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Existence of nonoscillatory solutions of second-order differential equations with mixed neutral term

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Abstract

In this study, we aim to contribute to the increasing interest in functional differential equations by obtaining new existence theorems for non-oscillatory solutions of second-order neutral differential equations involving positive and negative terms which have not been performed in previous studies. We consider different cases for the ranges of the neutral coefficients, by utilizing the Banach contraction mapping principle. The applicability of the results is illustrated by several examples in the last section.

Keywords. Neutral differential equations, Fixed point, Nonoscillatory solution.

2010 Mathematics Subject Classification. 34K11, 34K40, 34A12.

1. Introduction

Differential equations (DEs) with retarded and advanced arguments, known as mixed DEs, appear in many studies in both natural sciences and engineering, for example in the problems of optimal control theory [17], deceleration of neutrons in nuclear reactors [18], models for economic dynamics [19], nerve conduction theory [7], and in a spatial lattice identification of moving waves [14]. Therefore, researches on the properties of solutions of mixed neutral differential equations (NDEs) are of great value regardless of the theory of DEs or their practical applications.

It is a well-known fact that the investigation of oscillatory solutions is very important because of the large number of applications in practical problems. Besides, investigating the existence of non-oscillatory solutions has equally importance. Because when we establish the existence theorems for non-oscillatory solutions completely, the nonexistence criteria for oscillatory solutions are also determined. The existence of non-oscillatory solutions for ordinary DEs or dynamic equations on time scales has been researched by many scientists, see e.g. [10, 11, 16]. Meanwhile, the problem of the existence of non-oscillatory solutions of the NDEs has been studied extensively in the last years. We refer the reader to the papers [1–3, 6, 9, 20, 22, 23] and references cited therein for recent results on this topic.

In 2005, Zhang et al. [21] dealt with the existence of non-oscillatory solutions of the first order neutral delayed DEs with variable coefficients. The authors obtained sufficient conditions for the existence of non-oscillatory solutions turning on the some different intervals of neutral coefficients. Candan [4] and Mansouri et. al. [15] discussed finding existence criteria for non-oscillatory solutions of first-order NDEs of mixed type, by utilizing Banach's fixed point theorem. In [8], Kong considered a first order mixed NDE involving variable neutral coefficients with their different ranges and established several new existence theorems for non-oscillatory solutions. By using the Banach contraction principle, Candan [5] presented some conditions which ensure the existence of non-oscillatory solutions to a higher order NDE with variable coefficients. In [13], Li and Sun obtained several new theorems for non-oscillatory solutions of higher order NDEs by Schauder–Tychonoff fixed point theorem. Moreover, existence theorems for non-oscillatory solutions of second order mixed type NDEs with positive and negative terms were studied by Li et. al in [12].

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In this study, by defining a neutral term that includes both delayed and advanced arguments of the form

$$\mathcal{Z}(t) = y(t) + \mathcal{R}_1(t)\mathcal{F}_1\big(y\big(t-\zeta_1(t)\big)\big) + \mathcal{R}_2(t)\mathcal{F}_2\big(y\big(t+\zeta_2(t)\big)\big), \tag{N}$$

we deal with the existence of non-oscillatory solutions of second-order nonlinear mixed type NDEs of the form

$$\Big(a(t)\mathcal{Z}'(t)\Big)' + \sum_{i=1}^{n}\Big(G_i(t)y(t-g_i(t))\Big) - \sum_{i=1}^{m}\Big(H_i(t)y(t+h_i(t))\Big) = \psi(t), \tag{E}$$

for $t \geq t_0 > 0$, where the following requirements are always supposed to hold:

(i) $n \geq 1$ and $m \geq 1$ are natural numbers, $\psi, \mathcal{R}_i : [t_0,\infty) \to \mathbb{R}$ are continuous functions for $i = 1,2$ and $a \in C([t_0,\infty),(0,\infty))$ with $\int_{t_0}^{\infty} 1/a(\eta)\mathrm{d}\eta < \infty$;

(ii) $\mathcal{F}_1, \mathcal{F}_2 : [t_0,\infty) \to [0,\infty)$ are continuous functions and there exist two positive constants $\mathcal{A}_1$ and $\mathcal{A}_2$ such that

$$0 \leq \mathcal{F}_1(\xi) \leq \mathcal{A}_1\xi, \qquad 0 \leq \mathcal{F}_2(\xi) \leq \mathcal{A}_2\xi.$$

(iii) $G_j, H_k : [t_0,\infty) \to [0,\infty)$ are continuous functions such that not all of the $G_j(t)$ and $H_k(t)$ vanish in a neighborhood of infinity for $j = 1,2,...,n$ and $k = 1,2,...,m$;

(iv) $\zeta_1(t) > 0$, $\zeta_2(t) > 0$, $g_j(t) \geq 0$ with $t - \zeta_1(t)$ and $t - g_j(t)$ are increasing functions for $j = 1,2,...,n$ and $h_k(t) \geq 0$ for $k = 1,2,...,m$.

The purpose of this study is to obtain some new sufficient conditions that ensure the existence of non-oscillatory solutions of NDE (E), by utilizing the Banach contraction mapping principle. To set up our main results, we consider different cases for the ranges of the neutral coefficients $\mathcal{R}_1(t)$ and $\mathcal{R}_2(t)$.

Let $\alpha = \max\{\zeta_1(t), g_1(t), g_2(t), ..., g_n(t)\}$. By a solution of the NDE (E) we understand a function $y \in C\left([T_x-\alpha,\infty),\mathbb{R}\right)$ such that $\mathcal{Z}, a\mathcal{Z}' \in C^1\left([T_x,\infty),\mathbb{R}\right)$ and satisfies NDE (E) on $([T_x,\infty),\mathbb{R})$. As usual, such a nontrivial solution of (E) is said to be *oscillatory* if it is neither eventually negative nor eventually positive, and otherwise it is called *non-oscillatory.*

2. Main results

Theorem 2.1. *Suppose that* $0 \leq \mathcal{R}_1(t) \leq r_1 < 1$, $0 \leq \mathcal{R}_2(t) \leq r_2 < 1 - r_1$ *and*

$$\int_{t_0}^{\infty} \frac{1}{a(\eta)} \int_{t_0}^{\eta} \left(\sum_{i=1}^{n} G_i(\vartheta)\right) \mathrm{d}\vartheta \mathrm{d}\eta < \infty, \tag{2.1}$$

$$\int_{t_0}^{\infty} \frac{1}{a(\eta)} \int_{t_0}^{\eta} \left(\sum_{i=1}^{m} H_i(\vartheta)\right) \mathrm{d}\vartheta \mathrm{d}\eta < \infty, \qquad \int_{t_0}^{\infty} \frac{1}{a(\eta)} \int_{t_0}^{\eta} \big|\psi(\vartheta)\big| \mathrm{d}\vartheta \mathrm{d}\eta < \infty. \tag{2.2}$$

Then Eq. (E) *has one bounded non-oscillatory solution.*

Proof. In view of (2.1) and (2.2), a $t_1 > t_0$ can be chosen with

$$t_1 \geq t_0 + \max\left\{\sup_{t\geq t_0}\zeta_1(t), \sup_{t\geq t_0} g_1(t), \sup_{t\geq t_0} g_2(t), ..., \sup_{t\geq t_0} g_n(t)\right\}, \tag{2.3}$$

sufficiently large such that

$$\int_{t_1}^{\infty} \frac{1}{a(\eta)} \int_{t_1}^{\eta} \left(\sum_{i=1}^{n} G_i(\vartheta)\mathcal{E}_2 + \big|\psi(\vartheta)\big|\right) \mathrm{d}\vartheta \mathrm{d}\eta \leq \mathcal{E}_2 - \gamma, \tag{2.4}$$

$$\int_{t_1}^{\infty} \frac{1}{a(\eta)} \int_{t_1}^{\eta} \left(\sum_{i=1}^{m} H_i(\vartheta)\mathcal{E}_2 + \big|\psi(\vartheta)\big|\right) \mathrm{d}\vartheta \mathrm{d}\eta \leq \gamma - \mathcal{E}_1 - (r_1\mathcal{A}_1 + r_2\mathcal{A}_2)\mathcal{E}_2, \tag{2.5}$$

and

$$\int_{t_1}^{\infty} \frac{1}{a(\eta)} \int_{t_1}^{\eta} \left(\sum_{i=1}^{n} G_i(\vartheta) + \sum_{i=1}^{m} H_i(\vartheta) \right) \mathrm{d}\vartheta \mathrm{d}\eta \leq 1 - r_1\mathcal{A}_1 - r_2\mathcal{A}_2 - \frac{\mathcal{E}_1}{\mathcal{E}_2}, \tag{2.6}$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are positive constants such that

$$\mathcal{E}_1 + (r_1\mathcal{A}_1 + r_2\mathcal{A}_2)\mathcal{E}_2 < \mathcal{E}_2 \quad \text{and} \quad \gamma \in \big(\mathcal{E}_1 + (r_1\mathcal{A}_1 + r_2\mathcal{A}_2)\mathcal{E}_2, \mathcal{E}_2\big).$$

Let Ω be the set of all continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Then, Ω is a complete metric space. Set

$$\Psi = \Big\{ y \in \Omega : \mathcal{E}_1 \leq y(t) \leq \mathcal{E}_2, t \geq t_0 \Big\}.$$

Obviously, Ψ is a bounded, closed and convex sub-set of Ω. Consider the operator $\mathcal{T} : \Psi \to \Omega$ as follows:

$$(\mathcal{T}y)(t) = \begin{cases} \gamma - \mathcal{R}_1(t)\mathcal{F}_1\big(y\big(t-\zeta_1(t)\big)\big) - \mathcal{R}_2(t)\mathcal{F}_2\big(y\big(t+\zeta_2(t)\big)\big) \\ + \int_t^\infty \frac{1}{a(\eta)} \int_{t_1}^\eta \Big[\sum_{i=1}^n G_i(\vartheta) y(\vartheta - g_i(\vartheta)) \\ - \sum_{i=1}^m H_i(\vartheta) y(\vartheta + h_i(\vartheta)) - \psi(\vartheta) \Big] \mathrm{d}\vartheta \mathrm{d}\eta, \qquad t \geq t_1, \\ (\mathcal{T}y)(t_1), \qquad t_0 \leq t \leq t_1. \end{cases} \tag{2.7}$$

Clearly, $\mathcal{T}y$ is continuous. Meanwhile, for any $y \in \Psi$ and $t \geq t_1$, from condition (ii) and inequality (2.4), we have

$$\begin{aligned} (\mathcal{T}y)(t) &\leq \gamma + \int_t^\infty \frac{1}{a(\eta)} \int_{t_1}^\eta \left(\sum_{i=1}^n G_i(\vartheta) y(\vartheta - g_i(\vartheta)) - \psi(\vartheta) \right) \mathrm{d}\vartheta \mathrm{d}\eta \\ &\leq \gamma + \int_{t_1}^\infty \frac{1}{a(\eta)} \int_{t_1}^\eta \left(\sum_{i=1}^n G_i(\vartheta)\mathcal{E}_2 + \big|\psi(\vartheta)\big| \right) \mathrm{d}\vartheta \mathrm{d}\eta \leq \mathcal{E}_2, \end{aligned} \tag{2.8}$$

and from (2.5), we see that

$$\begin{aligned} (\mathcal{T}y)(t) &\geq \gamma - \mathcal{R}_1(t)\mathcal{F}_1\big(y\big(t-\zeta_1(t)\big)\big) - \mathcal{R}_2(t)\mathcal{F}_2\big(y\big(t+\zeta_2(t)\big)\big) \\ &\quad - \int_t^\infty \frac{1}{a(\eta)} \int_{t_1}^\eta \left(\sum_{i=1}^m H_i(\vartheta) y(\vartheta + h_i(\vartheta)) + \psi(\vartheta) \right) \mathrm{d}\vartheta \mathrm{d}\eta \\ &\geq \gamma - (r_1\mathcal{A}_1 + r_2\mathcal{A}_2)\mathcal{E}_2 \\ &\quad - \int_{t_1}^\infty \frac{1}{a(\eta)} \int_{t_1}^\eta \left(\sum_{i=1}^m H_i(\vartheta)\mathcal{E}_2 + \big|\psi(\vartheta)\big| \right) \mathrm{d}\vartheta \mathrm{d}\eta \geq \mathcal{E}_1. \end{aligned} \tag{2.9}$$

The inequalities (2.8) and (2.9) imply that $\mathcal{T}\Psi \subset \Psi$. So, in order to apply the contraction mapping principle, it is sufficient to signify that $\mathcal{T}$ is a contraction mapping on Ψ. Thus, for every $y_1, y_2 \in \Psi$ and $t \geq t_1$, we have

$$\begin{aligned} \big|(\mathcal{T}y_1)(t) - (\mathcal{T}y_2)(t)\big| &\leq \mathcal{R}_1(t)\big|\mathcal{F}_1\big(y_1\big(t-\zeta_1(t)\big)\big) - \mathcal{F}_1\big(y_2\big(t-\zeta_1(t)\big)\big)\big| \\ &+ \mathcal{R}_2(t)\big|\mathcal{F}_2\big(y_1\big(t+\zeta_2(t)\big)\big) - \mathcal{F}_2\big(y_2\big(t+\zeta_2(t)\big)\big)\big| \\ &+ \int_t^\infty \frac{1}{a(\eta)} \int_{t_1}^\eta \left(\sum_{i=1}^n G_i(\vartheta)\big|y_1(\vartheta - g_i(\vartheta)) - y_2(\vartheta - g_i(\vartheta))\big| \right. \\ &+ \left. \sum_{i=1}^m H_i(\vartheta)\big|y_1(\vartheta + h_i(\vartheta)) - y_2(\vartheta + h_i(\vartheta))\big| \right) \mathrm{d}\vartheta \mathrm{d}\eta, \end{aligned}$$

which implies that

$$\begin{aligned} \big|(\mathcal{T}y_1)(t)-(\mathcal{T}y_2)(t)\big| &\leq \big\|y_1-y_2\big\|\left[r_1\mathcal{A}_1+r_2\mathcal{A}_2+\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)+\sum_{i=1}^{m}H_i(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta\right] \\ &\leq \left[1-\frac{\mathcal{E}_1}{\mathcal{E}_2}\right]\big\|y_1-y_2\big\| = \mathcal{C}_1\big\|y_1-y_2\big\|. \end{aligned}$$

This shows with the sup norm that

$$\big|\mathcal{T}y_1-\mathcal{T}y_2\big| \leq \mathcal{C}_1\big\|y_1-y_2\big\|.$$

Since $\mathcal{C}_1 < 1$, $\mathcal{T}$ is a contraction mapping on Ψ. Therefore, there exists a unique bounded non-oscillatory solution, clearly a positive solution of Eq. (E) such that $y \in \Psi$ of $\mathcal{T}y = y$. □

Theorem 2.2. *Assume that (2.1) and (2.2) hold,* $0 \leq \mathcal{R}_1(t) \leq r_1 < 1$ *and* $r_1 - 1 < r_2 \leq \mathcal{R}_2(t) \leq 0$. *Then Eq. (E) has one bounded non-oscillatory solution.*

Proof. In view of (2.1) and (2.2), a sufficiently large $t_1 > t_0$ can be chosen satisfying (2.3) such that

$$\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)\mathcal{E}_4+\big|\psi(\vartheta)\big|\right)\mathrm{d}\vartheta\mathrm{d}\eta \leq (1+r_2\mathcal{A}_2)\mathcal{E}_4-\gamma, \tag{2.10}$$

$$\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{m}H_i(\vartheta)\mathcal{E}_4+\big|\psi(\vartheta)\big|\right)\mathrm{d}\vartheta\mathrm{d}\eta \leq \gamma-\mathcal{E}_3-r_1\mathcal{A}_1\mathcal{E}_4, \tag{2.11}$$

and

$$\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)+\sum_{i=1}^{m}H_i(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta \leq 1-r_1\mathcal{A}_1+r_2\mathcal{A}_2-\frac{\mathcal{E}_3}{\mathcal{E}_4}, \tag{2.12}$$

where $\mathcal{E}_3$ and $\mathcal{E}_4$ are positive constants such that

$$r_1\mathcal{A}_1\mathcal{E}_4+\mathcal{E}_3 < (1+r_2\mathcal{A}_2)\mathcal{E}_4 \quad \text{and} \quad \gamma \in \big(r_1\mathcal{A}_1\mathcal{E}_4+\mathcal{E}_3, (1+r_2\mathcal{A}_2)\mathcal{E}_4\big).$$

Let Ω be the set of all continuous and bounded functions on $[t_0, \infty)$ with the supremum norm. Then, Ω is a complete metric space. Set

$$\Psi = \Big\{y \in \Omega : \mathcal{E}_3 \leq y(t) \leq \mathcal{E}_4, t \geq t_0\Big\}.$$

Obviously, Ψ is a bounded, closed and convex subset of Ω. Consider the operator $\mathcal{T} : \Psi \to \Omega$ defined in (2.7). For any $y \in \Psi$ and $t \geq t_1$, from condition (ii) and inequality (2.10), we have

$$\begin{aligned} (\mathcal{T}y)(t) &\leq \gamma-\mathcal{R}_2(t)\mathcal{F}_2\big(y\big(t+\zeta_2(t)\big)\big)+\int_{t}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)y(\vartheta-g_i(\vartheta))-\psi(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta \\ &\leq \gamma-r_2\mathcal{A}_2\mathcal{E}_4+\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)\mathcal{E}_4+\big|\psi(\vartheta)\big|\right)\mathrm{d}\vartheta\mathrm{d}\eta \leq \mathcal{E}_4, \end{aligned} \tag{2.13}$$

and from (2.11), we see that

$$\begin{aligned} (\mathcal{T}y)(t) &\geq \gamma-\mathcal{R}_1(t)\mathcal{F}_1\big(y\big(t-\zeta_1(t)\big)\big)-\int_{t}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{m}H_i(\vartheta)y(\vartheta+h_i(\vartheta))+\psi(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta \\ &\geq \gamma-r_1\mathcal{A}_1\mathcal{E}_4-\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{m}H_i(\vartheta)\mathcal{E}_4+\big|\psi(\vartheta)\big|\right)\mathrm{d}\vartheta\mathrm{d}\eta \geq \mathcal{E}_3. \end{aligned} \tag{2.14}$$

These last two inequalities imply that $\mathcal{T}\Psi \subset \Psi$. So, in order to apply the contraction mapping principle, it is sufficient to show that $\mathcal{T}$ is a contraction mapping on Ψ. Thus, for every $y_1, y_2 \in \Psi$ and $t \geq t_1$, we have

$$\begin{aligned}\big|(\mathcal{T}y_1)(t)-(\mathcal{T}y_2)(t)\big| \leq \mathcal{R}_1(t)\big|\mathcal{F}_1\big(y_1\big(t-\zeta_1(t)\big)\big)-\mathcal{F}_1\big(y_2\big(t-\zeta_1(t)\big)\big)\big| - \mathcal{R}_2(t)\big|\mathcal{F}_2\big(y_1\big(t+\zeta_2(t)\big)\big)-\mathcal{F}_2\big(y_2\big(t+\zeta_2(t)\big)\big)\big| \\ + \int_t^\infty \frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n} G_i(\vartheta)\big|y_1(\vartheta-g_i(\vartheta))-y_2(\vartheta-g_i(\vartheta))\big| + \sum_{i=1}^{m} H_i(\vartheta)\big|y_1(\vartheta+h_i(\vartheta))-y_2(\vartheta+h_i(\vartheta))\big|\right)\mathrm{d}\vartheta\mathrm{d}\eta,\end{aligned}$$

which implies that

$$\begin{aligned}\big|(\mathcal{T}y_1)(t)-(\mathcal{T}y_2)(t)\big| &\leq \big\|y_1-y_2\big\|\left[r_1\mathcal{A}_1 - r_2\mathcal{A}_2 + \int_t^\infty \frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n} G_i(\vartheta)+\sum_{i=1}^{m} H_i(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta\right] \\ &\leq \left[1-\frac{\mathcal{E}_3}{\mathcal{E}_4}\right]\big\|y_1-y_2\big\| = \mathcal{C}_2\big\|y_1-y_2\big\|.\end{aligned}$$

This shows with the sup norm that

$$\big|\mathcal{T}y_1-\mathcal{T}y_2\big| \leq \mathcal{C}_2\big\|y_1-y_2\big\|.$$

Since $\mathcal{C}_2 < 1$, $\mathcal{T}$ is a contraction mapping on Ψ. Therefore, there exists a unique bounded non-oscillatory solution, clearly a positive solution of Eq. (E) such that $y \in \Psi$ of $\mathcal{T}y = y$. □

Theorem 2.3. *Suppose that (2.1) and (2.2) hold, the inverse functions of $t-\zeta_1(t)$ and $\mathcal{F}_1$ exist with $\big(t-\zeta_1(t)\big)^{-1} = \phi(t) \geq t$, and there exist two positive constants $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\mathcal{B}_1\xi \leq \mathcal{F}_1^{-1}(\xi) \leq \mathcal{B}_2\xi$. If $1 < r_1 \leq \mathcal{R}_1(t) \leq r_* < \infty$ and $0 \leq \mathcal{R}_2(t) \leq r_2 < r_1 - 1$, then Eq. (E) has one bounded non-oscillatory solution.*

Proof. In view of (2.1) and (2.2), a $t_1 > t_0$ can be chosen with

$$\phi(t_1) - g_i(\phi(t_1)) \geq t_0, \quad i = 1, 2, ..., n, \tag{2.15}$$

sufficiently large such that

$$\int_{t_1}^\infty \frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n} G_i(\vartheta)\mathcal{E}_6 + \big|\psi(\vartheta)\big|\right)\mathrm{d}\vartheta\mathrm{d}\eta \leq \frac{r_1\mathcal{E}_6}{\mathcal{B}_2} - \gamma, \tag{2.16}$$

$$\int_{t_1}^\infty \frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{m} H_i(\vartheta)\mathcal{E}_6 + \big|\psi(\vartheta)\big|\right)\mathrm{d}\vartheta\mathrm{d}\eta \leq \gamma - (1+r_2\mathcal{A}_2)\mathcal{E}_6 - \frac{r_*\mathcal{E}_5}{\mathcal{B}_1}, \tag{2.17}$$

and

$$\int_{t_1}^\infty \frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n} G_i(\vartheta) + \sum_{i=1}^{m} H_i(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta \leq \frac{r_1}{\mathcal{B}_2} - (1+r_2\mathcal{A}_2) - \frac{r_*\mathcal{E}_5}{\mathcal{B}_1\mathcal{E}_6}, \tag{2.18}$$

where $\mathcal{E}_5$ and $\mathcal{E}_6$ are positive constants such that

$$(1+r_2\mathcal{A}_2)\mathcal{E}_6 + \frac{r_*\mathcal{E}_5}{\mathcal{B}_1} < \frac{r_1\mathcal{E}_6}{\mathcal{B}_2} \text{ and } \gamma \in \left((1+r_2\mathcal{A}_2)\mathcal{E}_6 + \frac{r_*\mathcal{E}_5}{\mathcal{B}_1}, \frac{r_1\mathcal{E}_6}{\mathcal{B}_2}\right). \tag{2.19}$$

With the supremum norm, let Ω be the set of all bounded and continuous functions on $[t_0, \infty)$. Then, Ω is a complete metric space. Set

$$\Psi = \Big\{y \in \Omega : \mathcal{E}_5 \leq y(t) \leq \mathcal{E}_6, t \geq t_0\Big\}.$$

Consider the operator $\mathcal{T}:\Psi\to\Omega$ as follows:

$$(\mathcal{T}y)(t)=\begin{cases}\mathcal{F}_1^{-1}\Big\{\frac{1}{\mathcal{R}_1(\phi(t))}\Big(\gamma-y(\phi(t))\\ -\mathcal{R}_2(\phi(t))\mathcal{F}_2\big(y(\phi(t))+\zeta_2(\phi(t))\big)\\ +\int_{\phi(t)}^{\infty}\frac{1}{a(\eta)}\int_{\phi(t_1)}^{\eta}\Big[\sum_{i=1}^{n}G_i(\vartheta)y(\vartheta-g_i(\vartheta))\\ -\sum_{i=1}^{m}H_i(\vartheta)y(\vartheta+h_i(\vartheta))-\psi(\vartheta)\Big]\mathrm{d}\vartheta\mathrm{d}\eta\Big)\Big\}, & t\geq t_1,\\ \mathcal{T}(y)(t_1), & t_0\leq t\leq t_1.\end{cases}\tag{2.20}$$

Obviously, $\mathcal{T}y$ is continuous. Meanwhile, for any $y\in\Psi$ and $t\geq t_1$, from (2.16) and (2.17), it follows that

$$\begin{aligned}(\mathcal{T}y)(t) &\leq \frac{\mathcal{B}_2}{\mathcal{R}_1(\phi(t))}\left(\gamma+\int_{\phi(t)}^{\infty}\frac{1}{a(\eta)}\int_{\phi(t_1)}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)y(\vartheta-g_i(\vartheta))-\psi(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta\right)\\ &\leq \frac{\mathcal{B}_2}{r_1}\left(\gamma+\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)\mathcal{E}_6+\big|\psi(\vartheta)\big|\right)\mathrm{d}\vartheta\mathrm{d}\eta\right)\leq\mathcal{E}_6,\end{aligned}$$

and

$$\begin{aligned}(\mathcal{T}y)(t) &\geq \mathcal{B}_1\Bigg(\frac{1}{\mathcal{R}_1(\phi(t))}\Big[\gamma-y(\phi(t))-\mathcal{R}_2(\phi(t))\mathcal{F}_2\big(y(\phi(t))+\zeta_2(\phi(t))\big)\\ &\quad-\int_{\phi(t)}^{\infty}\frac{1}{a(\eta)}\int_{\phi(t_1)}^{\eta}\left(\sum_{i=1}^{m}H_i(\vartheta)y(\vartheta+h_i(\vartheta))+\psi(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta\Big]\Bigg)\\ &\geq \frac{\mathcal{B}_1}{r_*}\Bigg(\gamma-\mathcal{E}_6-r_2\mathcal{A}_2\mathcal{E}_6\\ &\quad-\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{m}H_i(\vartheta)y(\vartheta+h_i(\vartheta))+\big|\psi(\vartheta)\big|\right)\mathrm{d}\vartheta\mathrm{d}\eta\Bigg)\geq\mathcal{E}_5.\end{aligned}$$

This means that $\mathcal{T}\Psi\subset\Psi$. So, it is sufficient to show that $\mathcal{T}$ is a contraction mapping on Ψ. Thus, for every $y_1,y_2\in\Psi$ and $t\geq t_1$, we have

$$\begin{aligned}\big|(\mathcal{T}y_1)(t)-(\mathcal{T}y_2)(t)\big| &\leq \frac{\mathcal{B}_2}{\mathcal{R}_1(\phi(t))}\Big\{\big|y_1(\phi(t))-y_2(\phi(t))\big|+\mathcal{R}_2(\phi(t))\mathcal{A}_2\big|y_1\big[\phi(t)+\zeta_2(\phi(t))\big]-y_2\big[\phi(t)+\zeta_2(\phi(t))\big]\big|\\ &+\int_{\phi(t)}^{\infty}\frac{1}{a(\eta)}\int_{\phi(t_1)}^{\eta}\Big(G_i(\vartheta)\big|y_1(\vartheta-g_i(\vartheta))-y_2(\vartheta-g_i(\vartheta))\big|\\ &+H_i(\vartheta)\big|y_1(\vartheta+h_i(\vartheta))-y_2(\vartheta+h_i(\vartheta))\big|\Big)\mathrm{d}\vartheta\mathrm{d}\eta\Big\},\end{aligned}$$

or

$$\begin{aligned}\big|(\mathcal{T}y_1)(t)-(\mathcal{T}y_2)(t)\big| &\leq \|y_1-y_2\|\frac{\mathcal{B}_2}{r_1}\left(1+r_2\mathcal{A}_2+\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)+\sum_{i=1}^{m}H_i(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta\right)\\ &\leq \|y_1-y_2\|\left[1-\frac{\mathcal{B}_2r_*\mathcal{E}_5}{\mathcal{B}_1r_1\mathcal{E}_6}\right]=\mathcal{C}_3\big\|y_1-y_2\big\|.\end{aligned}$$

This shows with the sup norm that

$$\big|\mathcal{T}y_1-\mathcal{T}y_2\big|\leq\mathcal{C}_3\big\|y_1-y_2\big\|.$$

Since $\mathcal{C}_3 < 1$ from (2.19), $\mathcal{T}$ is a contraction mapping on Ψ. Therefore, there exists a unique bounded non-oscillatory solution, clearly a positive solution of Eq. (E) such that $y \in \Psi$ of $\mathcal{T}y = y$. □

Theorem 2.4. *Suppose that (2.1) and (2.2) hold, the inverse functions of $t - \zeta_1(t)$ and $\mathcal{F}_1$ exist with $\big(t - \zeta_1(t)\big)^{-1} = \phi(t) \geq t$, and there exist two positive constants $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\mathcal{B}_1\xi \leq \mathcal{F}_1^{-1}(\xi) \leq \mathcal{B}_2\xi$. If $1 < r_1 \leq \mathcal{R}_1(t) \leq r_* < \infty$ and $1 - r_1 < r_2 \leq \mathcal{R}_2(t) \leq 0$, then Eq. (E) has one bounded non-oscillatory solution.*

Proof. In view of (2.1) and (2.2), a $t_1 \geq t_0$ can be chosen sufficiently large satisfying (2.15) such that

$$\int_{t_1}^{\infty} \frac{1}{a(\eta)} \int_{t_1}^{\eta} \left(\sum_{i=1}^{n} G_i(\vartheta)\mathcal{E}_8 + \big|\psi(\vartheta)\big| \right) \mathrm{d}\vartheta \mathrm{d}\eta \leq \left(\frac{r_1}{\mathcal{B}_2} + r_2\mathcal{A}_2 \right) \mathcal{E}_8 - \gamma, \tag{2.21}$$

$$\int_{t_1}^{\infty} \frac{1}{a(\eta)} \int_{t_1}^{\eta} \left(\sum_{i=1}^{m} H_i(\vartheta)\mathcal{E}_8 + \big|\psi(\vartheta)\big| \right) \mathrm{d}\vartheta \mathrm{d}\eta \leq \gamma - \mathcal{E}_8 - \frac{r_*}{\mathcal{B}_1}\mathcal{E}_7, \tag{2.22}$$

and

$$\int_{t_1}^{\infty} \frac{1}{a(\eta)} \int_{t_1}^{\eta} \left(\sum_{i=1}^{n} G_i(\vartheta) + \sum_{i=1}^{m} H_i(\vartheta) \right) \mathrm{d}\vartheta \mathrm{d}\eta \leq \frac{r_1}{\mathcal{B}_2} - 1 + r_2\mathcal{A}_2 - \frac{r_*\mathcal{E}_7}{\mathcal{B}_1\mathcal{E}_8}, \tag{2.23}$$

where $\mathcal{E}_7$ and $\mathcal{E}_8$ are positive constants such that

$$\mathcal{E}_8 + \frac{r_*\mathcal{E}_7}{\mathcal{B}_1} < \left(\frac{r_1}{\mathcal{B}_2} + r_2\mathcal{A}_2 \right) \mathcal{E}_8 \text{ and } \ \gamma \in \left(\mathcal{E}_8 + \frac{r_*\mathcal{E}_7}{\mathcal{B}_1}, \left(\frac{r_1}{\mathcal{B}_2} + r_2\mathcal{A}_2 \right) \mathcal{E}_8 \right). \tag{2.24}$$

With the supremum norm, let Ω be the set of all bounded and continuous functions on $[t_0, \infty)$. Set

$$\Psi = \Big\{ y \in \Omega : \mathcal{E}_7 \leq y(t) \leq \mathcal{E}_8, t \geq t_0 \Big\}.$$

Consider the operator $\mathcal{T} : \Psi \to \Omega$ defined in (2.20). Then, for any $y \in \Psi$, from (2.21) and (2.22) for $t \geq t_1$, we have

$$\begin{aligned}
(\mathcal{T}y)(t) \ &\leq \ \mathcal{B}_2 \Bigg(\frac{1}{\mathcal{R}_1(\phi(t))} \Big[\gamma - \mathcal{R}_2(\phi(t))\mathcal{F}_2\big(y(\phi(t)) + \zeta_2(\phi(t))\big) \\
&\quad + \int_{\phi(t)}^{\infty} \frac{1}{a(\eta)} \int_{\phi(t_1)}^{\eta} \left(\sum_{i=1}^{n} G_i(\vartheta) y(\vartheta - g_i(\vartheta)) - \psi(\vartheta) \right) \mathrm{d}\vartheta \mathrm{d}\eta \Big] \Bigg) \\
&\leq \ \frac{\mathcal{B}_2}{r_1} \left(\gamma - r_2\mathcal{A}_2\mathcal{E}_8 + \int_{t_1}^{\infty} \frac{1}{a(\eta)} \int_{t_1}^{\eta} \left(\sum_{i=1}^{n} G_i(\vartheta)\mathcal{E}_8 + \big|\psi(\vartheta)\big| \right) \mathrm{d}\vartheta \mathrm{d}\eta \right) \leq \mathcal{E}_8,
\end{aligned}$$

and

$$\begin{aligned}
(\mathcal{T}y)(t) \ &\geq \ \mathcal{B}_1 \Bigg(\frac{1}{\mathcal{R}_1(\phi(t))} \Big[\gamma - y(\phi(t)) \\
&\quad - \int_{\phi(t)}^{\infty} \frac{1}{a(\eta)} \int_{\phi(t_1)}^{\eta} \left(\sum_{i=1}^{m} H_i(\vartheta) y(\vartheta + h_i(\vartheta)) + \psi(\vartheta) \right) \mathrm{d}\vartheta \mathrm{d}\eta \Big] \Bigg) \\
&\geq \ \frac{\mathcal{B}_1}{r_*} \left(\gamma - \mathcal{E}_8 - \int_{t_1}^{\infty} \frac{1}{a(\eta)} \int_{t_1}^{\eta} \left(\sum_{i=1}^{m} H_i(\vartheta)\mathcal{E}_8 + \big|\psi(\vartheta)\big| \right) \mathrm{d}\vartheta \mathrm{d}\eta \right) \geq \mathcal{E}_7.
\end{aligned}$$

Using the above one can conclude $\mathcal{T}\Psi \subset \Psi$. On the other hand, for every $y_1, y_2 \in \Psi$ and $t \geq t_1$, we have

$$\begin{aligned}\big|(\mathcal{T}y_1)(t) - (\mathcal{T}y_2)(t)\big| &\leq \|y_1 - y_2\| \frac{\mathcal{B}_2}{r_1}\left(1 - r_2\mathcal{A}_2 + \int_{\phi(t)}^{\infty} \frac{1}{a(\eta)} \int_{\phi(t_1)}^{\eta} \left(\sum_{i=1}^{n} G_i(\vartheta) + \sum_{i=1}^{m} H_i(\vartheta)\right) \mathrm{d}\vartheta \mathrm{d}\eta\right)\\ &\leq \|y_1 - y_2\| \frac{\mathcal{B}_2}{r_1}\left(1 - r_2\mathcal{A}_2 + \int_{t_1}^{\infty} \frac{1}{a(\eta)} \int_{t_1}^{\eta} \left(\sum_{i=1}^{n} G_i(\vartheta) + \sum_{i=1}^{m} H_i(\vartheta)\right) \mathrm{d}\vartheta \mathrm{d}\eta\right)\\ &\leq \|y_1 - y_2\| \left[1 - \frac{\mathcal{B}_2 r_* \mathcal{E}_7}{\mathcal{B}_1 r_1 \mathcal{E}_8}\right] = \mathcal{C}_4\big\|y_1 - y_2\big\|,\end{aligned}$$

this shows with the sup norm that

$$\big|\mathcal{T}y_1 - \mathcal{T}y_2\big| \leq \mathcal{C}_4\big\|y_1 - y_2\big\|.$$

Since $\mathcal{C}_4 < 1$ from (2.24), $\mathcal{T}$ is a contraction mapping on Ψ. Therefore, there exists a unique bounded non-oscillatory solution, clearly a positive solution of Eq. (E) such that $y \in \Psi$ of $\mathcal{T}y = y$. □

Theorem 2.5. *Suppose that* (2.1) *and* (2.2) *hold,* $-1 < r_1 \leq \mathcal{R}_1(t) \leq 0$ *and* $0 \leq \mathcal{R}_2(t) \leq r_2 < 1 + r_1$. *Then Eq.* (E) *has one bounded non-oscillatory solution.*

Proof. Due to (2.1) and (2.2), we can chose a $t_1 > t_0$ sufficiently large satisfying (2.3) such that

$$\int_{t_1}^{\infty} \frac{1}{a(\eta)} \int_{t_1}^{\eta} \left(\sum_{i=1}^{n} G_i(\vartheta)\mathcal{E}_{10} + \big|\psi(\vartheta)\big|\right) \mathrm{d}\vartheta \mathrm{d}\eta \leq (1 + r_1\mathcal{A}_1)\mathcal{E}_{10} - \gamma, \tag{2.25}$$

$$\int_{t_1}^{\infty} \frac{1}{a(\eta)} \int_{t_1}^{\eta} \left(\sum_{i=1}^{m} H_i(\vartheta)\mathcal{E}_{10} + \big|\psi(\vartheta)\big|\right) \mathrm{d}\vartheta \mathrm{d}\eta \leq \gamma - r_2\mathcal{A}_2\mathcal{E}_{10} - \mathcal{E}_9, \tag{2.26}$$

and

$$\int_{t_1}^{\infty} \frac{1}{a(\eta)} \int_{t_1}^{\eta} \left(\sum_{i=1}^{n} G_i(\vartheta) + \sum_{i=1}^{m} H_i(\vartheta)\right) \mathrm{d}\vartheta \mathrm{d}\eta \leq 1 + r_1\mathcal{A}_1 - r_2\mathcal{A}_2 - \frac{\mathcal{E}_9}{\mathcal{E}_{10}}, \tag{2.27}$$

where $\mathcal{E}_9$ and $\mathcal{E}_{10}$ are positive constants such that,

$$r_2\mathcal{A}_2\mathcal{E}_{10} + \mathcal{E}_9 < (1 + r_1\mathcal{A}_1)\mathcal{E}_{10} \quad \text{and} \quad \gamma \in \big(r_2\mathcal{A}_2\mathcal{E}_{10} + \mathcal{E}_9, (1 + r_1\mathcal{A}_1)\mathcal{E}_{10}\big).$$

With the supremum norm, let Ω be the set of all bounded and continuous functions on $[t_0, \infty)$. Set

$$\Psi = \Big\{y \in \Omega : \mathcal{E}_9 \leq y(t) \leq \mathcal{E}_{10}, t \geq t_0\Big\}.$$

Consider the operator $\mathcal{T} : \Psi \to \Omega$ as follows:

$$\big(\mathcal{T}y\big)(t) = \begin{cases} \gamma - \mathcal{R}_1(t)\mathcal{F}_1\big(y\big(t - \zeta_1(t)\big) - \mathcal{R}_2(t)\mathcal{F}_2\big(y\big(t + \zeta_2(t)\big)\big) \\ + \int_{t_1}^{\infty} \frac{1}{a(\eta)} \int_{t_1}^{\eta} \Big[\sum_{i=1}^{n} G_i(\vartheta)y(\vartheta - g_i(\vartheta)) \\ - \sum_{i=1}^{m} H_i(\vartheta)y(\vartheta + h_i(\vartheta)) - \psi(\vartheta)\Big]\mathrm{d}\vartheta \mathrm{d}\eta, \quad t \geq t_1, \\ \big(\mathcal{T}y\big)(t_1), \quad t_0 \leq t \leq t_1. \end{cases} \tag{2.28}$$

Clearly $\mathcal{T}y$ is continuous. For any $y \in \Psi$ and $t \geq t_1$, we obtain from (2.25) and (2.26) that

$$\begin{aligned}\big(\mathcal{T}y\big)(t) &\leq \gamma - \mathcal{R}_1(t)\mathcal{F}_1\big(y\big(t + \zeta_1(t)\big)\big) + \int_{t}^{\infty} \frac{1}{a(\eta)} \int_{t_1}^{\eta} \left(\sum_{i=1}^{n} G_i(\vartheta)y(\vartheta - g_i(\vartheta)) - \psi(\vartheta)\right) \mathrm{d}\vartheta \mathrm{d}\eta\\ &\leq \gamma - r_1\mathcal{A}_1\mathcal{E}_{10} + \int_{t_1}^{\infty} \frac{1}{a(\eta)} \int_{t_1}^{\eta} \left(\sum_{i=1}^{n} G_i(\vartheta)\mathcal{E}_{10} + \big|\psi(\vartheta)\big|\right) \mathrm{d}\vartheta \mathrm{d}\eta\\ &\leq \mathcal{E}_{10},\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}(\mathcal{T}y)(t) &\geq \gamma - \mathcal{R}_2(t)\mathcal{F}_2\big(y(t-\zeta_2(t))\big) - \int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{m}H_i(\vartheta)y(\vartheta+h_i(\vartheta))+\psi(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta\\ &\geq \gamma - r_2\mathcal{A}_2\mathcal{E}_{10} - \int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{m}H_i(\vartheta)\mathcal{E}_{10}+\big|\psi(\vartheta)\big|\right)\mathrm{d}\vartheta\mathrm{d}\eta\\ &\geq \mathcal{E}_9.\end{aligned} \tag{2.30}$$

So $\mathcal{T}\Psi \subset \Psi$. On the other hand, for every $y_1, y_2 \in \Psi$ and $t \geq t_1$, we get from (2.27) that

$$\begin{aligned}\big|(\mathcal{T}y_1)(t)-(\mathcal{T}y_2)(t)\big| &\leq \big\|y_1-y_2\big\|\Bigg(-r_1\mathcal{A}_1+r_2\mathcal{A}_2\\ &+ \int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)+\sum_{i=1}^{m}H_i(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta\Bigg)\\ &\leq \big\|y_1-y_2\big\|\left[1-\frac{\mathcal{E}_9}{\mathcal{E}_{10}}\right] = \mathcal{C}_5\big\|y_1-y_2\big\|,\end{aligned}$$

this shows with the sup norm that

$$\big|\mathcal{T}y_1-\mathcal{T}y_2\big| \leq \mathcal{C}_5\big\|y_1-y_2\big\|.$$

Since $\mathcal{C}_5 < 1$, then $\mathcal{T}$ is a contraction mapping on Ψ. Therefore, there exists a unique bounded non-oscillatory solution, in fact a positive solution of Eq. (E) such that $y \in \Psi$ of $\mathcal{T}y = y$. □

Theorem 2.6. *Assume that* (2.1) *and* (2.2) *hold,* $-1 < r_1 \leq \mathcal{R}_1(t) \leq 0$ *and* $-1-r_1 < r_2 \leq \mathcal{R}_2(t) \leq 0$. *Then Eq.* (E) *admits one bounded non-oscillatory solution.*

Proof. Due to (2.1) and (2.2), we can choose a $t_1 > t_0$ sufficiently large satisfying (2.3) such that

$$\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)\mathcal{E}_{12}+\big|\psi(\vartheta)\big|\right)\mathrm{d}\vartheta\mathrm{d}\eta \leq (1+r_1\mathcal{A}_1+r_2\mathcal{A}_2)\mathcal{E}_{12}-\gamma, \tag{2.31}$$

$$\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{m}H_i(\vartheta)\mathcal{E}_{12}+\big|\psi(\vartheta)\big|\right)\mathrm{d}\vartheta\mathrm{d}\eta \leq \gamma-\mathcal{E}_{11}, \tag{2.32}$$

and

$$\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)+\sum_{i=1}^{m}H_i(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta \leq 1+r_1\mathcal{A}_1+r_2\mathcal{A}_2-\frac{\mathcal{E}_{11}}{\mathcal{E}_{12}}, \tag{2.33}$$

where $\mathcal{E}_{11}$ and $\mathcal{E}_{12}$ are positive constants such that

$$\mathcal{E}_{11} < (1+r_1\mathcal{A}_1+r_2\mathcal{A}_2)\mathcal{E}_{12} \quad \text{and} \quad \gamma \in \big(\mathcal{E}_{11}, (1+r_1\mathcal{A}_1+r_2\mathcal{A}_2)\mathcal{E}_{12}\big).$$

With the sup norm, let Ω be the set of all bounded and continuous functions on $[t_0,\infty)$. Then Ω is a complete metric space. Set

$$\Psi = \Big\{y \in \Omega : \mathcal{E}_{11} \leq y(t) \leq \mathcal{E}_{12}, t \geq t_0\Big\}.$$

Consider the operator $\mathcal{T} : \Psi \to \Omega$ by (2.28). Then for any $y \in \Psi$ and $t \geq t_1$, from (2.31) and (2.32), we obtain

$$\begin{aligned}(\mathcal{T}y)(t) &\leq \gamma - \mathcal{R}_1(t)\mathcal{F}_1\big(y(t-\zeta_1(t))\big) - \mathcal{R}_2(t)\mathcal{F}_2\big(y(t+\zeta_2(t))\big) + \int_{t}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)y(\vartheta-g_i(\vartheta))-\psi(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta\\ &\leq \gamma - r_1\mathcal{A}_1\mathcal{E}_{12} - r_2\mathcal{A}_2\mathcal{E}_{12} + \int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)\mathcal{E}_{12}+|\psi(\vartheta)|\right)\mathrm{d}\vartheta\mathrm{d}\eta \leq \mathcal{E}_{12},\end{aligned} \tag{2.34}$$

and

$$\begin{aligned}(\mathcal{T}y)(t) &\geq \gamma - \int_t^\infty \frac{1}{a(\eta)} \int_{t_1}^\eta \left(\sum_{i=1}^m H_i(\vartheta) y(\vartheta + h_i(\vartheta)) + \psi(\vartheta) \right) \mathrm{d}\vartheta \mathrm{d}\eta \\ &\geq \gamma - \int_{t_1}^\infty \frac{1}{a(\eta)} \int_{t_1}^\eta \left(\sum_{i=1}^m H_i(\vartheta) \mathcal{E}_{12} + \big|\psi(\vartheta)\big| \right) \mathrm{d}\vartheta \mathrm{d}\eta \geq \mathcal{E}_{11}. \end{aligned} \tag{2.35}$$

So $\mathcal{T}\Psi \subset \Psi$. On the other hand, for every $y_1, y_2 \in \Psi$ and $t \geq t_1$, we get

$$\begin{aligned}\big|(\mathcal{T}y_1)(t) - (\mathcal{T}y_2)(t)\big| &\leq \big\|y_1 - y_2\big\| \left(-r_1\mathcal{A}_1 - r_2\mathcal{A}_2 + \int_t^\infty \frac{1}{a(\eta)} \int_{t_1}^\eta \left(\sum_{i=1}^n G_i(\vartheta) + \sum_{i=1}^m H_i(\vartheta) \right) \mathrm{d}\vartheta \mathrm{d}\eta \right) \\ &\leq \big\|y_1 - y_2\big\| \left[1 - \frac{\mathcal{E}_{11}}{\mathcal{E}_{12}} \right] = \mathcal{C}_6 \big\|y_1 - y_2\big\|. \end{aligned}$$

This implies that

$$\big|\mathcal{T}y_1 - \mathcal{T}y_2\big| \leq \mathcal{C}_6 \big\|y_1 - y_2\big\|,$$

i.e., $\mathcal{T}$ is a contraction mapping on Ψ. Therefore, there exists a unique bounded non-oscillatory solution, in fact a positive solution of Eq. (E) such that $y \in \Psi$ of $\mathcal{T}y = y$. □

Theorem 2.7. *Suppose that* (2.1) *and* (2.2) *hold, the inverse functions of* $t - \zeta_1(t)$ *and* $\mathcal{F}_1$ *exist with* $\big(t - \zeta_1(t)\big)^{-1} = \phi(t) \geq t$, *and there exist two positive real numbers* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *such that* $\mathcal{B}_1\xi \leq \mathcal{F}_1^{-1}(\xi) \leq \mathcal{B}_2\xi$. *If* $-\infty < r_{**} \leq \mathcal{R}_1(t) \leq r_1 < -1$ *and* $0 \leq \mathcal{R}_2(t) \leq r_2 < -r_1 - 1$, *then Eq.* (E) *admits one bounded non-oscillatory solution.*

Proof. From (2.1) and (2.2), we can chose a $t_1 > t_0$ sufficiently large satisfying (2.15) such that

$$\int_{t_1}^\infty \frac{1}{a(\eta)} \int_{t_1}^\eta \left(\sum_{i=1}^n G_i(\vartheta) \mathcal{E}_{14} + \big|\psi(\vartheta)\big| \right) \mathrm{d}\vartheta \mathrm{d}\eta \leq \frac{r_{**}}{\mathcal{B}_1} \mathcal{E}_{13} + \gamma, \tag{2.36}$$

$$\int_{t_1}^\infty \frac{1}{a(\eta)} \int_{t_1}^\eta \left(\sum_{i=1}^m H_i(\vartheta) \mathcal{E}_{14} + \big|\psi(\vartheta)\big| \right) \mathrm{d}\vartheta \mathrm{d}\eta \leq -\left(1 + r_2\mathcal{A}_2 + \frac{r_1}{\mathcal{B}_2} \right) \mathcal{E}_{14} - \gamma, \tag{2.37}$$

and

$$\int_{t_1}^\infty \frac{1}{a(\eta)} \int_{t_1}^\eta \left(\sum_{i=1}^n G_i(\vartheta) + \sum_{i=1}^m H_i(\vartheta) \right) \mathrm{d}\vartheta \mathrm{d}\eta \leq \frac{r_{**}\mathcal{E}_{13}}{\mathcal{B}_1\mathcal{E}_{14}} - \left(1 + r_2\mathcal{A}_2 + \frac{r_1}{\mathcal{B}_2} \right) \tag{2.38}$$

where $\mathcal{E}_{13}$ and $\mathcal{E}_{14}$ are positive constants such that

$$-\frac{r_{**}\mathcal{E}_{13}}{\mathcal{B}_1} < -\left(1 + \frac{r_1}{\mathcal{B}_2} + r_2\mathcal{A}_2 \right) \mathcal{E}_{14} \text{ and } \gamma \in \left(-\frac{r_{**}\mathcal{E}_{13}}{\mathcal{B}_1}, -\left(1 + r_2\mathcal{A}_2 + \frac{r_1}{\mathcal{B}_2} \right) \mathcal{E}_{14} \right).$$

With the sup norm, let Ω be the set of all bounded and continuous functions on $[t_0, \infty)$. Set

$$\Psi = \Big\{ y \in \Omega : \mathcal{E}_{13} \leq y(t) \leq \mathcal{E}_{14}, t \geq t_0 \Big\}.$$

Consider the operator $\mathcal{T} : \Psi \to \Omega$ as follows:

$$(\mathcal{T}y)(t) = \begin{cases} \mathcal{F}_1^{-1} \Big\{ \frac{-1}{\mathcal{R}_1(\phi(t))} \Big(\gamma + y(\phi(t)) \\ +\mathcal{R}_2(\phi(t)) \mathcal{F}_2\big(y(\phi(t)) + \zeta_2(\phi(t)) \big) \\ - \int_{\phi(t)}^\infty \frac{1}{a(\eta)} \int_{\phi(t_1)}^\eta \Big[\sum_{i=1}^n G_i(\vartheta) y(\vartheta - g_i(\vartheta)) \\ - \sum_{i=1}^m H_i(\vartheta) y(\vartheta + h_i(\vartheta)) - \psi(\vartheta) \Big] \mathrm{d}\vartheta \mathrm{d}\eta \Big) \Big\}, \quad t_1 \leq t, \\ \mathcal{T}(y)(t_1), \quad t_0 \leq t \leq t_1. \end{cases} \tag{2.39}$$

It is easy to see that $\mathcal{T}y$ is continuous. Meanwhile, for any $y \in \Psi$ and $t \geq t_1$, from (2.36) and (2.37), it follows that

$$\begin{aligned}(\mathcal{T}y)(t) &\leq \mathcal{B}_2\Bigg(\frac{-1}{\mathcal{R}_1(\phi(t))}\Big[\gamma + y(\phi(t)) + \mathcal{R}_2(\phi(t))\mathcal{F}_2(y(\phi(t)) + \zeta_2(\phi(t)) \\ &\quad + \int_{\phi(t)}^{\infty}\frac{1}{a(\eta)}\int_{\phi(t_1)}^{\eta}\left(\sum_{i=1}^{m}H_i(\vartheta)y(\vartheta + h_i(\vartheta)) + \psi(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta\Big]\Bigg) \\ &\leq -\frac{\mathcal{B}_2}{r_1}\Bigg(\gamma + \mathcal{E}_{14} + r_2\mathcal{A}_2\mathcal{E}_{14} \\ &\quad + \int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{m}H_i(\vartheta)\mathcal{E}_{14} + |\psi(\vartheta)|\right)\mathrm{d}\vartheta\mathrm{d}\eta\Bigg) \leq \mathcal{E}_{14},\end{aligned} \tag{2.40}$$

and

$$\begin{aligned}(\mathcal{T}y)(t) &\geq \mathcal{B}_1\left(\frac{-1}{\mathcal{R}_1(\phi(t))}\left(\gamma - \int_{\phi(t)}^{\infty}\frac{1}{a(\eta)}\int_{\phi(t_1)}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)y(\vartheta - g_i(\vartheta)) - \psi(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta\right)\right) \\ &\geq -\frac{\mathcal{B}_1}{r_{**}}\left(\gamma - \int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)\mathcal{E}_{14} + |\psi(\vartheta)|\right)\mathrm{d}\vartheta\mathrm{d}\eta\right) \geq \mathcal{E}_{13}.\end{aligned} \tag{2.41}$$

This means that $\mathcal{T}\Psi \subset \Psi$. On the other hand, for every $y_1, y_2 \in \Psi$ and $t \geq t_1$,

$$\begin{aligned}\big|(\mathcal{T}y_1)(t) - (\mathcal{T}y_2)(t)\big| &\leq \|y_1 - y_2\|\frac{-\mathcal{B}_2}{r_1}\left(1 + r_2\mathcal{A}_2 + \int_{\phi(t)}^{\infty}\frac{1}{a(\eta)}\int_{\phi(t_1)}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta) + \sum_{i=1}^{m}H_i(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta\right) \\ &\leq \|y_1 - y_2\|\left[1 - \frac{\mathcal{B}_2 r_{**}\mathcal{E}_{13}}{\mathcal{B}_1 r_1 \mathcal{E}_{14}}\right] = \mathcal{C}_7\|y_1 - y_2\|,\end{aligned}$$

this implies that

$$\big|\mathcal{T}y_1 - \mathcal{T}y_2\big| \leq \mathcal{C}_7\|y_1 - y_2\|.$$

Since $\mathcal{C}_7 < 1$ by (2.38), then $\mathcal{T}$ is a contraction mapping on Ψ. Therefore, there exists a unique bounded non-oscillatory solution of Eq. (E) such that $y \in \Psi$ of $\mathcal{T}y = y$. □

Theorem 2.8. *Suppose that* (2.1) *and* (2.2) *hold, the inverse functions of* $t - \zeta_1(t)$ *and* $\mathcal{F}_1$ *exist with* $\big(t - \zeta_1(t)\big)^{-1} = \phi(t) \geq t$, *and there exist two positive real numbers* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *such that* $\mathcal{B}_1\xi \leq \mathcal{F}_1^{-1}(\xi) \leq \mathcal{B}_2\xi$. *If* $-\infty < r_{**} \leq \mathcal{R}_1(t) \leq r_1 < -1$ *and* $r_1 + 1 < r_2 \leq \mathcal{R}_2(t) < 0$, *then Eq.* (E) *admits one bounded non-oscillatory solution.*

Proof. Due to (2.1) and (2.2), we can chose a sufficiently large $t_1 > t_0$ that satisfies (2.15) such that

$$\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)\mathcal{E}_{16} + |\psi(\vartheta)|\right)\mathrm{d}\vartheta\mathrm{d}\eta \leq \frac{r_{**}}{\mathcal{B}_1}\mathcal{E}_{15} + r_2\mathcal{A}_2\mathcal{E}_{16} + \gamma, \tag{2.42}$$

$$\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{m}H_i(\vartheta)\mathcal{E}_{16} + |\psi(\vartheta)|\right)\mathrm{d}\vartheta\mathrm{d}\eta \leq -\left(1 + \frac{r_1}{\mathcal{B}_2}\right)\mathcal{E}_{16} - \gamma, \tag{2.43}$$

and

$$\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta) + \sum_{i=1}^{m}H_i(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta \leq r_2\mathcal{A}_2 + \frac{r_{**}\mathcal{E}_{15}}{\mathcal{B}_1\mathcal{E}_{16}} - 1 - \frac{r_1}{\mathcal{B}_2}, \tag{2.44}$$

where $\mathcal{E}_{15}$ and $\mathcal{E}_{16}$ are positive constants such that

$$-\left(r_2\mathcal{A}_2\mathcal{E}_{16}+\frac{r_{**}\mathcal{E}_{15}}{\mathcal{B}_1}\right)<-\left(1+\frac{r_1}{\mathcal{B}_2}\right)\mathcal{E}_{16}\quad\text{and}\quad\gamma\in\left(-\left(r_2\mathcal{A}_2\mathcal{E}_{16}+\frac{r_{**}\mathcal{E}_{15}}{\mathcal{B}_1}\right),-\left(1+\frac{r_1}{\mathcal{B}_2}\right)\mathcal{E}_{16}\right).$$

With the sup norm, let Ω be the set of all bounded and continuous functions on $[t_0,\infty)$. Set

$$\Psi=\Big\{y\in\Omega:\mathcal{E}_{15}\le y(t)\le\mathcal{E}_{16},t\ge t_0\Big\}.$$

Consider the operator $\mathcal{T}:\Psi\to\Omega$ as (2.39). For $y\in\Psi$ and $t\ge t_1$, from (2.42) and (2.43),

$$\begin{aligned}(\mathcal{T}y)(t)&\le\mathcal{B}_2\left(\frac{-1}{\mathcal{R}_1(\phi(t))}\Big[\gamma+y(\phi(t))+\int_{\phi(t)}^{\infty}\frac{1}{a(\eta)}\int_{\phi(t_1)}^{\eta}\left(\sum_{i=1}^{m}H_i(\vartheta)y(\vartheta+h_i(\vartheta))+\psi(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta\Big]\right)\\&\le\frac{-\mathcal{B}_2}{r_1}\left(\gamma+\mathcal{E}_{16}+\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{m}H_i(\vartheta)\mathcal{E}_{16}+\big|\psi(\vartheta)\big|\right)\mathrm{d}\vartheta\mathrm{d}\eta\right)\\&\le\mathcal{E}_{16},\end{aligned}\tag{2.45}$$

and

$$\begin{aligned}(\mathcal{T}y)(t)&\ge\mathcal{B}_1\left(\frac{-1}{\mathcal{R}_1(\phi(t))}\Big[\gamma+\mathcal{R}_2(\phi(t))\mathcal{F}_2\big(y(\phi(t))+\zeta_2(\phi(t))\big)\right.\\&\qquad\left.-\int_{\phi(t)}^{\infty}\frac{1}{a(\eta)}\int_{\phi(t_1)}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)y(\vartheta-g_i(\vartheta))-\psi(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta\Big]\right)\\&\ge\frac{-\mathcal{B}_1}{r_{**}}\left(\gamma+r_2\mathcal{A}_2\mathcal{E}_{16}-\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)\mathcal{E}_{16}+\big|\psi(\vartheta)\big|\right)\mathrm{d}\vartheta\mathrm{d}\eta\right)\\&\ge\mathcal{E}_{15},\end{aligned}\tag{2.46}$$

this means that $\mathcal{T}\Psi\subset\Psi$. On the other hand, for every $y_1,y_2\in\Psi$ and $t\ge t_1$, we get from (2.44) that

$$\begin{aligned}\big|(\mathcal{T}y_1)(t)-(\mathcal{T}y_2)(t)\big|&\le\|y_1-y_2\|\frac{-\mathcal{B}_2}{r_1}\left(1-r_2\mathcal{A}_2+\int_{\phi(t)}^{\infty}\frac{1}{a(\eta)}\int_{\phi(t_1)}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)+\sum_{i=1}^{m}H_i(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta\right)\\&\le\|y_1-y_2\|\frac{-\mathcal{B}_2}{r_1}\left(1-r_2\mathcal{A}_2+\int_{t_1}^{\infty}\frac{1}{a(\eta)}\int_{t_1}^{\eta}\left(\sum_{i=1}^{n}G_i(\vartheta)+\sum_{i=1}^{m}H_i(\vartheta)\right)\mathrm{d}\vartheta\mathrm{d}\eta\right)\\&\le\|y_1-y_2\|\left[1-\frac{\mathcal{B}_2r_{**}\mathcal{E}_{15}}{\mathcal{B}_1r_1\mathcal{E}_{16}}\right]=\mathcal{C}_8\|y_1-y_2\|,\end{aligned}$$

this implies that

$$\big|\mathcal{T}y_1-\mathcal{T}y_2\big|\le\mathcal{C}_8\big\|y_1-y_2\big\|.$$

Since $\mathcal{C}_8<1$, then $\mathcal{T}$ is a contraction mapping on Ψ. Hence, there exists a unique bounded non-oscillatory solution, in fact an eventually positive solution of (E) such that $y\in\Psi$ of $\mathcal{T}y=y$. □

3. Applications and a Remark

We give two interesting examples that illustrate the versatility of ours results, in this section. Different illustrative examples can easily be constructed for other theorems, similarly.

Example 3.1. Considerr the NDE

$$\begin{aligned}&\left[\exp(t)\left(y(t)-\frac{2}{3}y(t-6\pi)+\left[\frac{1}{3}-\exp\left(-3t\right)\right]y(t+7\pi)\right)'\right]'\\&+5\exp\left(-2t\right)y\left(t-\frac{7\pi}{2}\right)+3\exp\left(-2t\right)y\left(t-2\pi\right)\\&-8\exp\left(-2t\right)y\left(t+8\pi\right)=-18\exp\left(-2t\right).\end{aligned}\tag{3.1}$$

Noting that

$$\begin{aligned}&a(t)=\exp\left(t\right),\quad \mathcal{R}_1(t)=-\frac{2}{3},\quad \mathcal{R}_2(t)=\frac{1}{3}-\exp\left(-3t\right),\\&G_1(t)=5\exp\left(-2t\right),\quad G_2(t)=3\exp\left(-2t\right),\quad H_1(t)=8\exp\left(-2t\right).\end{aligned}$$

A direct computation ensures that all the conditions of Theorem 2.5 are fulfilled. Actually, $y(t)=3+\sin t$ is such a non-oscillatory solution of Eq. (3.1).

Example 3.2. Consider the NDE

$$\begin{aligned}&\left[\exp(t)\left(y(t)-\left[\frac{4}{5}-\exp\left(-2t\right)\right]y\left(\frac{t}{2}\right)-\left[\frac{1}{10}+\exp\left(-2t\right)\right]y(2t)\right)'\right]'\\&+\frac{6}{5}\exp\left(-3t\right)y\left(\frac{t}{4}\right)+30\exp\left(-4t\right)y\left(\frac{t}{2}\right)-\frac{6}{5}\exp\left(-\frac{t}{2}\right)y\left(\frac{3t}{2}\right)\\&-30\exp\left(-\frac{t}{2}\right)y\left(\frac{7t}{4}\right)=2\exp\left(-t\right)+6\exp\left(-2t\right)-\frac{156}{5}\exp\left(-\frac{t}{2}\right).\end{aligned}\tag{3.2}$$

Note that

$$\begin{aligned}&\mathcal{R}_1(t)=-\frac{4}{5}+\exp\left(-2t\right),\quad \mathcal{R}_2(t)=-\frac{1}{10}-\exp\left(-2t\right),\\&G_1(t)=\frac{6}{5}\exp\left(-3t\right),\quad G_2(t)=30\exp\left(-4t\right),\quad H_1(t)=\frac{6}{5}\exp\left(-\frac{t}{2}\right),\\&H_2(t)=30\exp\left(-\frac{t}{2}\right),\quad a(t)=\exp\left(t\right).\end{aligned}$$

With a direct calculation, one can see that all the conditions of Theorem 2.6 are fulfilled. Indeed, $y(t)=1+\exp\left(-2t\right)$ is such a non-oscillatory, actually positive, solution of Eq. (3.2).

Remark 3.1. It should be pointed out that existence theorems presented in [5, 12, 13] fail to apply to the equations (3.1) and (3.2), because of the structure of functions $G_1(t)$ and $G_2(t)$ in (3.1), and there exist variable deviating arguments in (3.2).

4. Conclusion

This paper contains some sufficient conditions for the existence of non-oscillatory solutions of a comprehensive class of second order functional DEs with a mixed neutral term. By considering different cases for the ranges of the neutral coefficient functions, we utilize the Banach contraction mapping principle to prove our results.

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] T. Candan and R. Dahiya, *Existence of nonoscillatory solutions of higher order neutral differential equations with distributed deviating arguments*, J. Frank. Inst., *347* (2010), 1309–1316.

[2] T. Candan, *Existence of nonoscillatory solutions to first order neutral differential equations*, Appl. Math. Lett., *26* (2013), 1182–1186.

[3] T. Candan, *Nonoscillatory solutions of higher order differential and delay differential equations with forcing term*, Appl. Math. Lett., *39* (2015), 67–72.

[4] T. Candan, *Existence of nonoscillatory solutions to first order neutral differential equations*, Electron. J. Differ. Equ., *39* (2016), 1–11.

[5] T. Candan, *Existence of nonoscillatory solutions of higher-order nonlinear mixed neutral differential equations*, Dyn. Syst. Appl., *27*(4) (2018), 743–755.

[6] M. P. Chen, J. S. Yu, and Z. C. Wang, *Nonoscillatory solutions of neutral delay differential equations*, Bull. Austral. Math. Soc., *48*(3) (1993), 475–483.

[7] H. Chi, J. Bell, and B. Hassard, *Numerical solution of a nonlinear advance-delay differential equation from nerve conduction theory*, J. Math. Biol., *24* (1986), 583–601.

[8] F. Kong, *Existence of non-oscillatory solutions of a kind of first-order neutral differential equation*, Math. Commun., *22* (2017), 151–164.

[9] M. R. S. Kulenović and S. Hadžiomerspahić, *Existence of nonoscillatory solution of second order linear neutral delay equation*, J. Math. Anal. Appl., *228* (1998), 436–448.

[10] T. Kusano and M. Naito, *Unbounded nonoscillatory solutions of nonlinear ordinary differential equations of arbitrary order*, Hiroshima Math. J., *18* (1988), 361–372.

[11] H. Li, Z. Han, and Y. Wang, *Nonoscillatory solutions for super-linear Emden Fowler type dynamic equations on time scales*, Electron. J. Qual. Theory Differ. Equ., *53* (2015), 1–13.

[12] H. Li, Z. Han, and Y. Sun, *Existence of non-oscillatory solutions for second-order mixed neutral differential equations with positive and negative terms*, Int. J. Dyn. Syst. Differ. Equ., *7*(3) (2017), 259–271.

[13] H. Li and S. Sun, *Nonoscillation of higher order mixed differential equations with distributed delays*, Rev. Real Acad. Cienc. Exactas Fis. Nat. - A: Mat., *113*(3) (2019), 2617–2625.

[14] J. Mallet-Paret, *The global structure of traveling waves in spatially discrete dynamical systems*, J. Dyn. Differ. Equ., *11* (1999), 49–127.

[15] B. Mansouri, A. Ardjouni, and A. Djoudi, *Existence and uniqueness of nonoscillatory solutions of first-order neutral differential equations by using Banach's theorem*, Proc. Inst. Math. Mech., *45*(1) (2019), 15–30.

[16] M. Naito and K. Yano, *Positive solutions of higher order ordinary differential equations with general nonlinearities*, J. Math. Anal. Appl., *250* (2000), 27–48.

[17] L. Pontryagin, R. Gamkreledze, and E. Mischenko, *The Mathematical Theory of Optimal Processes*, Interscience, New York, 1962.

[18] M. Slater and H. S. Wilf, *A class of linear differential-difference equations*, Pacific J. Math., *10* (1960), 1419–1427.

[19] A. Rustichini, *Hopf bifurcation for functional differential equations of mixed type*, J. Dyn. Differ. Equ., *10* (1989), 145–177.

[20] H. Ye, J. Yin, and C. Jin, *Nonoscillatory solutions for a nonlinear neutral delay differential equation*, Appl. Math. Comput., *235* (2014), 283–291.

[21] W. Zhang, W. Feng, J. Yan, and J. Song, *Existence of nonoscillatory solutions of first-order linear neutral delay differential equations*, Comput. Math. Appl., *49* (2005), 1021–1027.

[22] Y. Zhou and B. G. Zhang, *Existence of nonoscillatory solutions of higher-order neutral differential equations with positive and negative coefficients*, Appl. Math. Lett., *12*(15) (2002), 867–874.

[23] Y. Zhou, *Existence for nonoscillatory solutions of second order nonlinear differential equations*, J. Math. Anal. Appl., *331* (2007), 91–96.