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Analytical solutions of the fractional (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation

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Abstract

This paper addresses the (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation with fractional derivative definition. Initially, conformable derivative definitions and their features are presented. Then, by submitting $\exp(-\varphi(\xi))$-expansion, generalized (G'/G)-expansion and Modified Kudryashov methods, exact solutions of this equation are generated. The 3D, contour, and 2D surfaces, as well as the related contour plot surfaces of some acquired data, are used to draw the physical aspect of the obtained findings. The physical meaning of the geometrical structures for some of these solutions is discussed. For the observation of the physical activities of the problem, achieved exact solutions are vital. The acquired results can help to demonstrate the physical application of the investigated models and other nonlinear physical models found in mathematical physics. Therefore, it would appear that these approaches might yield noteworthy results in producing the exact solutions to fractional differential equations in a wide range.

Keywords. Modified Kudryashov method, Generalized (G'/G)-expansion method, Exp($-\varphi(\xi)$)-expansion method, Fractional (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation, Conformable derivative.

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1. Introduction

Fractional differential equations arise in many different branches of social and basic sciences and engineering. They have gained prominence in recent years due to their crucial role in various domains involving complicated physical processes, from control theory and electrical circuits to wave propagation. Especially, they appear in electrodynamics of complex medium, electrical networks, signal and image processing, and electrodynamics, including porous flow, surface water flow, land sliding, faulting, circled fuel reactor, seismic waves, compressional and shear waves, wave motion, and distribution, transmission lines (see [10, 13–15, 18, 24–28, 31] and the references therein). They are used for modeling, analyzing, and designing many engineering problems. Since they more clearly illustrate nonlinear physical features and serve as a roadmap for future work, solutions to these equations have been among the most amazing in the related areas.

In order to compute these solutions and better understand the fundamental characteristics of physical structures in varied contexts, several authors have employed a variety of techniques. As a result, analytical methods have been developed and it has been shown that no single technique can be used to solve all types of nonlinear problems with precision. Therefore, many different methods have emerged, some of which are sub-equation method [2, 22, 32], Sardar sub-equation method [34], sine-cosine method [36, 39], extended tanh-coth expansion method [4, 35], the extended sinh-Gordon equation expansion method [37], simple equation method and its modification [11, 12], modified (G'/G)-expansion method [5, 6, 40], etc.

High-dimensional fractional partial differential equations have attracted academics' curiosity greatly in recent years. They also appear in modeling many phenomena in biology, chemistry, physics, engineering, mechanics, economy, and

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many different branches. There are some derivative definitions for fractional differential equations. Some of these are AtanganaBaleanu [20], Riemann-Liouville [17], Caputo [38], and conformable [3] approaches. The Riemann-Liouville fractional derivative approach is the result of combining the fractional derivative approaches defined by two famous mathematicians, Riemann and Liouville, which are frequently used today. In addition to these derivatives, the conformable fractional derivative approach is also included in the literature; this approach is preferred by many mathematicians because of its simplicity and reliability. Especially, Boiti–Leon–Manna–Pempinelli equation is among them, and it is an important equation to describe incompressible liquid in fluid mechanics. Due to their unpredictable features, the significance of these equations has become crucial in studying their analytical solutions.

First, we pay attention to the new (3+1)-dimensional BoitiLeonMannaPempinelli (BLMP) equation proposed by Wazwaz [41]

$$(u_x + u_y + u_z)_t + (u_x + u_y + u_z)_{xxx} + (u_x(u_x + u_y + u_z))_x = 0. \tag{1.1}$$

Then, this equation was reduced and a (3+1)-dimensional BLMP equation was formed, consisting of potential derivatives of u_y and u_z [33]

$$(u_y + u_z)_t + (u_y + u_z)_{xxx} + (u_x(u_y + u_z))_x = 0. \tag{1.2}$$

In this research, the following fractional (3+1)–dimensional BLMP equation

$$\mathscr{D}_t^{\omega}(u_y + u_z) + (u_y + u_z)_{xxx} + (u_x(u_y + u_z))_x = 0, \tag{1.3}$$

is considered. The main contribution of the present work is to find analytical solutions to the above fractional differential equation, frequently used in applied mathematics to express mathematical representations of practical issues as an excellent tool to express the memory and inherited properties of many substances and processes. Conformable fractional derivative via $\exp(-\varphi(\xi))$-expansion, generalized (G'/G)-expansion, and modified Kudryashov methods are used to obtain analytical solutions. Moreover, theoretical results are supported by numerical experiments. On the other hand, the results are also shown by graphs, and the physical meaning of the geometrical structures for some of these solutions is discussed. The achieved exact solutions are vital for observing the physical activities of the problem. The acquired results can help to demonstrate the physical application of the investigated models and other nonlinear physical models found in mathematical physics.

To obtain the above-mentioned novelties, the structure of the paper is designed as follows. Section 2 is reserved for the basic definitions. The exp(-$\varphi(\xi)$)-expansion method is explained in section 3, (G'/G)-expansion method is introduced in section 4. The modified Kudryashov method is introduced in section 5. The solutions of the governing equation are given in section 6. Finally, the paper concludes in section 7.

2. Basic Definitions

Definition 1. A function's conformable derivative, $g : [0, \infty) \to \mathbb{R}, t > 0, \omega \in (0, 1)$ of order ω is defined as follows:

$$\mathscr{D}_t^{\omega}(g)(t) = \lim_{\beta \to 0} \frac{g(t + \beta t^{1-\omega}) - g(t)}{\beta}. \tag{2.1}$$

Also, if g is ω-differentiable in some range $(0, k)$, where $k > 0$ and the $\lim_{t\to 0^+} \mathscr{D}_t^{\omega}(g)(t)$ exists, then definition becomes

$$\mathscr{D}_t^{\omega}(g)(0) = \lim_{t \to 0^+} \mathscr{D}_t^{\omega}(g)(t). \tag{2.2}$$

Lemma 1. For $0 < \omega \leq 1$, let g_1 and g_2 be ω-differentiable at $t > 0$ [16, 23, 29]. Then,

- $\mathscr{D}_t^{\omega}(t^{h_1}) = h_1 t^{h_1 - \omega}$, $h_1 \in \mathbb{R}$,

- $\mathscr{D}_t^{\omega}(h_1 g_1 + h_2 g_2) = h_1 \mathscr{D}_t^{\omega}(g_1) + h_2 \mathscr{D}_t^{\omega}(g_2),\ h_1, h_2 \in \mathbb{R}$,
- $\mathscr{D}_t^{\omega}(\frac{g_1}{g_2}) = \frac{g_1.\mathscr{D}_t^{\omega}(g_2) - g_2.\mathscr{D}_t^{\omega}(g_1)}{g_2^2}$,
- $\mathscr{D}_t^{\omega}(g_1.g_2) = g_1.\mathscr{D}_t^{\omega}(g_2) + g_2.\mathscr{D}_t^{\omega}(g_1)$,
- Since w is also differentiable, $\mathscr{D}_t^{\omega}(g_1)(t) = t^{1-\omega}\frac{dg_1(t)}{dt}$,
- $\mathscr{D}_t^{\omega}(C) = 0$, where C is a constant.

3. The exp($-\varphi(\xi)$)-expansion method

Consider the nonlinear evolution equation given in the following form:

$$\mathscr{P}(u, \mathscr{D}_t^{\omega}, \mathscr{D}_x u, \mathscr{D}_y u, \mathscr{D}_x^2 u, \mathscr{D}_y^2 u, \cdots) = 0. \tag{3.1}$$

Here, $\mathscr{D}_t^{\omega}$ stands for the arbitrary order conformable derivative operator. $\mathscr{P}$ is a polynomial in $u(x, y, \cdots, t)$ and its derivatives, $u = u(x, y, \cdots, t)$ is an unknown function and the subscripts denote partial derivatives. When employing the $\exp(-\varphi(\xi))$-expansion approach [1, 19, 21] to get solitary wave solutions of Eq. 3.1, it is imperative to execute the following actions.

- By using a compound variable called ξ, we combine the actual variables $x, y, \cdots, t$

$$\xi = kx + wy + \cdots + c\frac{t^{\omega}}{\omega}, \quad u(x, y, \cdots, t) = u(\xi). \tag{3.2}$$

- Eq. 3.1 is reduced to the following ordinary differential equation

$$\mathscr{K}(u(\xi), u'(\xi), u''(\xi), \cdots) = 0. \tag{3.3}$$

- The exact solutions can be built as the following finite series:

$$u(\xi) = \sum_{r=0}^{N} a_r (exp(-\varphi(\xi)))^r, \quad a_N \neq 0, \quad 0 \leq r \leq N. \tag{3.4}$$

- $\varphi = \varphi(\xi)$ satisfies the following ordinary differential equation

$$\varphi'(\xi) = exp(-\varphi(\xi)) + \eta exp(\varphi(\xi)) + \lambda. \tag{3.5}$$

- Depending on the relevant parameters, Eq. 3.5 has the following solutions
  for $\lambda^2 - 4\eta > 0$ and $\eta \neq 0$,

$$u_1(\xi) = \ln\left(\frac{-\sqrt{(\lambda^2 - 4\eta)}\tanh(\frac{\sqrt{(\lambda^2-4\eta)}}{2}(\xi + h)) - \lambda}{2\eta}\right), \tag{3.6}$$

  for $\lambda^2 - 4\eta < 0$ and $\eta \neq 0$,

$$u_2(\xi) = \ln\left(\frac{\sqrt{(4\eta - \lambda^2)}\tan(\frac{\sqrt{(4\eta-\lambda^2)}}{2}(\xi + h)) - \lambda}{2\eta}\right), \tag{3.7}$$

for $\lambda^2 - 4\eta > 0$, $\lambda \neq 0$ and $\eta = 0$,

$$u_3(\xi) = -\ln\left(\frac{\lambda}{\sinh(\lambda(h+\xi)) + \cosh(\lambda(h+\xi)) - 1}\right), \tag{3.8}$$

for $\lambda^2 - 4\eta = 0$, $\lambda \neq 0$, and $\eta \neq 0$,

$$u_4(\xi) = \ln(-\frac{2(\lambda(\xi+h)+2)}{\lambda^2(\xi+h)}), \tag{3.9}$$

for $\lambda^2 - 4\eta = 0$, $\lambda = 0$, and $\eta = 0$,

$$u_5(\xi) = \ln(\xi + h), \tag{3.10}$$

where h is an integration constant.

- The positive integer N in Eq. 3.4 is calculated by taking into account the homogeneous balance between the largest nonlinear terms and the highest order derivatives of $u(\xi)$ in Eq. 3.3. Substituting Eq. 3.4 with Eq. 3.5 into Eq. 3.3 and collecting all terms with the same powers of exp(-φ) together, the left-hand side of Eq. 3.3 is converted into a polynomial. We get a series of algebraic equations in terms of $B_r(r = 0, 1, 2, \cdots, N), c, \lambda$ and η. After setting each coefficient of this polynomial to zero, the solutions obtained by solving the system of algebraic equations and then substituting the outcomes into equation Eq. 3.4 result in solutions of Eq. 3.3.

4. Generalized (G'/G)-expansion method

We provide a thorough explanation of our method in this section as (G'/G). Now, let us say that a nonlinear equation is given by [42, 43]

$$\mathscr{B}(n, n_t, n_x, n_y, n_{xx}, n_{yy}, \cdots) = 0. \tag{4.1}$$

We seek its solutions in a more general form

$$n = a_0 + \sum_{i=1}^{N} a_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i, \quad (i = 1, 2, ..., N), \tag{4.2}$$

with $G(\xi)$ that satisfies the following ordinary differential equation:

$$\xi = kx + wy + sz + h\frac{t^\omega}{\omega}, \quad G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{4.3}$$

where λ and μ are real constants. The following are the main steps we conduct using the (G'/G)-expansion approach to better explain.

- Identify the number N.
- The algebraic equation relating to N may be obtained by substituting Eq. 4.2 and Eq. 4.3 into Eq. 4.1 and balancing the highest order derivative term with the nonlinear components in Eq. 4.1.
- Solve the system of nonlinear partial differential equations.
- The explicit expressions can be found via symbolic computation of a_0, $a_i(i = 1, 2, \cdots, N)$ and ξ.
- Depending on the solution $G(\xi)$ of Eq. 4.1, we can derive a number of basic solutions of Eq. 4.1 by substituting the findings acquired in the aforementioned phases of Eq. 4.3.

5. Modified Kudryashov method

Take the following nonlinear partial differential equation

$$\mathscr{F}_\omega\left(v, \frac{\partial v}{\partial \xi}, \frac{\partial^2 v}{\partial \xi^2}, \cdots, \frac{\partial^\omega v}{\partial t^\omega}, \frac{\partial^{2\omega} v}{\partial t^{2\omega}}\right) = 0, \tag{5.1}$$

where $v = v(x, y, z, t)$ is the transformation

$$\xi = kx + wy + sz + c\frac{t^\omega}{\omega}, \quad v(x, y, z, t) = v(\xi), \tag{5.2}$$

will transform Eq. 5.2 in the form of an ordinary differential equation

$$\mathscr{F}(v(\xi), v'(\xi), v''(\xi), \cdots) = 0. \tag{5.3}$$

Suppose the solution to Eq. 5.3 has the structure of

$$v(\xi) = \sum_{r=0}^{N} B_r \varphi^r(\xi), \quad B_N \neq 0. \tag{5.4}$$

In the Eq. 5.4, the function $\varphi(\xi)$ satisfies the ODE

$$\varphi'(\xi) = \log(a)\varphi(\xi)(\varphi(\xi) - 1). \tag{5.5}$$

The solution to the Eq. 5.5 is provided by

$$\varphi(\xi) = \frac{1}{da^\xi + 1}, \quad a > 0, \quad a \neq 0, \tag{5.6}$$

where d is constant. Substituting 5.4 and 5.5 into Eq. 5.3 results in the polynomial in $\varphi^r(\xi)$, where $(r = 0, 1, 2, \cdots, N)$. A set of algebraic equations in k, c and B_r are produced by setting all of the coefficients of $\varphi^r(\xi)$ zero [30]. After resolving this system, we obtain the equation's unsolved variables. Last but not least, by entering these values into 5.4 and Eq. 5.6, the exact solutions to Eq. 5.1 are obtained.

6. Solutions for the governing equation

Consider the following (3+1)-dimensional BLMP equation which is the fractional version of Eq. 1.2

$$\mathscr{D}_t^\omega(u_y + u_z) + (u_y + u_z)_{xxx} + (u_x(u_y + u_z))_x = 0. \tag{6.1}$$

Letting the transformations $u(x, y, z, t) = u(\xi)$, $\xi = kx + wy + sz + c\frac{t^\omega}{\omega}$ and integrating we obtain

$$csu' + cwu' + k^3su^{(3)} + k^3u^{(3)}w + k^2s\left(u'\right)^2 + k^2w\left(u'\right)^2 = 0. \tag{6.2}$$

Balancing $u^{(3)} = N + 3$, $(u')^2 = 2(N + 1)$ one get $N = 1$. If we substitute it in Eq. 5.4 and Eq. 3.4, the results are given in the following subsections

6.1. **Analytical solutions by exp($-\varphi(\xi)$)-expansion method.** Since $N = 1$, when we substitute Eq. 3.4, the series of sums comes as follows:

$$u = a_0 + a_1 exp(-\varphi(\xi)). \tag{6.3}$$

The following algebraic system of equations is formed when combined with Eq. 3.5.

$$\begin{aligned}
&-a_1c\mu s - a_1c\mu w - a_1\lambda^2k^3\mu s - 2a_1k^3\mu^2 s - a_1\lambda^2k^3\mu w - 2a_1k^3\mu^2 w + a_1^2k^2\mu^2 s + a_1^2k^2\mu^2 w = 0,\\
&-a_1c\lambda s - a_1c\lambda w + a_1\lambda^3k^3(-s) - 8a_1\lambda k^3\mu s - a_1\lambda^3k^3 w - 8a_1\lambda k^3\mu w + 2a_1^2\lambda k^2\mu s + 2a_1^2\lambda k^2\mu w = 0,\\
&-a_1cs - a_1cw - 7a_1\lambda^2k^3 s - 8a_1k^3\mu s - 7a_1\lambda^2k^3 w - 8a_1k^3\mu w + a_1^2\lambda^2k^2 s + 2a_1^2k^2\mu s + a_1^2\lambda^2k^2 w + 2a_1^2k^2\mu w = 0,\\
&-12a_1\lambda k^3 s - 12a_1\lambda k^3 w + 2a_1^2\lambda k^2 s + 2a_1^2\lambda k^2 w = 0,\\
&-6a_1k^3 s - 6a_1k^3 w + a_1^2k^2 s + a_1^2k^2 w = 0.
\end{aligned}$$

Here, we obtain one case and one set of solutions for a_0, a_1, and c.

Case 1.

$$a_1 = 6k, c = 4k^3\mu - k^3\lambda^2. \tag{6.4}$$

Set 1.

For $\lambda^2 - 4\mu > 0$, $\mu \neq 0$,

$$u_1(x,y,z,t) = a_0 + \frac{12k\mu}{-\sqrt{\lambda^2-4\mu}\tanh\left(\frac{1}{2}(h+A_1)\sqrt{\lambda^2-4\mu}\right)-\lambda}, \tag{6.5}$$

for $\lambda^2 - 4\mu < 0$, $\mu \neq 0$,

$$u_2(x,y,z,t) = a_0 + \frac{12k\mu}{\sqrt{4\mu-\lambda^2}\tan\left(\frac{1}{2}(h+A_1)\sqrt{4\mu-\lambda^2}\right)-\lambda}, \tag{6.6}$$

for $\lambda^2 - 4\mu > 0$, $\lambda \neq 0$ and $\mu = 0$,

$$u_3(x,y,z,t) = a_0 + \frac{6k\lambda}{\sinh(\lambda(h+A_1))+\cosh(\lambda(h+A_1))-1}, \tag{6.7}$$

for $\lambda^2 - 4\mu = 0$,$\lambda \neq 0$ and $\mu \neq 0$,

$$u_4(x,y,z,t) = a_0 + 3k\lambda\left(\frac{2\omega}{\omega(\lambda(h+kx+sz+wy)+2) - k^3\lambda\left(\lambda^2-4\mu\right)t^\omega} - 1\right), \tag{6.8}$$

for $\lambda^2 - 4\mu = 0$,$\lambda = 0$ and $\mu = 0$,

$$u_5(x,y,z,t) = a_0 + \frac{6k}{h+A_1}, \tag{6.9}$$

where $A_1 = \left(-\frac{k^3\left(\lambda^2-4\mu\right)t^\omega}{\omega} + kx + sz + wy\right)$.

6.2. **Analytical solutions by the generalized (G'/G)-expansion method.** Since $N = 1$, when we substitute Eq. 4.2, the series of sums comes as follows:

$$n = a_0 + a_1 \left(\frac{G'(\xi)}{G(\xi)} \right), \quad a_1 \neq 0. \tag{6.10}$$

The following algebraic system of equations is formed when combined with Eq. 4.3

$$\begin{aligned}
&-a_1 h\mu s - a_1 h\mu w - a_1 \lambda^2 k^3 \mu s - 2a_1 k^3 \mu^2 s - a_1 \lambda^2 k^3 \mu w - 2a_1 k^3 \mu^2 w + a_1^2 k^2 \mu^2 s + a_1^2 k^2 \mu^2 w = 0,\\
&-a_1 hs - a_1 hw - 7a_1 \lambda^2 k^3 s - 8a_1 k^3 \mu s - 7a_1 \lambda^2 k^3 w - 8a_1 k^3 \mu w + a_1^2 \lambda^2 k^2 s + 2a_1^2 k^2 \mu s + a_1^2 \lambda^2 k^2 w + 2a_1^2 k^2 \mu w = 0,\\
&-a_1 h\lambda s - a_1 h\lambda w + a_1 \lambda^3 k^3 (-s) - 8a_1 \lambda k^3 \mu s - a_1 \lambda^3 k^3 w - 8a_1 \lambda k^3 \mu w + 2a_1^2 \lambda k^2 \mu s + 2a_1^2 \lambda k^2 \mu w = 0,\\
&-12a_1 \lambda k^3 s - 12a_1 \lambda k^3 w + 2a_1^2 \lambda k^2 s + 2a_1^2 \lambda k^2 w = 0,\\
&-6a_1 k^3 s - 6a_1 k^3 w + a_1^2 k^2 s + a_1^2 k^2 w = 0.
\end{aligned}$$

Here, we obtain one case and one set of solutions for a_0, a_1, and h.

Case 2.

$$a_1 = 6k, h = 4k^3\mu - k^3\lambda^2. \tag{6.11}$$

Set 2.

For $\lambda^2 - 4\mu > 0$,

$$n_1(x, y, z, t) \quad = \quad a_0 + 6k \left(\vartheta_1 - \frac{\lambda}{2} \right), \tag{6.12}$$

for $\lambda^2 - 4\mu < 0$,

$$n_2(x, y, z, t) \quad = \quad a_0 + 6k \left(\vartheta_2 - \frac{\lambda}{2} \right), \tag{6.13}$$

for $\lambda^2 - 4\mu = 0$,

$$n_3(x, y, z, t) \quad = \quad a_0 + 6k \left(\frac{c_2}{c_2 (\xi) + c_1} - \frac{\lambda}{2} \right). \tag{6.14}$$

Here, c_1 and c_2 are arbitrary constants, where,

$$\begin{aligned}
\vartheta_1 &= \frac{\sqrt{\lambda^2 - 4\mu} \left(c_2 \sinh \left(\frac{1}{2}\xi\sqrt{\lambda^2 - 4\mu} \right) + c_1 \cosh \left(\frac{1}{2}\xi\sqrt{\lambda^2 - 4\mu} \right) \right)}{2 \left(c_1 \sinh \left(\frac{1}{2}\xi\sqrt{\lambda^2 - 4\mu} \right) + c_2 \cosh \left(\frac{1}{2}\xi\sqrt{\lambda^2 - 4\mu} \right) \right)},\\
\vartheta_2 &= \frac{\sqrt{4\mu - \lambda^2} \left(c_2 \cos \left(\frac{1}{2}\xi\sqrt{4\mu - \lambda^2} \right) - c_1 \sin \left(\frac{1}{2}\xi\sqrt{4\mu - \lambda^2} \right) \right)}{2 \left(c_2 \sin \left(\frac{1}{2}\xi\sqrt{4\mu - \lambda^2} \right) + c_1 \cos \left(\frac{1}{2}\xi\sqrt{4\mu - \lambda^2} \right) \right)},\\
\xi &= \left(\frac{t^\omega \left(4k^3\mu - k^3\lambda^2 \right)}{\omega} + kx + sz + wy \right).
\end{aligned}$$

6.3. **Analytical solutions by the modified Kudryashov method.** Since $N = 1$, when we substitute Eq. 5.4, the series of sums comes as follows:

$$v = B_0 + B_1\varphi(\xi), \quad B_1 \neq 0. \tag{6.15}$$

The following algebraic system of equations is formed when combined with Eq. 5.5

$$\begin{aligned}
&-B_1cs\log(a) - B_1cw\log(a) + B_1\left(-k^3\right)s\log^3(a) - B_1k^3w\log^3(a) = 0,\\
&-12B_1k^3s\log^3(a) - 12B_1k^3w\log^3(a) - 2B_1^2k^2s\log^2(a) - 2B_1^2k^2w\log^2(a) = 0,\\
&6B_1k^3s\log^3(a) + 6B_1k^3w\log^3(a) + B_1^2k^2s\log^2(a) + B_1^2k^2w\log^2(a) = 0,\\
&B_1cs\log(a) + B_1cw\log(a) + 7B_1k^3s\log^3(a) + 7B_1k^3w\log^3(a) + B_1^2k^2s\log^2(a) + B_1^2k^2w\log^2(a) = 0.
\end{aligned}$$

Here, we obtain one case and one set of solutions for B_0, B_1, and c.

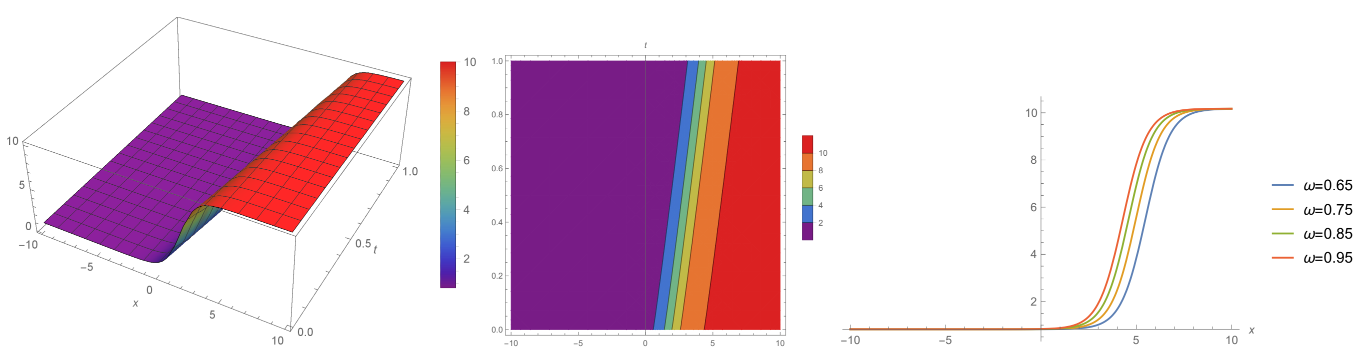
Case 3.

$$B_1 = -6k\log(a),\ c = -k^3\log^2(a),\ \xi = c\frac{t^\omega}{\omega} + kx + sz + wy. \tag{6.16}$$

If we insert these values into Eq. 6.15, using Eq. 5.6, we have the solution set as follows:

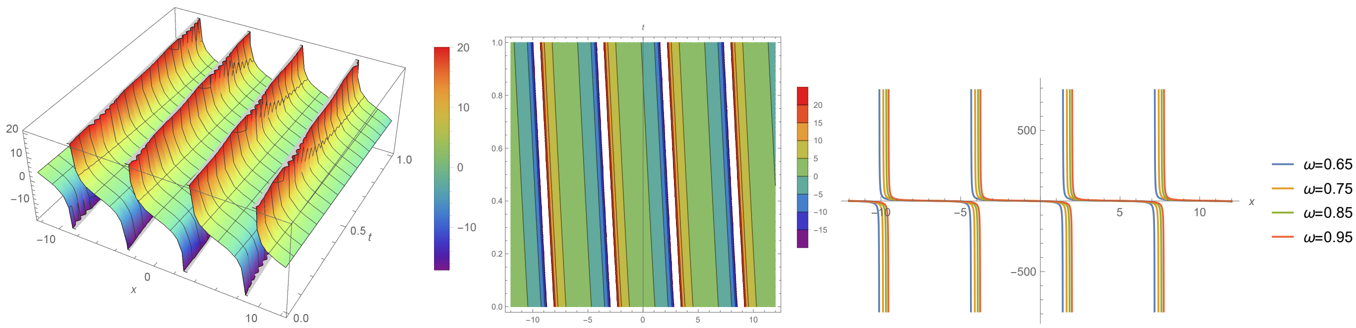
Set3.

$$v_1(x, y, z, t) \quad = \quad \frac{6k\log(a)}{da^{-\frac{k^3\log^2(a)t^\omega}{\omega} + kx + sz + wy} + 1} + B_0. \tag{6.17}$$



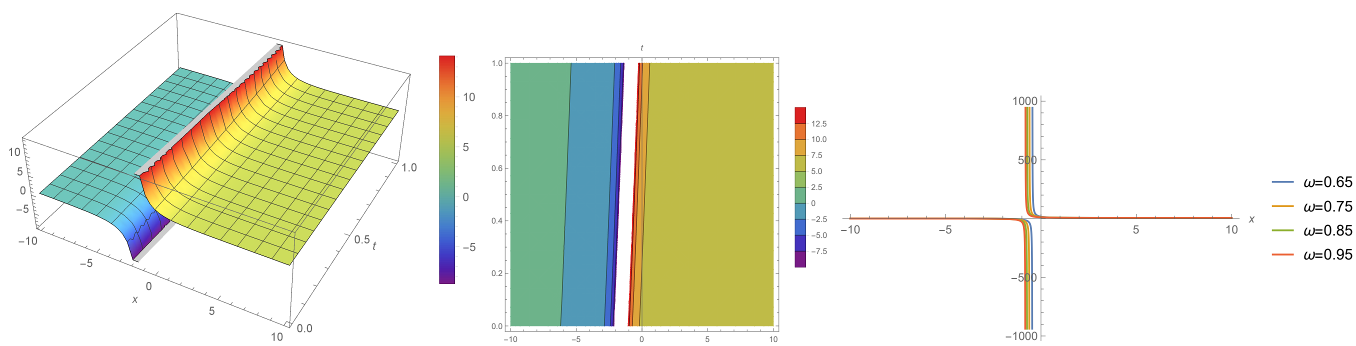
(A) (B) (C)

FIGURE 1. The 3D (A), contour (B) and 2D plots (C) of $\exp(-\varphi(\xi))$-expansion exact solution $u_1(x, y, z, t)$ of Eq. 6.5.



(A) (B) (C)

FIGURE 2. The 3D (A), contour (B) and 2D plots (C) of (G'/G)-expansion exact solution $n_2(x, y, z, t)$ of Eq. 6.13.



(A) (B) (C)

FIGURE 3. The 3D (A), contour (B) and 2D plots (C) of Modified Kudryashov exact solution $v_1(x, y, z, t)$ of Eq. 6.17.

Results and discussion

To investigate the behavior of solitons and evaluate the physical relevance of the obtained solutions, a variety of graphs which are plotted for $-10 \leq x \leq 10$, $-10 \leq y \leq 10$ are displayed in this work for Eq. 6.5, which is a dark soliton, Eq. 6.13, which is a periodic soliton and Eq. 6.17, which is a singular soliton. By choosing suitable values for unknown parameters, the 3D, contour, and 2D plots are presented in Figures 13 with the following values

- 1 with (a) and (b) for $k = -0.9, w = 0.8, s = -0.16, y = 0.01, z = -0.02, a_0 = 0.1, \mu = 0.25, \lambda = 2, h = 0.1$ and $\omega = 0.95$. Also for (c) $t = 0.97$.
- 2 with (a) and (b) for $c_1 = 5, c_2 = 7, k = w = 2, s = -1, y = 2, z = 1, a_0 = 0.1, \lambda = -0.25, \mu = 0.09$ and $\omega = 0.95$. Also for (c) $t = 0.9$.
- 3 with (a) and (b) for $a = 0.09, k = 0.36, w = 0.46, s = -0.16, y = 0.01, z = 0.02, B_0 = 0.1, d = -0.25$ and $\omega = 0.95$. Also for (c) $t = 0.99$.

The graphical representations reveal some new solutions that affirm the proposed approaches could be helpful for obtaining exact solutions to other types of equations.

7. CONCLUSION

In this study, $\exp(-\varphi(\xi))$-expansion, generalized (G'/G)-expansion, and modified Kudryashov methods were used to examine the soliton properties of the fractional (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation with conformable derivative. Then, for some of the solutions, 3D and 2D graphics were displayed to visualize the solutions with appropriate values. The accuracy of these approaches has been demonstrated by analytical results and graphical examples. Additionally, these solutions feature unique, significant physical characteristics that have been documented in the literature. The physical meaning of the geometrical structures for some of these solutions is discussed. For the observation of the physical activities of the problem, achieved exact solutions are vital. The proposed methods are believed to be powerful, effective, and may play an important role in describing the physical features of various nonlinear complex models. Therefore, these techniques can treat and solve some other fractional differential equations.

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