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Numerical solution of one-phase Stefan problem for the non-classical heat equation with a convective condition.

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Abstract

A numerical technique for the solution of the one-phase Stefan problem for the non-classical heat equation with a convective condition is discussed. This approach is based on a scheme introduced in [15]. The compatibility and convergence of the method are proven. Numerical examples round out the discussion.

Keywords. Stefan problem, Non-classical heat equation, Convective condition, Free boundary problem, Finite difference method.
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1. Introduction

A large number of problems in various areas of applied science appear as moving boundary or phase change problems. These can arise in heat conduction situations in conjunction with a change of phase and initial and moving boundary conditions and need to be solved in a time-dependent space domain with a moving boundary condition. Since the moving boundary is a function of time and its location has to be determined as a part of the solution, such problems are inherently nonlinear. In general, the nonlinearity associated with the moving boundary significantly complicates the analysis of this class of problems. A common example is the problem of melting ice that was first treated by Stefan [1] and after whom such problems are widely referred to as Stefan problems [4, 5, 12]. A simple model of the Fuzzy one-phase Stefan problem is studied in [13]. Some numerical methods have been applied to the heat equation and reaction-diffusion systems [7, 11]. Some fractional partial differential equations were studied in [1–3, 6]. The discussion of phase-change and free boundary problems appears in industrial processes and in other problems of technological interest because of its widespread use in various dimensions, some of which are mentioned in [8–10, 14, 16]. Motivated by [8] the following free boundary problem which we want to consider consists of numerical determining the temperature $U = U(x,t)$ and the free boundary $x = S(t)$ which satisfy the conditions:

$$U_t - U_{xx} = -F(U(0,t)), \quad 0 < x < S(t), 0 < t < T, \tag{1.1}$$

$$U_x(0,t) = g(t)[U(0,t) - f(t)], f(t) \geq 0, \quad 0 < t < T, \tag{1.2}$$

$$U(S(t),t) = 0, \quad 0 < t < T, \tag{1.3}$$

$$U(x,0) = \phi(x) \geq 0, \quad 0 < x \leq b, \tag{1.4}$$

$$U_x(S(t),t) = -\dot{S}(t), \quad 0 < t < T, \tag{1.5}$$

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$$S(0) = b > 0. \tag{1.6}$$

where $f, g \in C^0(R^+)$, $\phi \in C^1[0,b]$, $\phi(b) = 0$, $\dot{\phi}(0) = g(0)[\phi(0) - f(0)]$,and $F \in C^1(R^+)$. The function F referred to as the control function, is assumed to satisfy the condition $F(0) = 0$. The function $g(t)$ is the heat transfer coefficient and $f(t)$ is the temperature of the external fluid, both of which depend on time. The existence and uniqueness, local in time, of a solution for the problem (1.1)-(1.6) has been proved in [10]. The goal of this paper is to find numerical solution of a solution for the problem (1.1)-(1.6). We are only concerned with the variable space grid technique. These are reviewed in detail in the subsequent sections and applied to the one-phase Stefan problem given by Eqs. (1.1). The outline of this paper is as follows. Standard numerical techniques based on explicit finite-difference approximations are derived in section 2. The compatibility of the method is studied in section 3. The convergence of the method is studied in section 4 and finally, in section 5, we present the numerical result of two examples.

2. Numerical solution

In order to network the semi-infinite region, the number of distances between the fixed boundary $x = 0$, and the moving boundary $x = S(t)$ is considered constant and equal to N so that the moving boundary is always in the Nth network. By network lines and based on time changes, the following result was obtained. $\Delta x = \frac{S(t)}{N}$ and $x_i = i\Delta x$, for ith grid point similar to [15]

$$\left.\frac{\partial U}{\partial t}\right|_i = \left.\frac{\partial U}{\partial x}\right|_t \left.\frac{dx}{dt}\right|_i + \left.\frac{\partial U}{\partial t}\right|_x, \tag{2.1}$$

for the node x_i, the following expression was obtaned

$$\frac{dx_i}{dt} = i\frac{\Delta x_i}{dt} = \frac{i}{N} \times \frac{dS}{dt} = \frac{i\Delta x}{S(t)} \times \frac{dS(t)}{dt} = \frac{x_i}{S(t)} \times \frac{dS(t)}{dt}, \tag{2.2}$$

in which the suffices t, i, and x are to be kept constant during the differentiation processes. Thus, in the dimensionless model problem, the heat conduction equation (1.1) takes the form

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} + \frac{x_i \dot{S}(t)}{S(t)} \frac{\partial U}{\partial x} - F(U(0,t)),\ 0 < x < S(t),\ 0 < t < T. \tag{2.3}$$

We note that the grid size $\Delta x = \frac{S(t)}{N}$ waries with time t in each time step Δt, since N is constant. An explicit numerical solution based on finite difference to the problem (1.1)-(1.4) is obtained by substituting the time and temperature derivatives at the nodes (x_i, t_j) by the forward and central differences as following form

$$u_{i,j+1} = ru_{i-1,j} + \left(1 - 2r - \frac{kx_i \dot{s}_j}{hs_j}\right) u_{i,j} + \left(r + \frac{kx_i \dot{s}_j}{hs_j}\right) u_{i+1,j} - kF(u_{0,j}), \tag{2.4}$$

$j = 0, 1, \ldots, i = 0, 1, 2, \ldots, N-1$, since $\dot{s}_j = \frac{s_{j+1} - s_j}{k}$ then we have from (2.4)

$$u_{i,j+1} - \frac{i}{s_j}(u_{i+1,j} - u_{i,j})s_{j+1} = ru_{i-1,j} + (1-2r+i)u_{i,j} + (r-i)u_{i+1,j} - kF(u_{0,j}), \quad i = 0, 1, 2, \cdots, N-1, \tag{2.5}$$

$$u_{1,j} = (1 + hg_j)u_{0,j} + hg_j f_j, \quad j = 0, 1,\, 2 \cdots\ , \tag{2.6}$$

$$u_{i,j} = 0, i = N, \quad j = 0, 1,\, 2 \cdots\ , \tag{2.7}$$

$$u_{i,0} = h_i, \quad i = 0, 1, 2, \cdots\ . \tag{2.8}$$

In the above equations $u_{i,j} \approx U(x_i, t_j)$ and $s_j \approx S(t_j)$, $x_i = ih, (h \approx \Delta x)$, $t_j = jk$ is the grid size at the jth time step, $k = T/M$ is the time step, and $r = k/h^2$. By using the following three-term backward difference for the temperature gradient at the free interface ($x = S(t) = N\Delta x$).

$$\left.\frac{\partial u}{\partial x}\right|_{x=S(t)} = \frac{3u_{N,j} - 4u_{N-1,j} + u_{N-2,j}}{2h} + O(h^2). \tag{2.9}$$

For the Stefan condition (1.5), we can write:

$$\left(\frac{3u_{N,j} - 4u_{N-1,j} + u_{N-2,j}}{2h}\right) = -\frac{s_{j+1} - s_j}{k}, \tag{2.10}$$

or

$$s_{j+1} = s_j - \frac{k}{2h}\left(3u_{N,j} - 4u_{N-1,j} + u_{N-2,j}\right), \ j = 0, 1, 2, \cdots, \tag{2.11}$$

and for the condition (1.6), we can write $s_0 = b$.

3. Compatibility

We now show that the finite difference method is compatible with the one-phase Stephen problem for a non-classical thermal equation and convective boundary conditions. To make this goal by applying the finite difference method to equation (2.4), we have:

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \left(\frac{u_{i+1,j} - u_{i,j}}{h}\right)\frac{x_i \dot{s}_j}{s_j} - F(u_{0,j}), \tag{3.1}$$

for $j = 0, 1, 2, \cdots, i = 1, 2, \cdots, N-1$. By Taylor extension,

$$U_{i-1,j} = U_{i,j} - h\left(\frac{\partial U}{\partial x}\right)_{i,j} + \frac{h^2}{2!}\left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j} - \frac{h^3}{3!}\left(\frac{\partial^3 U}{\partial x^3}\right)_{i,j} + \frac{h^4}{4!}\left(\frac{\partial^4 U(x_i + \theta_1, t_j)}{\partial x^4}\right)_{i,j}, \tag{3.2}$$

$$U_{i+1,j} = U_{i,j} + h\left(\frac{\partial U}{\partial x}\right)_{i,j} + \frac{h^2}{2!}\left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j} + \frac{h^3}{3!}\left(\frac{\partial^3 U}{\partial x^3}\right)_{i,j} + \frac{h^4}{4!}\left(\frac{\partial^4 U(x_i + \theta_2, t_j)}{\partial x^4}\right)_{i,j}, \tag{3.3}$$

$$U_{i,j+1} = U_{i,j} + k\left(\frac{\partial U}{\partial t}\right)_{i,j} + \frac{k^2}{2!}\left(\frac{\partial^2 U(x_i, t_j + \theta_3)}{\partial t^2}\right)_{i,j}. \tag{3.4}$$

Hence

$$\begin{aligned} T_{i,j} =& \frac{1}{k}\left(U_{i,j} + k\left(\frac{\partial U}{\partial t}\right)_{i,j} + \frac{k^2}{2!}\left(\frac{\partial^2 U}{\partial t^2}\right)_{i,j} + \cdots - U_{i,j}\right) \\ & - \frac{1}{h^2}\left(U_{i,j} - h\left(\frac{\partial U}{\partial x}\right)_{i,j} + \frac{h^2}{2!}\left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j} - \frac{h^3}{3!}\left(\frac{\partial^3 U}{\partial x^3}\right)_{i,j} + \cdots - 2U_{i,j} \right. \\ & \left. + U_{i,j} + h\left(\frac{\partial U}{\partial x}\right)_{i,j} + \frac{h^2}{2!}\left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j} + \frac{h^3}{3!}\left(\frac{\partial^3 U}{\partial x^3}\right)_{i,j}\right) + \ldots \\ & - \frac{1}{h}\left(U_{i,j} + h\left(\frac{\partial U}{\partial x}\right)_{i,j} + \frac{h^2}{2!}\left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j} + \cdots - U_{i,j}\right)\frac{x_i \dot{s}_j}{S_j} + F(U_{0,j}), \end{aligned} \tag{3.5}$$

or equivalently

$$T_{i,j} = \left(\frac{\partial U}{\partial t}\right)_{i,j} - \left(\frac{\partial^2 U}{\partial x^2}\right)_{i,j} - \left(\frac{\partial U}{\partial x}\right)_{i,j} \times \frac{x_i \dot{S}_j}{S_j} + F(U_{0,j}) + O(h^2) + O(k). \tag{3.6}$$

Now, from (3.5), when $h \to 0$ and $k \to 0$ we have $T_{i,j} \to 0$. Therefore, the finite difference method is compatible with the non-classical heat problem

4. The discussion of numerical solution

In this section, we analyze the convergence of the explicit finite difference method for the problem (1.1)-(1.6). For this purpose, we have from (2.4)

$$u_{i,j+1} = u_{i,j} + r(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + \frac{k}{h}(u_{i+1,j} - u_{i,j})\frac{x_i \dot{s}_j}{s_j} - kF(u_{0,j}). \tag{4.1}$$

At the mesh points,

$$\begin{aligned} &u_{i,j} = U_{i,j} - e_{i,j}, u_{i-1,j} = U(x_i - h, t_j) - e_{i-1,j} = U_{i-1,j} - e_{i-1,j}, \\ &u_{i+1,j} = U(x_i + h, t_j) - e_{i+1,j} = U_{i+1,j} - e_{i+1,j}, \\ &u_{i,j+1} = U(x_i, t_j + k) - e_{i,j+1} = U_{i,j+1} - e_{i,j+1}. \end{aligned} \tag{4.2}$$

By replacing the expressions (4.2) in to (2.4) leads to

$$\begin{aligned} e_{i,j+1} &= e_{i,j} + \frac{k}{h^2}\left(e_{i+1,j} - 2e_{i,j} + e_{i-1,j}\right) + \frac{k}{h}\left(e_{i+1,j} - e_{i,j}\right)\frac{x_i \dot{s}_j}{s_j} \\ &+ U_{i,j+1} - U_{i,j} - \frac{k}{h^2}\left(U_{i+1,j} - 2U_{i,j} + U_{i-1,j}\right) - \frac{k}{h}\left(U_{i+1,j} - U_{i,j}\right)\frac{x_i \dot{s}_j}{s_j} + kF(U_{0,j} - e_{0,j}), \end{aligned} \tag{4.3}$$

or

$$\begin{aligned} e_{i,j+1} &= re_{i-1,j} + \left(1 - 2r - \frac{kx_i \dot{s}_j}{hs_j}\right)e_{i,j} + \left(r + \frac{kx_i \dot{s}_j}{hs_j}\right)e_{i+1,j} \\ &+ U_{i,j+1} - U_{i,j} - \frac{k}{h^2}\left(U_{i+1,j} - 2U_{i,j} + U_{i-1,j}\right) - \frac{k}{h}\left(U_{i+1,j} - U_{i,j}\right)\frac{x_i \dot{s}_j}{s_j} + kF(U_{0,j} - e_{0,j}). \end{aligned} \tag{4.4}$$

By Taylor's expansion,

$$\begin{aligned} U_{i,j+1} &= U(x_i, t_j + k) = U_{i,j} + k\left(\frac{\partial U}{\partial t}\right)(x_i, t_j + \theta_1 k), \\ U_{i+1,j} &= U(x_i + h, t_j) = U_{i,j} + h\left(\frac{\partial U}{\partial x}\right)_{i,j} + \frac{h^2}{2!}\left(\frac{\partial^2 U}{\partial x^2}\right)(x_i + \theta_2 h, t_j), \\ U_{i-1,j} &= U(x_i - h, t_j) = U_{i,j} - h\left(\frac{\partial U}{\partial x}\right)_{i,j} + \frac{h^2}{2!}\left(\frac{\partial^2 U}{\partial x^2}\right)(x_i + \theta_3 h, t_j), \end{aligned} \tag{4.5}$$

where , $0 < \theta_1 < 1$, $0 < \theta_2 < 1$ and $0 < \theta_3 < 1$. Substitution into (4.3) gives

$$e_{i,j+1} = re_{i-1,j} + \left(1 - 2r - \frac{kx_i \dot{s}_j}{hs_j}\right)e_{i,j} + \left(r + \frac{kx_i \dot{s}_j}{hs_j}\right)e_{i+1,j} - ke_{0,j}F'(\theta) \tag{4.6}$$

$$+k\left[\left(\frac{\partial U}{\partial t}\right)(x_i, t_j + \theta_1 k) - \left(\frac{\partial^2 U}{\partial x^2}\right)(x_i + \theta_4 h, t_j) - \frac{x_i \dot{s}_j}{s_j}\left(\frac{\partial U}{\partial x}\right)(x_i + \theta_5 h, t_j) + F(U_{0,j})\right], \tag{4.7}$$

where , $-1 < \theta_4 < 1$, $0 < \theta_5 < 1$ and θ is between $U_{0,j} - e_{0,j}$ and $U_{0,j}$. We have for the increasing free boundary $s(t)$, $r + \frac{kx_i \dot{s}_j}{hs_j} \geq 0$. For $k < \frac{1}{\frac{2}{h^2} + \frac{x_i \dot{s}_j}{hs_j}}, j = 0, 1, 2, ..., i = 1, 2, ...N - 1$, we have $1 - 2r - \frac{kx_i \dot{s}_j}{hs_j} > 0$. Suppose that E_j be the largest value of error $|e_{i,j}|$ along the jth time-row, N_1 is the largest absolute value of $e_{0,j}F'(\theta)$, and suppose that M_1 is the largest value in bracket in (4.7) for all values of i, j. So for each step size h and k, where $k < \frac{1}{\frac{2}{h^2} + \frac{x_i \dot{s}_j}{hs_j}}, j = 0, 1, 2, ..., i = 1, 2, ...N - 1$, we can write:

$$\begin{aligned} |e_{i,j+1}| &\leq r|e_{i-1,j}| + \left[1 - 2r - \frac{kx_i \dot{s}_j}{hs_j}\right]|e_{i,j}| + \left[r + \frac{kx_i \dot{s}_j}{hs_j}\right]|e_{i+1,j}| + kM_1 + kN_1 \\ &\leq rE_j + \left[1 - 2r - \frac{kx_i \dot{s}_j}{hs_j}\right]E_j + \left[r + \frac{kx_i \dot{s}_j}{hs_j}\right]E_j + k(M_1 + N_1) = E_j + k(M_1 + N_1), \end{aligned} \tag{4.8}$$

so we can write,

$$\begin{aligned} E_{j+1} &\leq E_j + k(M_1+N_1) \\ &\leq (E_{j-1} + k(M_1+N_1)) + k(M_1+N_1) \\ &= E_{j-1} + 2k(M_1+N_1) \\ &\leq \cdots \leq E_0 + jk(M_1+N_1). \end{aligned} \tag{4.9}$$

From which it follows that

$$E_j \leq E_0 + jk(M_1+N_1). \tag{4.10}$$

since the initial values for u and U are equal, therefore $E_0 = 0$. When $h \longrightarrow 0$, $k = rh^2$ also tends to zero and M_1 tends to

$$\left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} - \frac{x_i \dot{S}_j}{S_j}\frac{\partial U}{\partial x} + F(U(0,t))\right)_{i,j}. \tag{4.11}$$

Since U is an answer to equation (2.4) the limiting value of M_1 and therefore of E_j is zero. Since $|U_{i,j} - u_{i,j}| \leq E_j$ this proves that u converges to U as $h \longrightarrow 0$ when $k < \frac{1}{\frac{2}{h^2} + \frac{x_i \dot{s}_j}{h s_j}}$, $\quad j = 0,1,2,...,\quad i = 1,2,...N-1$.

Suppose that $s_{j+1} = S_{j+1} - e'_{j+1}$, $s_j = S_j - e'_j$, for the Stefan condition (2.11), we can write,

$$S_{j+1} - e'_{j+1} = S_j - e'_j - \frac{k}{2h}\left(3U_{N,j} - 3e_{N,j} - 4U_{N-1,j} + 4e_{N-1,j} + U_{N-2,j} - e_{N-2,j}\right),$$

hence

$$e'_{j+1} = e'_j + S_{j+1} - S_j - \frac{k}{2h}\left(3U_{N,j} - 3e_{N,j} - 4U_{N-1,j} + 4e_{N-1,j} + U_{N-2,j} - e_{N-2,j}\right). \tag{4.12}$$

By using Taylor's expansion, we have

$$\begin{aligned} e'_{j+1} &= e'_j + kS'(t_j + \theta_5 k) - \frac{3k}{2h}e_{N,j} + \frac{2k}{h}e_{N-1,j} - \frac{k}{2h}e_{N-2,j} \\ &-\frac{k}{2h}(3U_{N,j} - 4U_{N,j} + 4h(\frac{\partial U}{\partial x})(x_N - \theta_6 h, t_j) + U_{N,j} - 2h(\frac{\partial U}{\partial x})(x_N - \theta_7 h, t_j)), \end{aligned} \tag{4.13}$$

since $0 < \theta_i h < h, i = 6,7$ for sufficient small step size we can assume $\theta_6 \cong \theta_7$. Hence

$$e'_{j+1} = e'_j + k\left[S'(t_j + \theta_5 k) + \left(\frac{\partial U}{\partial x}\right)(x_N - \theta_6 h, t_j)\right] - \frac{3k}{2h}e_{N,j} + \frac{2k}{h}e_{N-1,j} - \frac{k}{2h}e_{N-2,j}, \tag{4.14}$$

then we have,

$$\left|e'_{j+1}\right| \leq \left|e'_j\right| + k\left[S'(t_j + \theta_5 k) + \left(\frac{\partial U}{\partial x}\right)(x_N - \theta_6 h, t_j)\right] + \frac{3k}{2h}\left|e_{N,j}\right| + \frac{2k}{h}\left|e_{N-1,j}\right| + \frac{k}{2h}\left|e_{N-2,j}\right|, \tag{4.15}$$

suppose that P is the largest value in the bracket, we can write:

$$\begin{aligned} &\left|e'_{j+1}\right| \leq \left|e'_j\right| + kP + \left(\frac{3k}{2h} + \frac{2k}{h} + \frac{k}{2h}\right)E_j = \left|e'_j\right| + kP + \left(\frac{4k}{h}\right)E_j \\ &\leq \left|e'_{j-1}\right| + 2kP + \left(\frac{4k}{h}\right)(E_{j-1} + 2k(M+N)) \\ &\leq \cdots \leq |e'_0| + jkP + \left(\frac{4k}{h}\right)(E_0 + jk(M+N)), \end{aligned}$$

Because the initial values for s, u and S, U are the same, i.e., $e'_0 = 0$ and $E_0 = 0$. When k tends to zero and P tends to $\left(\dot{S}(t) + \frac{\partial U}{\partial x}|_{x=S(t)}\right)_j$. Since (S, U) is a solution of problem (1.1)-(1.6) the limiting value of P is zero. When h and

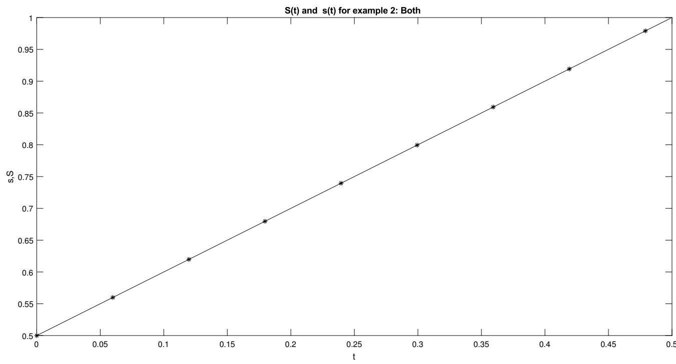

Figure 1. Comparison of the diagram of the moving boundary as predicted by the numerical with the exact solution.

Table 1. The absolute errors of the discrete solution S at some points.

t	Numerical solution	Exact solution	Error
0.0	0.5000	0.5000	0
0.0002	0.5003	0.5002	0.1694e-4
0.0005	0.5005	0.5005	0.2321e-4
0.0007	0.5008	0.5007	0.2747e-4
0.0010	0.5010	0.5010	0.3054e-4
0.0013	0.5013	0.5013	0.3273e-4

k tend to zero also M and N tend to zero. Hence $|e'_{j+1}|$ tends to zero. As $|S_{j+1} - s_{j+1}| \leq e'_{j+1}$, this proves that s converges to S as h and k tend to zero.

5. Numerical Result

In this section, two examples are considered to show the efficiency of the method. All calculations were performed in double precision arithmetic using MATLAB software.

Example 5.1. Consider the one-phase Stefane problem with thermal convection condition,

$$U_t - U_{xx} = 0,\ 0 < x < S(t), \quad 0 < t < 0.5,$$
$$U_x(0,t) = -2U(0,t) + exp(t+0.5) - 2, \quad 0 < t < 0.5,$$
$$U(S(t),t) = 0, \quad 0 < t < 0.5,$$
$$U_x(S(t),t) = -\dot{S}(t), \quad 0 < t < 0.5,$$
$$U(x,0) = exp(0.5 - x) - 1, \quad 0 < x \leq 0.5,$$
$$S(0) = 0.5,$$

where $g(t) = -2,\ f(t) = \frac{1}{2}exp(t+0.5) - 1,\ F(W) = 0$ and $T = 0.5$. The exact solution is given by

$$U(x,t) = exp(t + 0.5 - x) - 1,\ S(t) = t + 0.5,$$

This problem is solved by the method (2.5) and (2.8) with $M = 5001$ and $N = 11$. In Figure 1, we have plotted the continuous moving boundary $S(t)$ and its discrete solution s. The absolute errors of the discrete solution (s,u) are tabulated in Tables 1, 2.

TABLE 2. The absolute errors of the discrete solution U at $x = 0.2480$

t	Numerical solution	Exact solution	Error
0.0	0.2866	0.2866	0
0.0002	0.2867	0.2869	0.1708e-3
0.0005	0.2869	0.2872	0.3357e-3
0.0007	0.2871	0.2876	0.5001e-3
0.0010	0.2872	0.2879	0.6646e-3
0.0013	0.2874	0.2882	0.8293e-3

TABLE 3. The absolute errors of the discrete solution S at some points.

t	Numerical solution	Exact solution	Error
0.0	0.5000	0.5000	0
0.0002	0.5002	0.5002	0.1670e-4
0.0005	0.5005	0.5005	0.2275e-4
0.0007	0.5007	0.5007	0.2680e-4
0.0010	0.5010	0.5010	0.2965e-4
0.0013	0.5013	0.5013	0.3164e-4

TABLE 4. The absolute errors of the discrete solution U at $x = 0.2480$

t	Numerical solution	Exact solution	Error
0.0	0.2520	0.2520	0
0.0002	0.2521	0.2522	0.1317e-3
0.0005	0.2522	0.2525	0.2587e-3
0.0007	0.2524	0.2527	0.3854e-3
0.0010	0.2525	0.2530	0.5120e-3
0.0013	0.2526	0.2532	0.6387e-3

Example 5.2.

$$\begin{aligned}
&U_t - U_{xx} = 1, \quad 0 < x < S(t), 0 < t < 0.5,\\
&U_x(0,t) = -2(U(0,t) - t), \quad 0 < t < 0.5,\\
&U(S(t),t) = 0, \quad 0 < t < 0.5,\\
&U_x(S(t),t) = -\dot{s}(t), \quad 0 < t < 0.5,\\
&U(x,0) = \frac{1}{2} - x, \quad 0 < x \leq 0.5,\\
&S(0) = 0.5,
\end{aligned}$$

where $g(t) = -2$, $f(t) = t$, $F(W) = -1$ and $T = 0.5$. The exact solution to the problem is

$$U(x,t) = t - x + 0.5, \quad S(t) = t + 0.5.$$

This problem is solved by the method (2.5) and (2.8) with $M = 5001$ and $N = 11$. In Figure 2, we have plotted the continuous moving boundary $S(t)$ and its discrete solution s. The absolute errors of the discrete solution (s,u) are tabulated in Tables 3, 4.

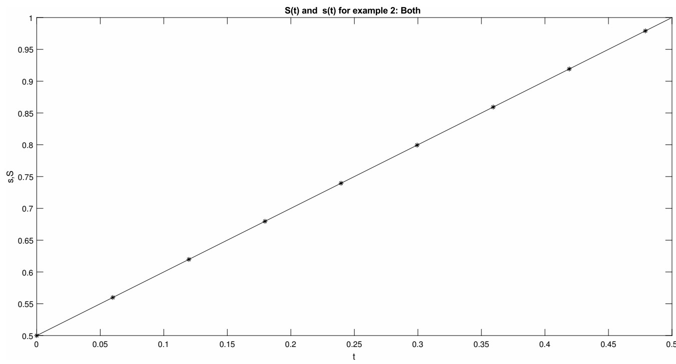

FIGURE 2. Comparison of the diagram of the moving boundary as predicted by the numerical with the exact solution.

6. Conclusions

In this paper, we derived a numerical method based on explicit finite-difference approximation for a one-phase Stefan problem for the non-classical heat equation with a convective condition. We have studied the convergence and compatibility of the method. We have given two examples to show the efficiency of the method.

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