß}_1(\varpi)\|\mathtt{z}_1-\hat{\mathtt{z}}_1\|_{\mathfrak{r}_1} \\ &+\frac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_1+\mathfrak{r}_2)}\left(|\mathfrak{T}_1|\sum_{i=1}^{p}c_i\mathfrak{z}_i^{\ell_1+\mathfrak{r}_2-1}+\mathfrak{d}^{\ell_1-\mathfrak{r}_1+\mathfrak{r}_2}\right)\hat{\text{ß}}_1(\varpi)\|\mathtt{z}_2-\hat{\mathtt{z}}_2\|_{\mathfrak{r}_2}. \end{aligned}$$

Similary, we have

$$
\begin{aligned}
\big\|\mathtt{z}_2(\mathtt{u},\varpi)-&\hat{\mathtt{z}}_2(\mathtt{u},\varpi)\big\|_{\mathfrak{r}_1}\\
&\leq L_2\epsilon_2+\frac{\Gamma(\mathfrak{r}_1)}{\Gamma(\ell_2+\mathfrak{r}_1)}\left(|\mathfrak{T}_1|\sum_{i=1}^{p}c_i'\mathfrak{z}_i^{\ell_2+\mathfrak{r}_1-1}+\mathfrak{d}^{\ell_2+\mathfrak{r}_1-\mathfrak{r}_2}\right)\mathfrak{K}_2(\varpi)\|\mathtt{z}_1-\hat{\mathtt{z}}_1\|_{\mathfrak{r}_1}\\
&\quad+\frac{\Gamma(\mathfrak{r}_2)}{\Gamma(\ell_1+\mathfrak{r}_2)}\left(|\mathfrak{T}_1|\sum_{i=1}^{p}c_i'\mathfrak{z}_i^{\ell_1+\mathfrak{r}_2-1}+\mathfrak{d}^{\ell_2}\right)\hat{\mathfrak{K}}_2(\varpi)\|\mathtt{z}_2-\hat{\mathtt{z}}_2\|_{\mathfrak{r}_2}.
\end{aligned}
$$

From (5.2), the above two inequalities can be written as

$$
\begin{cases}
(1-\sigma_1)\|\mathtt{z}_1-\hat{\mathtt{z}}_1\|_{\mathfrak{r}_1}-\eta_1\|\mathtt{z}_2-\hat{\mathtt{z}}_2\|_{\mathfrak{r}_2}\leq L_1\epsilon_1,\\
(1-\eta_2)\|\mathtt{z}_2-\hat{\mathtt{z}}_2\|_{\mathfrak{r}_2}-\sigma_2\|\mathtt{z}_1-\hat{\mathtt{z}}_1\|_{\mathfrak{r}_2}\leq L_2\epsilon_2,
\end{cases}\tag{5.5}
$$

Further, system (5.5) can be put into matrix form as

$$
\begin{pmatrix}1-\sigma_1 & -\eta_1\\ -\sigma_2 & 1-\eta_2\end{pmatrix}\begin{pmatrix}\|\mathtt{z}_1-\hat{\mathtt{z}}_1\|_{\mathfrak{r}_1}\\ \|\mathtt{z}_2-\hat{\mathtt{z}}_2\|_{\mathfrak{r}_2}\end{pmatrix}\leq\begin{pmatrix}L_1\epsilon_1\\ L_2\epsilon_2\end{pmatrix}.
$$

On solving above inequality, we obtain

$$
\begin{pmatrix}\|\mathtt{z}_1-\hat{\mathtt{z}}_1\|_{\mathfrak{r}_1}\\ \|\mathtt{z}_2-\hat{\mathtt{z}}_2\|_{\mathfrak{r}_2}\end{pmatrix}\leq\begin{pmatrix}\dfrac{1-\eta_2}{\wp} & \dfrac{\eta_1}{\wp}\\[2ex] \dfrac{\sigma_2}{\wp} & \dfrac{1-\sigma_1}{\wp}\end{pmatrix}\begin{pmatrix}L_1\epsilon_1\\ L_2\epsilon_2\end{pmatrix}.
$$

Therefore,

$$
\|\mathtt{z}_1-\hat{\mathtt{z}}_1\|_{\mathfrak{r}_1}+\|\mathtt{z}_2-\hat{\mathtt{z}}_2\|_{\mathfrak{r}_2}\leq\left[\frac{1-\eta_2+\sigma_2}{\wp}\right]L_1\epsilon_1+\left[\frac{1-\sigma_1+\eta_1}{\wp}\right]L_2\epsilon_2.
$$

Let $\epsilon=\max\{\epsilon_1,\epsilon_2\}$. Then, we get

$$
\|(\mathtt{z}_1,\mathtt{z}_2)-(\hat{\mathtt{z}}_1.\hat{\mathtt{z}}_2)\|_{C_{\mathfrak{r}_1,\mathfrak{r}_2}}\leq L\epsilon,
$$

where

$$
L=\frac{1}{\wp}\big[(1+\sigma_2-\eta_2)L_1+(1-\sigma_1+\eta_1)L_2\big].
$$

Further, if we can write

$$
\|(\mathtt{z}_1,\mathtt{z}_2)-(\hat{\mathtt{z}}_1.\hat{\mathtt{z}}_2)\|_{C_{\mathfrak{r}_1,\mathfrak{r}_2}}\leq L\psi(\epsilon),\ \text{where}\ \psi(0)=0,
$$

then the solutions of (1.1)-(1.2) are generalized Ulam-Hyers stable. □

## 6. Examples

Let $\Omega=(-\infty,0)$ be equipped with the usual $\sigma$-algebra consisting of Lebesgue measurable subsets of $(-\infty,0)$.

**Example 6.1.** *Consider the following system of random Hilfer fractional differential equations with nonlocal conditions*

$$
\begin{cases}
\mathtt{D}_{0^+}^{\frac{2}{3},\frac{1}{2}}\mathtt{z}_1(\mathtt{u},\varpi)=\dfrac{40e^{-t}}{9t+180}\dfrac{\|(\mathtt{u},\varpi)\|}{4+\|\mathtt{z}_1(\mathtt{u},\varpi)\|}+\dfrac{3}{\mathtt{u}^2+54}\cos\mathtt{z}_2(\mathtt{u},\varpi),\\[2ex]
\mathtt{D}_{0^+}^{\frac{2}{3},\frac{1}{2}}\mathtt{z}_2(\mathtt{u},\varpi)=\dfrac{1}{32}\sin\mathtt{z}_1(\mathtt{u},\varpi)+\dfrac{\arctan\mathtt{z}_2(\mathtt{u},\varpi)}{16+150\mathtt{u}^2},\ \ \mathtt{u}\in J':=(0,1],
\end{cases}\tag{6.1}
$$

*with nonlocal conditions*

$$
\begin{cases}
\left(I_{0^+}^{\frac{1}{6}}\mathtt{z}_1\right)(0,\varpi)=c_1\mathtt{z}_1(\tfrac{1}{2},\varpi)+c_2\mathtt{z}_1(\tfrac{1}{2},\varpi),\\[1ex]
\left(I_{0^+}^{\frac{1}{6}}\mathtt{z}_2\right)(0,\varpi)=c_1'\mathtt{z}_2(\tfrac{1}{2},\varpi)+c_2'\mathtt{z}_2(\tfrac{1}{2},\varpi),
\end{cases}\tag{6.2}
$$

where $\ell_1 = \ell_2 = \frac{2}{3}$, $\mathfrak{n}_1 = \mathfrak{n}_2 = \frac{1}{2}$, $\mathfrak{r}_1 = \mathfrak{r}_2 = \frac{5}{6}$ and $c_1 = c_2 = c'_1 = c'_2 = 1$, $b = 1$. Moreover, $\mathtt{z}_1(\frac{1}{2}, \varpi) = \mathtt{z}_2(\frac{1}{2}, \varpi) = \dfrac{1}{(1+\varpi^2)}$ and $|\mathfrak{T}_1| = |\mathfrak{T}_2| = \dfrac{1}{\Gamma(\frac{5}{6}) - 2(\frac{1}{2})^{\frac{5}{6}-1}} \approx 0,8959$.

Note that

$$\|f_1(\mathtt{u}, \mathtt{z}_1(\mathtt{u},\varpi), \mathtt{z}_2(\mathtt{u},\varpi), \varpi) - f_1(\mathtt{u}, \widetilde{\mathtt{z}}_1, \varpi), \widetilde{\mathtt{z}}_2(\mathtt{u},\varpi), \varpi)\| \leq \frac{1}{18}\|\mathtt{z}_1 - \widetilde{\mathtt{z}}_1\| + \frac{1}{18}\|\mathtt{z}_2 - \widetilde{\mathtt{z}}_2\|,$$

and

$$\|f_2(\mathtt{u}, \mathtt{z}_1(\mathtt{u},\varpi), \mathtt{z}_2(\mathtt{u},\varpi), \varpi) - f_2(\mathtt{u}, \widetilde{\mathtt{z}}_1, \varpi), \widetilde{\mathtt{z}}_2(\mathtt{u},\varpi), \varpi)\| \leq \frac{1}{16}\|\mathtt{z}_1 - \widetilde{\mathtt{z}}_1\| + \frac{1}{16}\|\mathtt{z}_2 - \widetilde{\mathtt{z}}_2\|,$$

where $\text{ß}_1(\varpi) = \text{ß}_2(\varpi) = \dfrac{1}{18}$ and $\hat{\text{ß}}_1(\varpi) = \hat{\text{ß}}_1(\varpi) = \dfrac{1}{16}$. Also the matrix given by

$$M(\varpi) := \begin{pmatrix} 0,4776 & 0,4776 \\ 0,0475 & 0,0475 \end{pmatrix},$$

converges to 0. Theorem 3.1 implies that the coupled random system (1.1) and (1.2) has a unique random solution defined on $[0,1] \times \Omega$.

**Example 6.2.** *Consider the following coupled random system of fractional differential equation*

$$\begin{cases} \mathtt{D}_{0^+}^{\frac{2}{3},\frac{1}{2}} \mathtt{z}_1(\mathtt{u},\varpi) = f(\mathtt{u}, \mathtt{z}_1(\mathtt{u},\varpi), \mathtt{z}_2(\mathtt{u},\varpi), \varpi), \\ \mathtt{D}_{0^+}^{\frac{2}{3},\frac{1}{2}} \mathtt{z}_2(\mathtt{u},\varpi) = g(\mathtt{u}, \mathtt{z}_1(\mathtt{u},\varpi), \mathtt{z}_2(\mathtt{u},\varpi), \varpi), \end{cases} \quad \mathtt{u} \in J' := (0,1], \tag{6.3}$$

*with nonlocal conditions*

$$\begin{cases} \left(I_{0^+}^{\frac{1}{6}} \mathtt{z}_1\right)(0,\varpi) = \sum_{i=1}^{2} c_i \mathtt{z}_1(\frac{3}{2}, \varpi), \\ \left(I_{0^+}^{\frac{1}{6}} \mathtt{z}_2\right)(0,\varpi) = \sum_{i=1}^{2} c'_i \mathtt{z}_2(\frac{1}{2}, \varpi), \end{cases} \tag{6.4}$$

where $\ell_1 = \ell_2 = \frac{2}{3}, \mathfrak{n}_1 = \mathfrak{n}_2 = \frac{1}{2}, \mathfrak{r}_1 = \mathfrak{r}_2 = \frac{5}{6}$ and $c_1 = c_2 = 4, c'_1 = c'_2 = 5, b = 4$. Morever $\mathtt{z}_1(\frac{3}{2}, \varpi) = \mathtt{z}_2(\frac{3}{2}, \varpi) = \dfrac{1}{(1+\varpi^2)}$, where

$$f(\mathtt{u}, \mathtt{z}_1(\mathtt{u},\varpi), \mathtt{z}_2(\mathtt{u},\varpi), \varpi) = \frac{\sqrt{\mathtt{u}}\mathtt{z}_1^2(\mathtt{u},\varpi)\sin(\mathtt{u})}{32e^{\mathtt{u}}(1+\|\mathtt{z}_1\|)} + \frac{\mathtt{z}_2^2(\mathtt{u},\varpi)\cos(\mathtt{z}_1(\mathtt{u},\varpi)\mathtt{z}_2(\mathtt{u},\varpi))}{16e^{\mathtt{u}}(1+\|\mathtt{z}_2\|^7)},$$

$$g(\mathtt{u}, \mathtt{z}_1(\mathtt{u},\varpi), \mathtt{z}_2(\mathtt{u},\varpi), \varpi) = \frac{\sqrt{\mathtt{u}}\mathtt{z}_1^2(\mathtt{u},\varpi) + \mathtt{z}_1(\mathtt{u},\varpi)\mathtt{z}_2(\mathtt{u},\varpi)\sin(\mathtt{u})}{128e^{\mathtt{u}}(1+\mathtt{u}^2)(1+\|\mathtt{z}_1\|+\|\mathtt{z}_2\|)} + \frac{\mathtt{z}_2(\mathtt{u},\varpi)e^{-\mathtt{z}_2^2(\mathtt{u},\varpi)}}{100(1+\mathtt{z}_1^2(\mathtt{u},\varpi))}.$$

Not that

$$\|f(\mathtt{u}, \mathtt{z}_1(\mathtt{u},\varpi), \mathtt{z}_2(\mathtt{u},\varpi), \varpi)\| \leq \frac{1}{32}\|\mathtt{z}_1\| + \frac{1}{16}\|\mathtt{z}_2\|,$$

and

$$\|g(\mathtt{u}, \mathtt{z}_1(\mathtt{u},\varpi), \mathtt{z}_2(\mathtt{u},\varpi), \varpi) = \frac{1}{128}\|\mathtt{z}_1\| + \frac{1}{100}\|\mathtt{z}_2\|.$$

We get $a_1(\varpi) = \dfrac{1}{32}, b_1(\varpi) = \dfrac{1}{16}, a_2(\varpi) = \dfrac{1}{128}, b_2(\varpi) = \dfrac{1}{100}$,

$$\begin{aligned} C_1 =& \max\left\{\frac{4^{5/6}}{32}\frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{2}{3}+\frac{5}{6})} + \frac{4^{2/3}}{64}\frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{2}{3}+\frac{5}{6})}, \ \frac{4^{5/6}}{8}\frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{2}{3}+\frac{5}{6})} + \frac{4^{2/3}}{100}\frac{\Gamma(\frac{5}{6})}{\Gamma(\frac{2}{3}+\frac{5}{6})}\right\} \\ =& \max\{0.449, 0.6928\} = 0.6928 < 1. \end{aligned}$$

As a consequence of Theorem 4.1, we conclude that $N$ has a fixed point, which is the random solution of problem (6.3) ans (6.4) defined on $[0,4] \times \Omega$.

## 7. Conclusion

The existence, uniqueness, compactness, and stability of coupled random fraction differential equations using the Hilfer fractional derivatives with nonlocal boundary conditions are explored in this study. In order to discuss the arguments, a few random fixed point theorems in a separable vector are used. The abstract results are supported by examples. In the future, we plan to study coupled random fractional differential equations involving Caputo-Fabrizio derivatives, iterative systems of random fractional order boundary value problems, and random fractional dynamic equations on time scales.

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