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Solving Abel's equations with the shifted Legendre polynomials

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Abstract

In this article, a numerical method is presented to solve Abel's equations. In the given method, the solution of the equation is found as a finite expansion of the shifted Legendre polynomials. To this end, the integral and differential parts of the equation are converted to vector-matrix representations. Therefore, the equation is converted to an algebraic system of the equations and by solving it, the solution of the equation is obtained. Further, the numerical example is given to illustrate the method's efficiency.

Keywords. Abel's equation, Integral equation, Caputo differential operator, Shifted Legendre polynomial.
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1. Introduction

Modeling many problems in engineering and applied sciences are transformed to integral equations. Abel's equations are one of the integral equations which are directly derived from problems of physics [2, 5]. Various types of mathematical methods are used to solve them. For example, the operational matrix of Bernstein's polynomials, collocation method, and Chebyshev wavelet methods are used to solve Abel's equations [4, 6, 8, 12]. We solve Abel's equation with the shifted Legendre polynomials with suitable convergence. We use the properties of fractional calculators and operational matrix [1, 7]. Some of the preliminaries and properties are expressed as follows. Section 2 lists the essential theorems. In the section 3, the solving method and an example are presented.

Definition 1.1. ([3]) The Riemann–Liouville fractional integral operator of order $\alpha \in \mathbb{R}_+$ is defined on $L_1[m, n]$ by

$$I_m^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_m^x (x-s)^{\alpha-1} f(s)ds\,, \quad m \le x \le n, \tag{1.1}$$

where $\Gamma(.)$ denotes the Gamma function.

Definition 1.2. ([3]) The Caputo differential operator of order $\alpha \in \mathbb{R}_+$ is defined on $L_1[m, n]$ by

$$D_m^\alpha f(x) = \frac{1}{\Gamma(\rho-\alpha)} \int_m^x (x-s)^{\rho-\alpha-1} f^{(\rho)}(s)ds\,, \quad m \le x \le n, \quad \rho = \lceil \alpha \rceil, \tag{1.2}$$

where $\lceil . \rceil$ denotes the ceiling function.

Example 1.3. ([3]) Consider $f(x) = (x-m)^\nu$ for some $\nu > -1$ and $\delta > 0$. Then

$$\begin{aligned} I_m^\alpha f(x) &= \frac{\Gamma(\nu+1)}{\Gamma(\nu+1+\alpha)}(x-m)^{\nu+\alpha}\,, \\ D_m^\alpha f(x) &= \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\alpha)}(x-m)^{\nu-\alpha}\,. \end{aligned} \tag{1.3}$$

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Definition 1.4. ([9]) The shifted Legendre polynomials are defined as

$$\phi(r) = [\phi_0(r), \phi_1(r), \phi_2(r), \dots], \tag{1.4}$$

with

$$\phi_i(r) = \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+k+1)}{\Gamma(k+1)k!(i-k)!} r^k, \quad i = 0, 1, 2, \dots. \tag{1.5}$$

We have $\phi(r) = \Phi R_r$, which Φ is the Legendre coefficient matrix and $R_r = [1, r, r^2, \dots]^T$.

Definition 1.5. ([3]) Hypergeometric function is defined by

$${}_pF_q\Big(a_1, a_2, \dots, a_p; b_1, b_2, \dots, b_q; z\Big) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \dots (a_p)_n}{(b_1)_n (b_2)_n \dots (b_q)_n} \cdot \frac{z^n}{n!}, \tag{1.6}$$

that $(a)_n = a(a+1)(a+2)...(a+n-1)$, $n = 1, 2, ...$ and $(a)_0 = 1$.

The parameters of the function must be such that the denominator of the fraction in the sentences of series not be zero.

2. Operational matrices of the shifted Legendre polynomials

We obtain operational matrices related to the shifted Legendre polynomials for some types of Abel's equations.

Theorem 2.1. *Let $\phi(r)$ is the shifted Legendre polynomial vector and $b \in (0,1)$, then*

$$\int_0^r \frac{\phi_i(t)}{(r-t)^b} dt = Q\phi(r), \tag{2.1}$$

with

$$Q = \Phi \mathbf{\Gamma} \mathbf{C},$$

where $\mathbf{\Gamma}$ is the diagonal matrix with array $\mathbf{\Gamma}_{m,m} = \dfrac{\Gamma(1-b)\Gamma(m+1)}{\Gamma(2-b+m)}$, $m = 0, 1, 2, ...$ and $\mathbf{C}$ is a matrix with array as follows

$$c_{m,i} = \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+k+1)}{\Gamma(k+1)(i-k)!k!(k+m+2-b)}. \tag{2.2}$$

Proof.

$$\begin{aligned}
\int_0^r \frac{\phi_i(t)}{(r-t)^b} dt &= \int_0^r \frac{\Phi R_t}{(r-t)^b} dt \\
&= \Phi \left[\int_0^r \frac{1}{(r-t)^b} dt, \int_0^r \frac{t}{(r-t)^b} dt, \int_0^r \frac{t^2}{(r-t)^b} dt, ..., \int_0^r \frac{t^m}{(r-t)^b} dt \right]^T \\
&= \Phi \left[\frac{\Gamma(1-b)\Gamma(1)}{\Gamma(1-b+1)} r^{1-b}, \frac{\Gamma(1-b)\Gamma(2)}{\Gamma(1-b+2)} r^{2-b}, ..., \frac{\Gamma(1-b)\Gamma(m+1)}{\Gamma(2-b+m)} r^{m+1-b} \right]^T = \Phi \mathbf{\Gamma} \mathbf{\Pi},
\end{aligned} \tag{2.3}$$

that $\mathbf{\Pi} = \left[r^{1-b}, r^{2-b}, ..., r^{m+1-b}\right]^T$. By approximation r^{m+1-b}, we have

$$r^{m+1-b} = \sum_{i=0}^{\infty} c_{m,i}\phi_i(r) = C_m \Phi R_r, \ C_m = [c_{m,0}, c_{m,1}, c_{m,2}, ...], \tag{2.4}$$

therefore

$$\mathbf{\Pi} = \left[C_0\Phi R_r, C_1\Phi R_r, C_2\Phi R_r, ..., C_m\Phi R_r, ...\right]^T = \mathbf{C}\Phi R_r, \ \mathbf{C} = [C_0, C_1, C_2, ...]^T, \tag{2.5}$$

with replace in (2.3) we will have

$$\int_0^r \frac{\phi_i(t)}{(r-t)^b}dt = \Phi\mathbf{\Gamma}\mathbf{C}\Phi R_r = Q\Phi R_r = Q\phi(r), \tag{2.6}$$

which cofficient $c_{m,i}$ is obtained with $< r^{m+1-b}, \phi_i(r) >$ as follows

$$c_{m,i} = \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+k+1)}{\Gamma(k+1)(i-k)!k!(k+m+2-b)}. \tag{2.7}$$

□

Theorem 2.2. *Let $\phi(r)$ is the shifted Legendre polynomial vector, then*

$$\int_r^1 \frac{\phi_i(t)}{(t-r)^b}dt = \Theta\phi(r), \tag{2.8}$$

that Θ is a matrix to form

$$\sum_{z=0}^{m} \frac{m!}{z!(m-z)!(m-b-z+1)}\Phi C_m. \tag{2.9}$$

Proof.

$$\begin{aligned}\int_r^1 \frac{\phi_i(t)}{(t-r)^b}dt &= \int_r^1 \frac{\Phi R_t}{(t-r)^b}dt \\ &= \Phi\left[\int_r^1 \frac{1}{(t-r)^b}dt, \int_r^1 \frac{t}{(t-r)^b}dt, \int_r^1 \frac{t^2}{(t-r)^b}dt, ..., \int_r^1 \frac{t^m}{(t-r)^b}dt\right]^T.\end{aligned} \tag{2.10}$$

We use variable substitution $u = t - r$, so

$$\int_r^1 \frac{t^m}{(t-r)^b}dt = \int_0^{1-r}(u+r)^m u^{-b}du = \int_0^{1-r}\sum_{z=0}^{m}\frac{m!}{z!(m-z)!}u^{m-z-b}r^z du = \sum_{z=0}^{m}\frac{m!}{z!(m-z)!}r^z\frac{(1-r)^{m-b-z+1}}{m-b-z+1}. \tag{2.11}$$

For approximation $r^z(1-r)^{m-b-z+1}$, we let

$$r^z(1-r)^{m-b-z+1} = \sum_{i=0}^{N} c_{m,i}\phi_i(r) = C_m\Phi R_r,\ \ C_m = [c_{m,0}, c_{m,1}, c_{m,2}, ...], \tag{2.12}$$

then (2.11) is as follows

$$\int_r^1 \frac{t^m}{(t-r)^b}dt = \sum_{z=0}^{m}\frac{m!}{z!(m-z)!(m-b-z+1)}C_m\Phi R_r, \tag{2.13}$$

and

$$\int_r^1 \frac{\phi_i(t)}{(t-r)^b}dt = \Phi\sum_{z=0}^{m}\frac{m!}{z!(m-z)!(m-b-z+1)}C_m\Phi R_r = \Theta\Phi R_r = \Theta\phi(r). \tag{2.14}$$

Cofficient $c_{m,i}$ is obtained with $< r^z(1-r)^{m-b-z+1}, \phi_i(r) >$ as follows

$$c_{m,i} = \sum_{k=0}^{i}\frac{(-1)^{i-k}\Gamma(i+k+1)\Gamma(z+k+1)\Gamma(m-b-z+2)}{\Gamma(k+1)(i-k)!k!\Gamma(k+m+3-b)}. \tag{2.15}$$

□

Theorem 2.3. *Let $\phi(r)$ is the shifted Legendre polynomial vector and $\beta > 0$, then*

$$I^{\beta}\phi_i(r) = \Lambda\phi(r), \tag{2.16}$$

that Λ is as follows

$$\Lambda = \sum_{k=0}^{i}\sum_{l=0}^{j} \frac{(-1)^{i-k+j-l}\Gamma(i+k+1)}{(i-k)!k!\Gamma(\beta+k+1)} \frac{\Gamma(j+l+1)}{\Gamma(l+1)(j-l)!l!(\beta+k+l+1)}. \tag{2.17}$$

Proof. Due to Definition 1.1 and Example 1.3, we have

$$I^{\beta}\phi_i(r) = I^{\beta}\sum_{k=0}^{i}\frac{(-1)^{i-k}\Gamma(i+k+1)}{\Gamma(k+1)(i-k)!k!}r^k = \sum_{k=0}^{i}\frac{(-1)^{i-k}\Gamma(i+k+1)}{\Gamma(k+1)(i-k)!k!}I^{\beta}r^k = \sum_{k=0}^{i}\frac{(-1)^{i-k}\Gamma(i+k+1)}{(i-k)!k!\Gamma(\beta+k+1)}r^{\beta+k}. \tag{2.18}$$

Let

$$r^{\beta+k} = \sum_{j=0}^{N} c_{i,j}\phi_j(r) = C\Phi R_r, \; C = [c_{i,0}, c_{i,1}, c_{i,2}, ...], \tag{2.19}$$

using $< r^{\beta+k}, \phi_j(r) >$ coefficient $c_{i,j}$ is obtained as follows

$$c_{i,j} = \sum_{l=0}^{j}\frac{(-1)^{j-l}\Gamma(j+l+1)}{\Gamma(l+1)(j-l)!l!(\beta+k+l+1)}, \tag{2.20}$$

therefore

$$\begin{aligned} I^{\beta}\phi_i(r) &= \sum_{k=0}^{i}\frac{(-1)^{i-k}\Gamma(i+k+1)}{(i-k)!k!\Gamma(\beta+k+1)}C\Phi R_r = \Lambda\Phi R_r = \Lambda\phi(r), \\ \Lambda &= \sum_{k=0}^{i}\sum_{l=0}^{j}\frac{(-1)^{i-k}\Gamma(i+k+1)}{(i-k)!k!\Gamma(\beta+k+1)}\frac{(-1)^{j-l}\Gamma(j+l+1)}{\Gamma(l+1)(j-l)!l!(\beta+k+l+1)}. \end{aligned} \tag{2.21}$$

□

3. Implementation of solving Abel's equations

In modeling of some natural phenomena, Abel's integral appears as follows [11],

$$s(y) = 2\int_y^1 (r^2 - y^2)^{(-b)}\varphi(r)rdr, \qquad y \in [0,1], \;\; b \in (0,1). \tag{3.1}$$

The other form of (3.1) is as follows

$$s(y) = \int_y^1 (r-y)^{(-b)}D^{\beta}\psi(r)dr, \qquad \psi(r) = \varphi(\sqrt{r}), \tag{3.2}$$

with the initial condition $\psi(0) = a$, where a is a constant. Suppose

$$D^{\beta}\psi(r) = \sum_{i=0}^{m} c_i\phi_i(r) = C^T\phi(r), \tag{3.3}$$

$$\psi(0) = a = A^T\phi(r), \tag{3.4}$$

$$s(y) = F^T\phi(y), \tag{3.5}$$

that C and $\phi(r)$ are vectors as follows

$$\phi(r) = [\phi_0(r), \phi_1(r), ..., \phi_m(r)], \quad C = [c_0, c_1, ..., c_m], \quad R_a = [1, a, a^2, a^3, ...], \tag{3.6}$$

which $\phi_i(r), i = 0, ..., m$, is the shifted Legendre function of degree i that is given by (1.5). We have $\phi(r) = \Phi R_r$ that Φ is Legendre coefficient matrix.
We obtain with integration (3.3)

$$\psi(r) = C^T I^\beta \phi(r) + A^T \phi(r), \tag{3.7}$$

using (3.3) and (3.2) is obtained

$$F^T \phi(y) = C^T \int_y^1 (r-y)^{(-b)} \phi(r) dr, \tag{3.8}$$

and due to Theorem 2.2,

$$\int_y^1 (r-y)^{(-b)} \phi(r) dr = \Theta \phi(r). \tag{3.9}$$

So from (3.8) and (3.9) is obtained

$$F^T = \Theta C^T, \tag{3.10}$$

and with substitution in (3.7) is given

$$\psi(r) = (F^T \Theta^{-1} I^\beta \phi(r) + A^T)\phi(r), \tag{3.11}$$

and thus the solution of (3.2) is obtained.

Example 3.1. ([10]) Let the first type of Abel's integral as follows

$$\int_y^1 \frac{D^{\frac{3}{4}}\psi(r)}{(r-y)^{\frac{1}{2}}} dr = I(y), \quad 0 < y < 1, \quad 0 \le r \le 1, \quad \text{with} \quad \psi(0) = 0, \tag{3.12}$$

where

$$I(y) = \frac{2}{\Gamma(\frac{9}{4})}\Big[\frac{4}{7}(1-y)^{\frac{1}{2}} + \frac{\sqrt{\pi} y^{\frac{7}{4}} \Gamma(-\frac{7}{4})}{\Gamma(-\frac{5}{4})} + \frac{20}{21} {}_2F_1(-\frac{3}{4}, \frac{1}{2}, \frac{1}{4}, y)\Big],$$

and the exact solution of Equation (3.12) is $\psi(r) = r^2$.

The shifted Legendre polynomials for $i = 6$ of Example 3.1 is

$$\phi(y) = \begin{bmatrix} 1 \\ -1+2y \\ 1-6y+6y^2 \\ -1+12y-30y^2+20y^3 \\ 1-20y+90y^2-140y^3+70y^4 \\ -1+30y-210y^2+560y^3-630y^4+252y^5 \end{bmatrix}.$$

For the right side of (3.12), we consider $I(y) = F^T\phi(y)$, so

$$F_i = \int_0^1 I(y)\phi_i(y) dy, \tag{3.13}$$

and

$$F^T = \begin{bmatrix} 1.286539 & -0.044836 & -0.226179 & -0.075085 & -0.027091 & -0.014647 \end{bmatrix}.$$

Due to Theorem 2.2

$$\Theta = \begin{bmatrix} 1.333333 & -0.461880 & -0.085184 & -0.033596 & -0.017316 & -0.010308 \\ 0.461880 & 0.571428 & -0.344265 & -0.079351 & -0.035529 & -0.019895 \\ -0.085183 & 0.344265 & 0.432900 & -0.283162 & -0.069695 & -0.032811 \\ 0.033596 & -0.079351 & 0.283162 & 0.363636 & -0.245819 & -0.062378 \\ -0.017316 & 0.035529 & -0.069695 & 0.245819 & 0.319891 & -0.220107 \\ 0.010308 & -0.019894 & 0.032811 & -0.062378 & 0.220107 & 0.288983 \end{bmatrix},$$

and according to Theorem 2.3

$$\Lambda = \begin{bmatrix} 0.621751 & 0.293701 & -0.025277 & 0.007871 & -0.003492 & 0.001858 \\ -0.293701 & 0.124350 & 0.172825 & -0.020271 & 0.007442 & -0.003667 \\ -0.025277 & -0.172825 & 0.081098 & 0.127109 & -0.016272 & 0.006373 \\ -0.007871 & -0.020271 & -0.127108 & 0.062262 & 0.102321 & -0.013645 \\ -0.003492 & -0.007442 & -0.016272 & -0.102321 & 0.051315 & 0.0865047 \\ -0.001858 & -0.003667 & -0.006374 & -0.013645 & -0.086504 & 0.044036 \end{bmatrix},$$

and by setting in (3.11), we obtain

$$\psi(r) = 0.01269042457 + 0.2556851715r - 0.6659055728r^2 + 4.304544937r^3 - 4.761160033r^4 + 1.879229689r^5,$$

that is the approximate solution of Equation (3.12).

Conclusion

We know that using orthogonal polynomials bases in approximating functions creates a sparse matrix. In this work, we apply the shifted Legendre polynomials to solve Abel's integral equation with fractional operators. This method gives the desired accuracy of the equation solution.

References

[1] S. Alavi, A. Haghighi, A. Yari, and F. Soltanian, *A numerical method for solving fractional optimal control problems using the operational matrix of Mott polynomials*, Comput. methods differ. equ., *10*(3) (2022), 755–773.

[2] K. E. Atkinson, *The Numerical Solution of Integral equations of Second Kind*, Cambridge University Press, Cambridge, 1997.

[3] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer-Verlag Berlin, Heidelberg, 2010.

[4] S. Dixit, R. K. Pandey, S. Kumar, and O. P. Singh, *Solution of the generalized Abel integral equation by using almost operational matrix*, Am. J. Comput. Math., *1* (2011), 226–234.

[5] R. B. Guenther, *Linear Integral Equations (Ram P. Kanwal)*, SIAM Review, *14* (1972), 669.

[6] M. Gulsu, Y. Ozturk, and M. Sezer, *On the solution of the Abel equation of the second kind by the shifted Chebyshev polynomials*, Appl. Math. Comput., *217* (2011), 4827–4833.

[7] K. Issa, B. Yisa, and J. Biazar, *Numerical solution of space fractional diffusion equation using shifted Gegenbauer polynomials*, Comput. methods differ. equ., *10*(2) (2022), 431–444.

[8] K. Sadri, A. Amini, and C. Cheng, *A new operational method to solve Abel's and generalized Abel's integral equations*, Appl. Math. Comput., *317* (2018), 49–67.

[9] J. Shen, T. Tang, and L. L. Wang, *Spectral Methods: Algorithms, Analysis and Applications*, Springer-Verlag Berlin, Heidelberg 2011.

[10] C. S. Singh, H. Singh, V. K. Singh, and O. P. Singh, *Fractional order operational matrix methods for fractional singular integro-differential equation*, Appl. Math. Model., *40* (2016), 10705–10718.

[11] O. P. Singh, V. K. Singh, and R. K. Pandey, *A stable numerical inversion of Abel's integral equation using almost Bernstein operational matrix*, Appl. Numer. Math., *62* (2012), 567–579.

[12] S. Sohrabi, *Comparison Chebyshev wavelets method with BPFs method for solving Abel's integral equation*, Ain Shams Eng. J., *2* (2011), 249–254.