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Numerical solution of third-Order boundary value problems using non-classical sinc-collocation method

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Abstract

In this work, a non-classical sinc-collocation method is used to find numerical solution of third-order boundary value problems. The novelty of this approach is based on using the weight functions in the traditional sinc-expansion. The properties of sinc-collocation are used to reduce the boundary value problems to a nonlinear system of algebraic equations which can be solved numerically. In addition, the convergence of the proposed method is discussed by preparing the theorems which show exponential convergence and guarantee its applicability. Several examples are solved and the numerical results show the efficiency and applicability of the method.

Keywords. Non-classical, Sinc collocation method, Third-order, Boundary value problem, Convergence.
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1. Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, for instance, in the deflection of a curved beam having a constant or varying cross section, aeroelasticity, electromagnetic waves, a three layer beam, the theory of thin film flow, and incompressible flows [14, 19]. Since the boundary value problems have wide applications in scientific research, therefore, faster and more accurate numerical solution of boundary value problems is very important.

Some researchers have studied and numerically solved third order boundary value problems using different methods with different boundary conditions, for instance, finite difference [20], modified Adomian decomposition [9], Reproducing kernel [14], nonpolynomial splines [6, 10], quintic splines [13], B-spline functions [4], Haar wavelets [8], sinc-collocation method [22] and boundary shape function methods [15].

In recent years a variety of numerical methods based on sinc approximation have been developed. Sinc methods were introduced by Frank stenger in [25, 26] and has been extended in [27]. In this paper we consider the following class of third order boundary value problems:

$$Lx = \varepsilon x'''(t) + p_1(t)x''(t) + p_2(t)x'(t) = g(t, x), \tag{1.1}$$

subject to one of the following boundary conditions

$$(i) \quad x(0) = \alpha, \quad x'(0) = \beta, \quad x(1) = \gamma, \tag{1.2}$$

$$(ii) \quad x(0) = \alpha, \quad x'(0) = \beta, \quad x'(1) = \gamma, \tag{1.3}$$

$$(iii) \quad x'(0) = \alpha, \quad x(1) = \beta, \quad x'(1) = \gamma, \tag{1.4}$$

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$$(iv) \quad x(0) = \alpha, \quad x(1) = \beta, \quad x'(1) = \gamma, \tag{1.5}$$

where p_1 and p_2 are continuous functions on $(0, 1)$, $0 < \varepsilon \le 1$, and the function $g(t, x)$ satisfies the Lipschitz condition: $|g(t, x) - g(t, x^*)| \le l|x - x^*|$ and α, β and γ are constants.

Here, we use a non-classical sinc-collocation method for the solution of problem (1.1) subject to one of the boundary conditions (1.2)-(1.5). The idea of employing nonclassical weight functions for the first time has been used by Shizgal for solving the Boltzmann equation and related problems [24]. Alipanah et.al used the nonclassical pseudospectral method to solve the brachistochrone problem [1]. Our method reduces the solution of equation (1.1) with each of the boundary conditions (1.2)-(1.5) to a set of algebraic equations. We used the non-classical sinc basis functions because of their better accuracy compared to the classic. The numerical results that we illustrate the high accuracy and robustness of the method.

The main difficulty of the problem (1.1) is when $0 < \varepsilon < 1$, in which case we call it the singular perturbed boundary value problem. Our method for solving such problems has much better accuracy compared to some other methods. Youssri et.al used the modified lucas polynomials method for the numerical treatment of second-order boundary value problems [30]. Khan et.al used the non-polynomial cubic spline method for the solution of higher order boundary value problems [11]. Taherkhani et.all used a pseudospectral sinc method for numerical investigation of the nonlinear time-fractional Klein-Gordon and Sine-Gordon equations [28]. Babolian et.al used a sinc-Galerkin technique for the numerical solution of a class of singular boundary value problems [2] and Eftekhari et.al used DE-sinc-collocation method for solving a class of second-order nonlinear BVPs [5] also Saadatmandi et.al used numerical calculation of fractional derivatives for the sinc functions via Legendre polynomials [21].

The paper is organized as follows: In section 2, we review some of the main properties of the sinc function that are necessary for the formulation of the discrete system. In section 3, we explain the interpolation using non-classical sinc functions. In section 4, the presented method is used to approximate the solution of problem (1.1) with each of the boundary conditions (1.2)-(1.5). In section 5, the error analysis of the method is discussed. In section 6, numerical examples are given to illustrate the efficiency of the presented method. The conclusion is presented in section 7.

2. **Sinc function preliminaries**

The sinc function on $\mathbb{R}$ is defined as follows

$$Sinc(t) = \begin{cases} \frac{sin(\pi t)}{\pi t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

For $h > 0$, the translated sinc functions is defined as follows [16]

$$S(j, h)(t) = Sinc\left(\frac{t - jh}{h}\right), \quad j = 0, \pm1, \pm2, \cdots . \tag{2.1}$$

The sinc functions are cardinal for the interpolating points $t_k = kh$, i.e,

$$S(j, h)(kh) = \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

Let g be a function defined on the real line then for $h \in \mathbb{R}^+$ the series

$$C(g, h)(t) = \sum_{j=-\infty}^{\infty} g(jh)S(j, h)(t),$$

is called the Whittaker cardinal expansion of g whenever this series converges. Obviously, the cardinal function interpolates g at the points $\{jh\}_{j=-\infty}^{\infty}$ and it is based on the infinite strip D_s in the complex plane

$$D_s = \{z = u + iv : |v| < d \le \pi/2\}.$$

Such an approximations can be constructed for infinite, semi-infinite and finite intervals [16]. To construct approximation on the interval $(0, 1)$, we employ the conformal map

$$\psi(w) = ln\left(\frac{w}{1-w}\right), \tag{2.2}$$

which maps the eye-shaped region

$$D_E = \left\{w = x + iy : \left|arg\left(\frac{w}{1-w}\right)\right| < d \leq \pi/2\right\},$$

onto D_s.
The sinc basis functions used in our numerical approach are given by the composition of the translated sinc functions $S(j, h)$ and the conformal map ψ as follows

$$S_j(w) = S(j, h)o\psi(w) = Sinc\left(\frac{\psi(w) - jh}{h}\right), \quad w \in D_E, \tag{2.3}$$

where

$$w = \psi^{-1}(z) = \frac{e^z}{1+e^z},$$

is an inverse mapping of $z = \psi(w)$.
For the evenly spaced knots $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the corresponding images $t_k \in (0, 1)$ which are real in D_E are as follows

$$t_k = \psi^{-1}(kh) = \frac{e^{kh}}{1+e^{kh}}, \quad k = 0, \pm 1, \pm 2, \ldots \tag{2.4}$$

3. Non-classical sinc function interpolation

Let $g(t)$ be a given function on the real line, we define the approximation of $g(t)$ with non-classical sinc as follows

$$g(t) \simeq \hat{g}(t) = \sum_{j=-\infty}^{\infty} \frac{W(t)}{W(jh)} g(jh) sinc\left(\frac{t-jh}{h}\right), \tag{3.1}$$

where $W(t)$ is a positive weight function. For the interpolating points $t_k = kh$, we have

$$\hat{g}(kh) = g(kh), \quad k = 0, \pm 1, \pm 2, \cdots.$$

Definition 3.1. Let $B(D_s)$ be the set of all analytical functions in D_s that satisfy

$$\int_{-d}^{d} |g(t+iv)|dv = O\left(|t|^a\right), \quad t \to \pm\infty, \quad 0 \leq a < 1,$$

and

$$N(g, D_s) = \lim_{v \to d^-} \left\{\int_{-\infty}^{\infty} |g(t+iv)|dt + \int_{-\infty}^{\infty} |g(t-iv)|dt\right\} < \infty.$$

In fact, $B(D_s)$ is the class of functions that the sinc approximation converges exponentially.

Theorem 3.2. *If $g \in B(D_s)$, $h \in \mathbb{R}$ and the weight function W is selected such that*

$$\frac{W(u)}{W(jh)} < c_1 < \infty, \tag{3.2}$$

then

$$g(u) - \hat{g}(u) = S_h(u)I(u),$$

where

$$S_h(u) = \frac{\sin\left(\frac{\pi u}{h}\right)}{2\pi i}, \quad i = \sqrt{-1},$$

$$I(u) = \int_{-\infty}^{\infty} \left(\frac{G(u, t - id^-)W(u)}{\sin(\pi(t - id^-)/h)W(t - id^-)} - \frac{G(u, t + id^-)W(u)}{\sin(\pi(t + id^-)/h)W(t + id^-)} \right) dt,$$

$$G(s, u \pm iv) = \frac{g(u \pm iv)}{(u - s \pm iv)}.$$

Moreover

$$||g - \hat{g}||_\infty \leq \frac{c_1 N(g, D_s)}{2\pi d \sinh\left(\frac{\pi d}{h}\right)} = O\left(e^{-\frac{\pi d}{h}}\right). \tag{3.3}$$

Proof. Let us define $f(z)$ as follows

$$f(z) = \frac{\sin\left(\frac{\pi u}{h}\right) g(z) W(u)}{(z - u)\sinh\left(\frac{\pi z}{h}\right) W(z)},$$

then the proof is similar with theorem 2.13 in [16]. □

Theorem 3.3. *Let $g \in B(D_s)$, condition (3.2) is satisfied and there are positive constants β_1, β_2, and c such that*

$$|g(u)| \leq c \begin{cases} e^{-\beta_1 |u|}, & u \in (-\infty, 0), \\ e^{-\beta_2 |u|}, & u \in [0, \infty). \end{cases}$$

Also let

$$N = \left[\left|\frac{\beta_1}{\beta_2} M + 1\right|\right], \tag{3.4}$$

and

$$h = \left(\frac{\pi d}{\beta_1 M}\right)^{\frac{1}{2}} \leq \frac{2\pi d}{ln(2)}, \tag{3.5}$$

then

$$||g - \hat{g}_{M,N}|| \leq k_1 c_1 M^{\frac{1}{2}} e^{-\sqrt{\pi d \beta_1 M}}, \tag{3.6}$$

where

$$\hat{g}_{M,N}(u) = \sum_{j=-M}^{N} g(jh) \frac{W(u)}{W(jh)} sinc\left(\frac{u - jh}{h}\right). \tag{3.7}$$

Proof. Using equations (3.1) and (3.7):

$$|g(u) - \hat{g}_{M,N}(u)| = \left| g(u) - \hat{g}(u) + \sum_{j=-\infty}^{-M-1} g(jh) \frac{W(u)}{W(jh)} sinc\left(\frac{u - jh}{h}\right) + \sum_{j=N+1}^{\infty} g(jh) \frac{W(u)}{W(jh)} sinc\left(\frac{u - jh}{h}\right) \right|.$$

Using triangular inequality it leads to

$$|g(u) - \hat{g}_{M,N}(u)| \leq |g(u) - \hat{g}(u)| + \left| \sum_{j=-\infty}^{-M-1} g(jh) \frac{W(u)}{W(jh)} sinc\left(\frac{u - jh}{h}\right) \right| + \left| \sum_{j=N+1}^{\infty} g(jh) \frac{W(u)}{W(jh)} sinc\left(\frac{u - jh}{h}\right) \right|.$$

Now by some computation, we obtain

$$\left| \sum_{j=-\infty}^{-M-1} g(jh) \frac{W(u)}{W(jh)} sinc\left(\frac{u - jh}{h}\right) \right| \leq c_1 \sum_{j=M+1}^{\infty} |g(-jh)| \leq c_1 c \sum_{j=M+1}^{\infty} e^{-\beta_1 jh}$$

$$= c_1 c \left(\frac{e^{-\beta_1 h(M+1)}}{1 - e^{-\beta_1 h}}\right) = c_1 c e^{-\beta_1 M h} \left(\frac{1}{e^{\beta_1 h} - 1}\right) \leq \frac{c_1 c}{\beta_1 h} e^{-\beta_1 M h}, \tag{3.8}$$

similarly,

$$\left|\sum_{j=N+1}^{\infty} g(jh)\frac{W(u)}{W(jh)} sinc\left(\frac{u-jh}{h}\right)\right| \leq \frac{c_1 c}{\beta_2 h} e^{-\beta_2 Nh}. \tag{3.9}$$

Using (3.3) and (3.8)-(3.9):

$$|g(u) - \hat{g}_{M,N}(u)| \leq \frac{c_1 c}{\beta_1 h} e^{-\beta_1 Mh} + \frac{c_1 c}{\beta_2 h} e^{-\beta_2 Nh} + \frac{2c_1 N(g, D_s)}{\pi d} e^{-\frac{\pi d}{h}}. \tag{3.10}$$

Finally using (3.4) and (3.5), we obtain

$$\begin{aligned} |g(u) - \hat{g}_{M,N}(u)| &\leq \left(\frac{c}{\beta_1} + \frac{c}{\beta_2} + \frac{2N(g, D_s)}{\pi d}\sqrt{\frac{\pi d}{\beta_1}}\right)\sqrt{\frac{\beta_1}{\pi d}} c_1 M^{\frac{1}{2}} e^{-\sqrt{\pi d \beta_1 M}} \\ &\equiv c_1 k_1 M^{\frac{1}{2}} e^{-\sqrt{\pi d \beta_1 M}}. \end{aligned}$$

□

Definition 3.4. Let D_E be a simply connected domain in $\mathbb{C}$ with boundary points a and b. Let ψ be a conformal map from D_E onto D_s with $\psi(a) = -\infty$ and $\psi(b) = \infty$. Also, we denote the inverse map of ψ by ϕ and we define

$$\Gamma = \{w \in \mathbb{C} : w = \phi(u), u \in \mathbb{R}\},$$

and

$$w_j = \phi(jh), \quad j = 0. \pm 1, \pm 2, \ldots.$$

Definition 3.5. Let $B(D_E)$ be the class of functions G which are analytic in D_E, and

$$\int_{\phi(L+u)} |G(w)dw| \to 0, \quad u \to \pm\infty,$$

where $L = \{iv : |v| < d\}$ and

$$N(G, D_E) \equiv \int_{\partial D_E} |G(w)dw| < \infty,$$

where ∂D_E is the boundary of D_E.

Theorem 3.6. *Suppose $G \in B(D_E)$ and the weight function W is selected such that $\frac{W(\xi)}{W(w_j)} < c_1$, then for all $\xi \in \Gamma$*

$$\varepsilon(G)(\xi) \equiv \frac{G(\xi)}{\psi'(\xi)} - \sum_{j=-\infty}^{\infty} \frac{G(w_j)W(\xi)}{\psi'(w_j)W(w_j)} sinc\left(\frac{\psi(\xi) - jh}{h}\right)$$

$$= \frac{\sin\left(\frac{\pi\psi(\xi)}{h}\right)}{2\pi i} \lim_{\gamma \to \partial D_E} \int_\gamma \frac{G(w)W(\xi)}{(\psi(w) - \psi(\xi))\sin(\pi\psi(w)/h)W(w)} dw,$$

moreover

$$||\varepsilon(G)||_\infty \leq \frac{c_1 N(G, D_E)}{2\pi d \sinh\left(\frac{\pi d}{h}\right)} \leq \frac{2c_1 N(G, D_E)}{\pi d} e^{-\frac{\pi d}{h}}. \tag{3.11}$$

Proof. Let us define the conformal rectangles $\phi(R_n)$, where

$$R_n = \left\{z \in \mathbb{C} : z = u + iv, |v| < v_n, \quad -\left(n + \frac{1}{2}\right)h < u < \left(n + \frac{1}{2}\right)h\right\},$$

and $v_n = d - \frac{1}{n}$. Note that $R_n \subset D_s$. Using the change of variables $z = \psi(w)$ and $\psi(\xi) = u$, we have

$$\frac{\sin\left(\frac{\pi u}{h}\right)}{2\pi i}\int_{\partial\psi(R_n)} \frac{G(w)W(\xi)}{(\psi(w) - \psi(\xi))\sin\left(\frac{\pi z}{h}\right)W(w)} dw = \frac{\sin\left(\frac{\pi u}{h}\right)}{2\pi i}\int_{\partial R_n} \frac{G(\phi(z))\phi'(z)W(\phi(u))}{(z - u)\sin\left(\frac{\pi z}{h}\right)W(\phi(z))} dz.$$

Similar to theorem 1.13 and 2.13 in [16] and theorem 3.2

$$\frac{\sin\left(\frac{\pi u}{h}\right)}{2\pi i}\int_{\partial R_n}\frac{G(\phi(z))\phi'(z)W(\phi(u))}{(z-u)\sin\left(\frac{\pi z}{h}\right)W(\phi(z))}dz = G(\phi(u))\phi'(u) - \sum_{j=-M}^{N} G(\phi(jh))\phi'(jh)\frac{W(\phi(u))}{W(\phi(jh))}sinc\left(\frac{u-jh}{h}\right)$$

given that $\psi(\xi)=u$, $\xi=\phi(u)$ and $\phi'(u)=\frac{1}{\psi'(\xi)}$ then

$$G(\phi(u))\phi'(u) - \sum_{j=-M}^{N} G(\phi(jh))\phi'(jh)\frac{W(\phi(u))}{W(\phi(jh))}sinc\left(\frac{u-jh}{h}\right) = \frac{G(\xi)}{\psi'(\xi)} - \sum_{j=-M}^{N}\frac{G(w_j)}{\psi'(w_j)}\frac{W(\xi)}{W(w_j)}sinc\left(\frac{\psi(\xi)-jh}{h}\right).$$

The proof of (3.11) is similar to the proof of (3.3) in theorem 3.2. □

Corollary 3.7. *Let $G\in B(D_E)$ and there are positive constants β_1, β_2, and c such that*

$$\left|\frac{G(t)}{\psi'(t)}\right| \le c\begin{cases} e^{-\beta_1|\psi(t)|}, & t\in\Gamma_a,\\ e^{-\beta_2|\psi(t)|}, & t\in\Gamma_b,\end{cases}$$

where

$$\Gamma_a = \{t\in\Gamma : \psi(t)\in(-\infty,0)\}, \tag{3.12}$$

and

$$\Gamma_b = \{t\in\Gamma : \psi(t)\in[0,\infty)\}. \tag{3.13}$$

If N and h are given in (3.4) and (3.5), and the weight function W is selected such that $\frac{W(t)}{W(t_j)} < c_1$, then for all $t\in\Gamma$,

$$\left|\frac{G(t)}{\psi'(t)} - \sum_{j=-M}^{N}\frac{G(t_j)W(t)}{\psi'(t_j)W(t_j)}sinc\left(\frac{\psi(t)-jh}{h}\right)\right| \le k_2c_1M^{\frac{1}{2}}e^{-\sqrt{\pi d\beta_1 M}}.$$

Proof. The proof is similar the theorem 3.3. □

Theorem 3.8. *Let $\psi' G\in B(D_E)$ and the weight function is selected such that $\frac{W(t)}{W(t_j)} < c_1$, then for all $t\in\Gamma$*

$$\left|G(t) - \sum_{j=-\infty}^{\infty} G(t_j)\frac{W(t)}{W(t_j)}sinc\left(\frac{\psi(t)-jh}{h}\right)\right| \le \frac{c_1N(\psi'G, D_E)}{2\pi d\sinh\left(\frac{\pi d}{h}\right)}. \tag{3.14}$$

Also, let there are positive constants β_1, β_2, and c such that

$$|G(t)| \le c\begin{cases} e^{-\beta_1|\psi(t)|}, & t\in\Gamma_a,\\ e^{-\beta_2|\psi(t)|}, & t\in\Gamma_b,\end{cases}$$

where Γ_a and Γ_b are defined in (3.12) and (3.13). If N and h are given as (3.4) and (3.5), then for all $t\in\Gamma$,

$$\left|G(t) - \sum_{j=-M}^{N} G(t_j)\frac{W(t)}{W(t_j)}sinc\left(\frac{\psi(t)-jh}{h}\right)\right| \le c_1k_3M^{1/2}e^{-\sqrt{\pi d\beta_1 M}}. \tag{3.15}$$

Proof. The relation (3.14) can be obtained using (3.11) in theorem 3.6 for $\psi' G$.
To prove (3.15), using triangular inequality

$$\left|G(t) - \sum_{j=-M}^{N} G(t_j)\frac{W(t)}{W(t_j)} sinc\left(\frac{\psi(t)-jh}{h}\right)\right| \leq \left|G(t) - \sum_{j=-\infty}^{\infty} G(t_j)\frac{W(t)}{W(t_j)} sinc\left(\frac{\psi(t)-jh}{h}\right)\right| + \left|\sum_{j=-\infty}^{-M-1} G(t_j)\frac{W(t)}{W(t_j)} sinc\left(\frac{\psi(t)-jh}{h}\right)\right| + \left|\sum_{j=N+1}^{\infty} G(t_j)\frac{W(t)}{W(t_j)} sinc\left(\frac{\psi(t)-jh}{h}\right)\right|,$$

and some computations, we have

$$\begin{aligned}\left|\sum_{j=-\infty}^{-M-1} G(t_j)\frac{W(t)}{W(t_j)} sinc\left(\frac{\psi(t)-jh}{h}\right)\right| &\leq c_1 \sum_{j=-\infty}^{-M-1} |G(t_j)| \\ &= c_1 \sum_{j=-\infty}^{-M-1} \left|G\left(\psi^{-1}(jh)\right)\right| = c_1 \sum_{j=M+1}^{\infty} \left|G\left(\psi^{-1}(-jh)\right)\right| \\ &\leq c_1 c \sum_{j=M+1}^{\infty} e^{-\beta_1 jh} \\ &\leq \frac{c_1 c}{\beta_1 h} e^{-\beta_1 Mh}.\end{aligned} \tag{3.16}$$

Similarly,

$$\left|\sum_{j=N+1}^{\infty} G(t_j)\frac{W(t)}{W(t_j)} sinc\left(\frac{\psi(t)-jh}{h}\right)\right| \leq \frac{c_1 c}{\beta_2 h} e^{-\beta_2 Nh}. \tag{3.17}$$

Using (3.14), (3.16), and (3.17)

$$\left|G(t) - \sum_{j=-M}^{N} G(t_j)\frac{W(t)}{W(t_j)} sinc\left(\frac{\psi(t)-jh}{h}\right)\right| \leq \frac{c_1 c}{\beta_1 h} e^{-\beta_1 Mh} + \frac{c_1 c}{\beta_2 h} e^{-\beta_2 Nh} + \frac{2c_1 N(\psi' G, D_E)}{\pi d} e^{-\frac{\pi d}{h}},$$

and finally from (3.4) and (3.5), we obtain

$$\begin{aligned}\left|G(t) - \sum_{j=-M}^{N} G(t_j)\frac{W(t)}{W(t_j)} sinc\left(\frac{\psi(t)-jh}{h}\right)\right| &\leq \left[\frac{c}{\beta_1} + \frac{c}{\beta_2} + \frac{2N(\psi' G, D_E)}{\pi d}\sqrt{\frac{\pi d}{\beta_1}}\right]\sqrt{\frac{\beta_1}{\pi d}} c_1 M^{1/2} e^{-\sqrt{\pi d \beta_1 M}} \\ &\equiv c_1 k_3 M^{1/2} e^{-\sqrt{\pi d \beta_1 M}}.\end{aligned}$$

□

Theorem 3.9. *Let ${\psi'}^2 G \in B(D_E)$ and the weight function is selected such that $\frac{W(t)}{W(t_j)} < c_1$, then for all $t \in \Gamma$*

$$\left|G(t) - \sum_{j=-\infty}^{\infty} \phi'(t_j) G(t_j)\frac{W(t)}{W(t_j)} \frac{sinc\left(\frac{\psi(t)-jh}{h}\right)}{\psi'(t)}\right| \leq \frac{c_1 N({\psi'}^2 G, D_E)}{2\pi d \sinh\left(\frac{\pi d}{h}\right)}. \tag{3.18}$$

Also, let there are positive constants β_1, β_2, and c such that

$$|(\psi' G)(t)| \leq c \begin{cases} e^{-\beta_1|\psi(t)|}, & t \in \Gamma_a, \\ e^{-\beta_2|\psi(t)|}, & t \in \Gamma_b, \end{cases}$$

where Γ_a and Γ_b are defined in (3.12) and (3.13). If N and h are given as (3.4) and (3.5), then for all $t \in \Gamma$,

$$\left| G(t) - \sum_{j=-M}^{N} \phi'(t_j) G(t_j) \frac{W(t)}{W(t_j)} \frac{sinc\left(\frac{\psi(t)-jh}{h}\right)}{\psi'(t)} \right| \leq c_1 k_4 M^{1/2} e^{-\sqrt{\pi d \beta_1 M}}. \tag{3.19}$$

Proof. Relation (3.18) can be proved using (3.11) in theorem 3.6 for ${\psi'}^2 G$.
To prove (3.19), using triangular inequality

$$\begin{aligned} \left| G(t) - \sum_{j=-M}^{N} G(t_j)\psi'(t_j) \frac{W(t)}{W(t_j)} \frac{sinc\left(\frac{\psi(t)-jh}{h}\right)}{\psi'(t)} \right| &\leq \left| G(t) - \sum_{j=-\infty}^{\infty} G(t_j)\psi'(t_j) \frac{W(t)}{W(t_j)} \frac{sinc\left(\frac{\psi(t)-jh}{h}\right)}{\psi'(t)} \right| \\ &+ \left| \sum_{j=-\infty}^{-M-1} G(t_j)\psi'(t_j) \frac{W(t)}{W(t_j)} \frac{sinc\left(\frac{\psi(t)-jh}{h}\right)}{\psi'(t)} \right| \\ &+ \left| \sum_{j=N+1}^{\infty} G(t_j)\psi'(t_j) \frac{W(t)}{W(t_j)} \frac{sinc\left(\frac{\psi(t)-jh}{h}\right)}{\psi'(t)} \right|, \end{aligned}$$

and some computations

$$\begin{aligned} \left| \sum_{j=-\infty}^{-M-1} G(t_j)\psi'(t_j) \frac{W(t)}{W(t_j)} \frac{sinc\left(\frac{\psi(t)-jh}{h}\right)}{\psi'(t)} \right| &\leq \frac{c_1}{4} \sum_{j=-\infty}^{-M-1} |(\psi' G)(t_j)| \\ &= \frac{c_1}{4} \sum_{j=-\infty}^{-M-1} \left|(\psi' G)\left(\psi^{-1}(jh)\right)\right| = \frac{c_1}{4} \sum_{j=M+1}^{\infty} \left|(\psi' G)\left(\psi^{-1}(-jh)\right)\right| \\ &\leq \frac{c_1 c}{4} \sum_{j=M+1}^{\infty} e^{-\beta_1 jh} \\ &\leq \frac{c_1 c}{4\beta_1 h} e^{-\beta_2 Mh}, \end{aligned} \tag{3.20}$$

similarly

$$\left| \sum_{j=N+1}^{\infty} G(t_j)\psi'(t_j) \frac{W(t)}{W(t_j)} \frac{sinc\left(\frac{\psi(t)-jh}{h}\right)}{\psi'(t)} \right| \leq \frac{c_1 c}{4\beta_2 h} e^{-\beta_2 Nh}, \tag{3.21}$$

and using (3.18), (3.20), and (3.21), we have

$$\left| G(t) - \sum_{j=-M}^{N} G(t_j)\psi'(t_j) \frac{W(t)}{W(t_j)} \frac{sinc\left(\frac{\psi(t)-jh}{h}\right)}{\psi'(t)} \right| \leq \frac{c_1 c}{4\beta_1 h} e^{-\beta_1 Mh} + \frac{c_1 c}{4\beta_2 h} e^{-\beta_2 Nh} + \frac{2c_1 N({\psi'}^2 G, D_E)}{\pi d} e^{-\frac{\pi d}{h}}.$$

Finally using (3.4) and (3.5):

$$\begin{aligned} \left| G(t) - \sum_{j=-M}^{N} G(t_j)\psi'(t_j) \frac{W(t)}{W(t_j)} \frac{sinc\left(\frac{\psi(t)-jh}{h}\right)}{\psi'(t)} \right| &\leq \left[\frac{c}{4\beta_1} + \frac{c}{4\beta_2} + \frac{2N({\psi'}^2 G, D_E)}{\pi d} \sqrt{\frac{\pi d}{\beta_1}} \right] \sqrt{\frac{\beta_1}{\pi d}} c_1 M^{1/2} e^{-\sqrt{\pi d \beta_1 M}} \\ &\equiv c_1 k_4 M^{1/2} e^{-\sqrt{\pi d \beta_1 M}}. \end{aligned}$$

□

Lemma 3.10. *Let ψ be a one-to-one conformal mapping of the simply connected domain D_E onto D_s. Then we have [16, 27]*

$$\delta_{j,k}^{(0)} = \Big(S(j,h)o\psi(t)\Big)\Big|_{t=t_k} = \begin{cases} 1, & j=k, \\ 0, & j\neq k, \end{cases} \tag{3.22}$$

$$\delta_{j,k}^{(1)} = h\frac{d}{d\phi}\Big(S(j,h)o\psi(t)\Big)\Big|_{t=t_k} = \begin{cases} 0, & j=k, \\ \frac{(-1)^{k-j}}{k-j}, & j\neq k, \end{cases} \tag{3.23}$$

$$\delta_{j,k}^{(2)} = h^2\frac{d^2}{d\phi^2}\Big(S(j,h)o\psi(t)\Big)\Big|_{t=t_k} = \begin{cases} \frac{-\pi^2}{3}, & j=k, \\ \frac{-2(-1)^{k-j}}{(k-j)^2}, & j\neq k. \end{cases} \tag{3.24}$$

$$\delta_{j,k}^{(3)} = h^3\frac{d^3}{d\phi^3}\Big(S(j,h)o\psi(t)\Big)\Big|_{t=t_k} = \begin{cases} 0, & j=k, \\ \frac{(-1)^{k-j}}{(k-j)^3}\left[6-\pi^2(k-j)^2\right], & j\neq k. \end{cases} \tag{3.25}$$

4. The non-classical sinc-collocation method

Consider the equation (1.1) connected to one of the boundary conditions (1.2)-(1.5). The translated sinc functions $S_k(t)$ are not differentiable at zero and one, so we define the new functions

$$\left\{\frac{S_k(t)}{\psi'(t)}\right\}_{k=-n}^{n}, \tag{4.1}$$

which are called the modified sinc basis functions. The new basis functions are satisfied in the relations

$$\lim_{t\longrightarrow 0}\frac{S_j(t)}{\psi'(t)} = \lim_{t\longrightarrow 1}\frac{S_j(t)}{\psi'(t)} = 0, \tag{4.2}$$

$$\lim_{t\longrightarrow 0}\left(\frac{S_j(t)}{\psi'(t)}\right)' = \lim_{t\longrightarrow 1}\left(\frac{S_j(t)}{\psi'(t)}\right)' = 0. \tag{4.3}$$

So, they are well defined and differentiable at 0 and 1 now. The solution of (1.1) along with each of the boundary conditions of (1.2)-(1.5) can be approximated by

$$x(t) \simeq x_{M,N}(t) = U_{M,N}(t) + v(t), \tag{4.4}$$

where

$$U_{M,N}(t) = \sum_{j=-M}^{N} c_j \frac{W(t)}{W(t_j)}\frac{S_j(t)}{\psi'(t)}, \tag{4.5}$$

since each of the boundary conditions of (1.2)-(1.5) are nonhomogeneous, in proportion to each of the boundary conditions (1.2)-(1.5), we have added polynomials $v(t)$ to the approximation solution, which are as follows

$$(i) \quad v(t) = (-2\gamma+\beta+2\alpha)t^3 + (3\gamma-2\beta-3\alpha)t^2 + \beta t + \alpha + A(t^3-t^2), \tag{4.6}$$

$$(ii) \quad v(t) = (\gamma+\beta+2\alpha)t^3 - (\gamma+2\beta+3\alpha)t^2 + \beta t + \alpha + A(3t^2-2t^3), \tag{4.7}$$

$$(iii) \quad v(t) = (\gamma-2\beta+\alpha)t^3 - (-\gamma+3\beta-2\alpha)t^2 + \alpha t + A(2t^3-3t^2+1), \tag{4.8}$$

$$(iv) \quad v(t) = (\gamma-2\beta+2\alpha)t^3 + (-\gamma+3\beta-3\alpha)t^2 + \alpha + A(t-2t^2+t^3). \tag{4.9}$$

Now the approximate solution $x_{M,N}(t)$, in each of the boundary conditions (1.2)-(1.5) holds as:

$$(i) \quad x_{M,N}(0)=\alpha, \quad x'_{M,N}(0)=\beta, \quad x_{M,N}(1)=\gamma, \tag{4.10}$$

$$(ii) \quad x_{M,N}(0)=\alpha, \quad x'_{M,N}(0)=\beta, \quad x'_{M,N}(1)=\gamma, \tag{4.11}$$

$$(iii) \quad x'_{M,N}(0)=\alpha, \quad x_{M,N}(1)=\beta, \quad x'_{M,N}(1)=\gamma, \tag{4.12}$$

$$(iv) \quad x_{M,N}(0)=\alpha, \quad x_{M,N}(1)=\beta, \quad x'_{M,N}(1)=\gamma. \tag{4.13}$$

Substituting $x_{M,N}$ from (4.4) into (1.1), multiplying both sides by $\frac{h^3}{\psi'^2}$ and discretizing the result at the sinc grid points t_k, $k=-M-1,\cdots,N$, we have

$$\begin{aligned} &h^3\sum_{j=-M}^{N}\left(\varepsilon\rho_{3,j}(t_k)+\frac{p_1(t_k)}{\psi'(t_k)}\rho_{2,j}(t_k)+\frac{p_2(t_k)}{\psi'(t_k)^2}\rho_{1,j}(t_k)\right)c_j+\frac{h^3}{\psi'(t_k)^2}\left(\varepsilon v'''(t_k)+p_1(t_k)v''(t_k)+p_2(t_k)v'(t_k)\right)\\ &=\frac{h^3}{\psi'(t_k)^2}g\left(t_k,\frac{c_k}{\psi'(t_k)}+v(t_k)\right), \quad k=-M-1,\cdots,N, \end{aligned} \tag{4.14}$$

where

$$\rho_{m,j}(t)=\frac{1}{(\psi'(t))^{m-1}}\frac{d^m}{dt^m}\left(\frac{W(t)S_j(t)}{W(t_j)\psi'(t)}\right), \tag{4.15}$$

$$\rho_{0,j}(t)=\frac{W(t)}{W(t_j)}S_j(t). \tag{4.16}$$

Simplifying (4.14) we obtain the following system of nonlinear equations with the unknowns $c_j, j=-M-1,\cdots,N$,

$$\begin{aligned} \sum_{j=-M}^{N}\Bigg[&\varepsilon\left(\frac{W(t_k)}{W(t_j)}\right)\delta_{jk}^{(3)}+h\left(\frac{W}{W(t_j)}\left(\frac{p_1}{\psi'}\right)+\frac{W'}{W(t_j)}\left(\frac{3\varepsilon}{\psi'}\right)\right)(t_k)\delta_{jk}^{(2)}\\ &+h^2\Bigg(\frac{W}{W(t_j)}\left(2\varepsilon\left(\frac{1}{\psi'}\right)\left(\frac{1}{\psi'}\right)''+\varepsilon\left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)'+\left(\frac{1}{\psi'}\right)\left(\frac{1}{\psi'}\right)'p_1+\left(\frac{1}{\psi'}\right)^2p_2\right)\\ &+\frac{W'}{W(t_j)}\left(3\varepsilon\left(\frac{1}{\psi'}\right)\left(\frac{1}{\psi'}\right)'+2\left(\frac{1}{\psi'}\right)^2p_1\right)+\frac{W''}{W(t_j)}\left(3\varepsilon\left(\frac{1}{\psi'}\right)^2\right)\Bigg)(t_k)\delta_{jk}^{(1)}\\ &+h^3\Bigg(\frac{W}{W(t_j)}\left(\varepsilon\left(\frac{1}{\psi'}\right)'''\left(\frac{1}{\psi'}\right)^2+\left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)''p_1+\left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)'p_2\right)\\ &+\frac{W'}{W(t_j)}\left(3\varepsilon\left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)''+2\left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)'p_1+\left(\frac{1}{\psi'}\right)^3p_2\right)\\ &+\frac{W''}{W(t_j)}\left(3\varepsilon\left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)'+\left(\frac{1}{\psi'}\right)^3p_1\right)+\varepsilon\frac{W'''}{W(t_j)}\left(\frac{1}{\psi'}\right)^3\Bigg)(t_k)\delta_{jk}^{(0)}\Bigg]c_j\\ &+\frac{h^3}{\psi'(t_k)^2}\left(\varepsilon v'''(t_k)+p_1(t_k)v''(t_k)+p_2(t_k)v'(t_k)\right)\\ &=\frac{h^3}{\psi'(t_k)^2}g\left(t_k,\frac{c_k}{\psi'(t_k)}+v(t_k)\right), \quad k=-M-1,\cdots,N, \end{aligned} \tag{4.17}$$

where we used $c_{-M-1}=0$. Then by solving the system of equations (4.17) using the Newtons method, the unknowns c_j, A, and $x_{M,N}(t)$ are obtained.

5. Error analysis

Let the differential equation (1.1) subject to one of the boundary conditions of (1.2)-(1.5) has a unique solution $x \in B(D_E)$, and $\frac{p_1}{\psi'}$ and $\frac{p_2}{\psi'^2}$ belong to $B(D_E)$.
Since $\delta^n_{jk} = (-1)^n \delta^n_{kj}, n = 0, 1, 2, 3$, the system of equations (4.17) for unknown coefficients $c_j, j = -M-1, \cdots, N$ can be written in matrix form. We use the following notations: let u be defined on Γ, then $D(u) = diag(u(t_{-M-1}), \ldots, u(t_N))$; let $I^{(n)}, n = 0, 1, 2, 3$ be the matrix $I^{(n)} = [\delta^{(n)}_{kj}]$, where $\delta^{(n)}_{kj}$ is (k, j)th element. Clearly, it follows from (4.6)-(4.9) that

$$(i) \quad v(t) = (-2\gamma + \beta + 2\alpha)t^3 + (3\gamma - 2\beta - 3\alpha)t^2 + \beta t + \alpha + x'(1)(t^3 - t^2), \tag{5.1}$$

$$(ii) \quad v(t) = (\gamma + \beta + 2\alpha)t^3 - (\gamma + 2\beta + 3\alpha)t^2 + \beta t + \alpha + x(1)(3t^2 - 2t^3), \tag{5.2}$$

$$(iii) \quad v(t) = (\gamma - 2\beta + \alpha)t^3 - (-\gamma + 3\beta - 2\alpha)t^2 + \alpha t + x(0)(2t^3 - 3t^2 + 1), \tag{5.3}$$

$$(iv) \quad v(t) = (\gamma - 2\beta + 2\alpha)t^3 + (-\gamma + 3\beta - 3\alpha)t^2 + \alpha + x'(0)(t - 2t^2 + t^3), \tag{5.4}$$

using the above notations, the system of equations (4.17) can be written as follows

$$\mathbf{Ac} + \mathbf{K}(\mathbf{c}) = \mathbf{q}, \tag{5.5}$$

where the matrix $\mathbf{A}$ and vectors $\mathbf{c}$, $\mathbf{K}(\mathbf{c})$, and $\mathbf{q}$ are given by

$$\mathbf{c} = [0, c_{-M}, \cdots, c_N],$$

$$\begin{aligned}
\mathbf{A} =\Bigg[& -\varepsilon D(W) I^{(3)} + hD\left[W\left(\frac{p_1}{\psi'}\right) + W'\left(\frac{3\varepsilon}{\psi'}\right)\right] I^{(2)} \\
& - h^2 D\Bigg(W\left(2\varepsilon\left(\frac{1}{\psi'}\right)\left(\frac{1}{\psi'}\right)'' + \varepsilon\left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)' + \left(\frac{1}{\psi'}\right)\left(\frac{1}{\psi'}\right)' p_1 + \left(\frac{1}{\psi'}\right)^2 p_2\right) \\
& + W'\left(3\varepsilon\left(\frac{1}{\psi'}\right)\left(\frac{1}{\psi'}\right)' + 2\left(\frac{1}{\psi'}\right)^2 p_1\right) + W''\left(3\varepsilon\left(\frac{1}{\psi'}\right)^2\right)\Bigg) I^{(1)} \\
& + h^3 D\Bigg(W\left(\varepsilon\left(\frac{1}{\psi'}\right)'''\left(\frac{1}{\psi'}\right)^2 + \left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)'' p_1 + \left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)' p_2\right) \\
& + W'\left(3\varepsilon\left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)'' + 2\left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)' p_1 + \left(\frac{1}{\psi'}\right)^3 p_2\right) \\
& + W''\left(3\varepsilon\left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)' + \left(\frac{1}{\psi'}\right)^3 p_1\right) + W'''\varepsilon\left(\frac{1}{\psi'}\right)^3\Bigg)\Bigg]\mathbf{f},
\end{aligned}$$

$$\mathbf{f} = \left[\frac{1}{W(t_{-M-1})}, \ldots, \frac{1}{W(t_N)}\right]^T,$$

$$\mathbf{q} = -h^3 D\left(\frac{1}{\psi'^2}\right)\mathbf{v},$$

$$\mathbf{v} = \left[\varepsilon v'''(t_{-M-1}) + p_1 v''(t_{-M-1}) + p_2 v'(t_{-M-1}), \ldots, \varepsilon v'''(t_N) + p_1 v''(t_N) + p_2 v'(t_N)\right]^T,$$

$$\mathbf{K}(\mathbf{c}) = -h^3 D\left(\frac{1}{\psi'^2}\right)\mathbf{g},$$

$$\mathbf{g} = \left[g(t_{-M-1}, v(t_{-M-1})), g\left(t_{-M}, \frac{c_{-M}}{\psi'(t_{-M})} + v(t_{-M-1})\right), \ldots, g\left(t_{N-1}, \frac{c_{N-1}}{\psi'(t_{N-1})} + v(t_{N-1})\right), g(t_N, v(t_N))\right].$$

To obtain a bound on the error $|x(t) - x_{M,N}(t)|$, we first need to find a bound on $\left|\left|\mathbf{A}\hat{\mathbf{x}^*} + \mathbf{K}(\hat{\mathbf{x}^*}) - \mathbf{q}\right|\right|$ where $\hat{\mathbf{x}^*}$ is a vector defined by

$$\hat{\mathbf{x}^*} = \left[\hat{x^*}_{-M-1}, \ldots, \hat{x^*}_N\right]^T,$$

with

$$\hat{x^*}_n = x_n^* \psi'(t_n) = (x - v)(t_n)\psi'(t_n).$$

Also, we need to obtain a bound on $\left|\left|A^{-1}\right|\right|$.

Lemma 5.1. *Let ${\psi'}^2 x^* \in B(D_E)$ and there are positive constants β_1, β_2, and c such that*

$$|(\psi' x^*)(t)| = |(\psi'(x - v))(t)| \leq c \begin{cases} e^{-\beta_1|\psi(t)|}, & t \in \Gamma_a, \\ e^{-\beta_2|\psi(t)|}, & t \in \Gamma_b, \end{cases} \tag{5.6}$$

where Γ_a and Γ_b are defined in (3.12) and (3.13). Let $g(t,x)$ satisfies the Lipschitz condition: $|g(t,x) - g(t,x^*)| \leq l|x - x^*|$ and N and h are satisfied in (3.4) and (3.5). If the weight function W is selected such that $\frac{W(t)}{W(t_j)} < c_1$, $\frac{W'(t)}{W(t_j)} < c_1$, $\frac{W''(t)}{W(t_j)} < c_1$ and $\frac{W'''(t)}{W(t_j)} < c_1$, then we have

$$\left|\left|\boldsymbol{A}\hat{\boldsymbol{x}^*} + \boldsymbol{K}(\hat{\boldsymbol{x}^*}) - \boldsymbol{q}\right|\right| \leq c_1 k_5 M^{\frac{1}{2}} e^{-\sqrt{\pi d \beta_1 M}}, \tag{5.7}$$

where $\boldsymbol{A}$, $\hat{\boldsymbol{x}^*}$, $\boldsymbol{K}$, and $\boldsymbol{q}$ are defined befor.

Proof. Let us define the kernels $T_n, n = 0, 1, 2, 3$ associated with the modified non-classical sinc function by

$$T_n(t, w) = \frac{1}{2\pi i(\psi')^{n-1}} \frac{\partial^n}{\partial t^n}\left(\frac{\sin\left(\frac{\pi\psi(t)}{h}\right) W(t)}{\psi'(t)(\psi(w) - \psi(t))W(w)}\right). \tag{5.8}$$

The series expansion for $\hat{x^*}(t) = x^*(t)\psi'(t)$ can be written as

$$x^*(t) - \sum_{j=-\infty}^{\infty} \hat{x^*}(t_j)\frac{\rho_{0,j}(t)}{\psi'(t)} = \int_{\partial D} \frac{T_0(t, w)}{\psi'(t)\sin\left(\frac{\pi\psi(w)}{h}\right)}\psi'(w)\hat{x^*}(w)dw, \tag{5.9}$$

where $\rho_{0,j}(t)$ is defined by (4.15). So it results

$$\frac{d^n}{dt^n}x^*(t) - \sum_{j=-\infty}^{\infty}(\psi'(t))^{n-1}\rho_{n,j}(t)\hat{x^*}(t_j) = \int_{\partial D}\frac{(\psi'(t))^{n-1}T_n(t, w)}{\sin\left(\frac{\pi\psi(w)}{h}\right)}\psi'(w)\hat{x^*}(w)dw, \quad n = 0, 1, 2, 3. \tag{5.10}$$

Let r_k denote the k-th component of the residual vector $\mathbf{r} = \mathbf{A}\hat{\mathbf{x}^*} + \mathbf{K}(\hat{\mathbf{x}^*}) - \mathbf{q}$. Then, replacing c_j with $\hat{x^*}(t_j)$ in (4.14) we obtain

$$\begin{aligned} r_k &= \{\mathbf{A}\hat{\mathbf{x}^*} + \mathbf{K}(\hat{\mathbf{x}^*}) - \mathbf{q}\}_k \qquad (5.11)\\ &= h^3 \sum_{j=-M}^{N}\left(\varepsilon\rho_{3,j}(t_k) + \frac{p_1(t_k)}{\psi'(t_k)}\rho_{2,j}(t_k) + \frac{p_2(t_k)}{\psi'(t_k)^2}\rho_{1,j}(t_k)\right)\hat{x^*}(t_j) - \frac{h^3}{\psi'(t_k)^2}g\left(t_k, \frac{\hat{x^*}(t_k)}{\psi'(t_k)} + v(t_k)\right)\\ &\quad + \frac{h^3}{\psi'(t_k)^2}\left(\varepsilon v'''(t_k) + p_1(t_k)v''(t_k) + p_2(t_k)v'(t_k)\right). \end{aligned}$$

Since $\frac{h^3}{\psi'^2}(Lx-g)=0$, by subtracting from (5.11) we have

$$\begin{aligned}
r_k &= \left\{\mathbf{A}\hat{\mathbf{x}^*} + \mathbf{K}(\hat{\mathbf{x}^*}) - \mathbf{q} - \frac{h^3}{\psi'(t)^2}(Lx-g)\right\}_k \\
&= h^3\sum_{j=-M}^{N}\left(\varepsilon\rho_{3,j}(t_k) + \frac{p_1(t_k)}{\psi'(t_k)}\rho_{2,j}(t_k) + \frac{p_2(t_k)}{\psi'(t_k)^2}\rho_{1,j}(t_k)\right)\hat{x^*}(t_j) \\
&\quad - \frac{h^3}{\psi'(t_k)^2}g\left(t_k, \frac{\hat{x^*}(t_k)}{\psi'(t_k)} + v(t_k)\right) + \frac{h^3}{\psi'(t_k)^2}\big(\varepsilon v'''(t_k) + p_1(t_k)v''(t_k) \\
&\quad + p_2(t_k)v'(t_k)\big) - \frac{h^3}{\psi'(t_k)^2}\left(\varepsilon x'''(t_k) + p_1(t_k)x''(t_k) + p_2(t_k)x'(t_k)\right) \\
&\quad + \frac{h^3}{\psi'(t_k)^2}g\left(t_k, x(t_k)\right) \\
&= r_k^{(1)} + r_k^{(2)} + r_k^{(3)},
\end{aligned}$$

where

$$\begin{aligned}
r_k^{(1)} &= -\frac{h^3}{\psi'(t_k)^2}\left(\varepsilon x'''(t_k) + p_1(t_k)x''(t_k) + p_2(t_k)x'(t_k) - \varepsilon v'''(t_k) - p_1(t_k)v''(t_k) - p_2(t_k)v'(t_k)\right) \\
&\quad + h^3\sum_{j=-\infty}^{\infty}\left(\varepsilon\rho_{3,j}(t_k) + \frac{p_1(t_k)}{\psi'(t_k)}\rho_{2,j}(t_k) + \frac{p_2(t_k)}{\psi'(t_k)^2}\rho_{1,j}(t_k)\right)\hat{x^*}(t_j) \\
&= -\frac{h^3}{\psi'(t_k)^2}\left(\varepsilon {x^*}'''(t_k) + p_1(t_k){x^*}''(t_k) + p_2(t_k){x^*}''(t_k)\right) \\
&\quad + h^3\sum_{j=-\infty}^{\infty}\left(\varepsilon\rho_{3,j}(t_k) + \frac{p_1(t_k)}{\psi'(t_k)}\rho_{2,j}(t_k) + \frac{p_2(t_k)}{\psi'(t_k)^2}\rho_{1,j}(t_k)\right)\hat{x^*}(t_j) \\
&= -h^3\int_{\partial D_E}\left(\varepsilon T_3(t_k,w) + \frac{p_1(t_k)}{\psi'(t_k)}T_2(t_k,w) + \frac{p_2(t_k)}{\psi'(t_k)^2}T_1(t_k,w)\right)\frac{\psi'(w)\hat{x^*}(w)}{\sin\left(\frac{\pi\psi(w)}{h}\right)}dw,
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
r_k^{(2)} &= -h^3\sum_{j=-\infty}^{-M-1}\left(\varepsilon\rho_{3,j}(t_k) + \frac{p_1(t_k)}{\psi'(t_k)}\rho_{2,j}(t_k) + \frac{p_2(t_k)}{\psi'(t_k)^2}\rho_{1,j}(t_k)\right)\hat{x^*}(t_j) \\
&\quad - h^3\sum_{j=N}^{\infty}\left(\varepsilon\rho_{3,j}(t_k) + \frac{p_1(t_k)}{\psi'(t_k)}\rho_{2,j}(t_k) + \frac{p_2(t_k)}{\psi'(t_k)^2}\rho_{1,j}(t_k)\right)\hat{x^*}(t_j),
\end{aligned}$$

$$r_k^{(3)} = \frac{h^3}{\psi'(t_k)^2}\left(g(t_k, x(t_k)) - g\left(t_k, \frac{\hat{x^*}(t_k)}{\psi'(t_k)} + v(t_k)\right)\right).$$

From (5.8) we can obtain

$$T_0(t_k,w) = 0, \quad T_1(t_k,w) = \frac{(-1)^k W(t_k)}{2ih(\psi(z)-kh)W(w)},$$

$$T_2(t_k,w) = \frac{(-1)^k}{2ih(\psi(w)-kh)^2}\left(2 + (\psi(w)-kh)\left(\frac{1}{\psi'}\right)'(t_k)\right)\frac{W(t_k)}{W(w)} + \frac{(-1)^k W'(t_k)}{2ih(\psi(w)-kh)W(w)}\left(\frac{2}{\psi'}\right)(t_k),$$

$$
\begin{aligned}
T_3(t_k,w) = {} & \frac{(-1)^k}{2ih(\psi(w)-kh)^3}\Bigg(6+(\psi(w)-kh)^2\left(-\left(\frac{\pi}{h}\right)^2+2\left(\frac{1}{\psi'}\right)\left(\frac{1}{\psi'}\right)''(t_k)\right) \\
& -(\psi(w)-kh)^2\left(\left(\frac{1}{\psi'}\right)'\right)^2(t_k)\Bigg)\frac{W(t_k)}{W(w)}+\frac{(-1)^k}{2ih(\psi(w)-kh)^2}\Bigg(\frac{6}{\psi'} \\
& +\frac{3}{\psi'}\left(\frac{1}{\psi'}\right)'(\psi(w)-kh)\Bigg)\frac{W'(t_k)}{W(w)}+\frac{3(-1)^k}{2ih(\psi(w)-kh)}\left(\frac{1}{\psi'}\right)^2\frac{W''(t_k)}{W(w)}.
\end{aligned}
$$

Also, we have the following bounds on $\frac{1}{\psi'}$ and its derivatives

$$
\left|\frac{1}{\psi'(t)}\right|\le\frac{1}{4},\quad \left|\left(\frac{1}{\psi'(t)}\right)'\right|\le 1,\quad \left|\left(\frac{1}{\psi'(t)}\right)''\right|\le 2,\quad \left|\left(\frac{1}{\psi'}\right)\left(\frac{1}{\psi'}\right)''\right|\le\frac{1}{2}. \tag{5.13}
$$

Now, since $|I\psi(w)|=d$ on ∂D_E, setting $u(w)=R\psi(w)$ and using relation (5.13) and lemmas assumptions on W we have

$$
|T_1(t_k,w)|\le\frac{c_1c_1'h^{-1}}{\big((u(w)-kh)^2+d^2\big)^{\frac{1}{2}}},\quad |T_2(t_k,w)|\le\frac{c_1c_2'h^{-1}}{\big((u(w)-kh)^2+d^2\big)^{\frac{1}{2}}},\quad |T_3(t_k,w)|\le\frac{c_1c_3'h^{-1}}{\big((u(w)-kh)^2+d^2\big)^{\frac{1}{2}}}.
$$

Then, we obtain

$$
h^3\left|\varepsilon T_3(t_k,w)+\frac{p_1(t_k)}{\psi'(t_k)}T_2(t_k,w)+\frac{p_2(t_k)}{\psi'(t_k)^2}T_1(t_k,w)\right|\le\frac{c_1c_2h}{\big((u(w)-kh)^2+d^2\big)^{\frac{1}{2}}}, \tag{5.14}
$$

where c_2 is a constant depending on (5.13), h, d, ε and on the bounds for the coefficients of the differential equation. Therefore, we have

$$
\begin{aligned}
\big|\big|\mathbf{A}\hat{\mathbf{x}^*}+\mathbf{K}(\hat{\mathbf{x}^*})-\mathbf{q}\big|\big| &= \left(\sum_{k=-M}^{N}|r_k|^2\right)^{\frac{1}{2}} \\
&\le\left(\sum_{k=-M}^{N}|r_k^{(1)}|^2\right)^{\frac{1}{2}}+\left(\sum_{k=-M}^{N}|r_k^{(2)}|^2\right)^{\frac{1}{2}}+\left(\sum_{k=-M}^{N}|r_k^{(3)}|^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{5.15}
$$

The first term at right-hand side satisfies

$$
\sum_{k=-M}^{N}|r_k^{(1)}|^2\le\sum_{k=-M}^{N}\left|\int_{\partial D_E}\frac{c_1c_2h}{\big((u(w)-kh)^2+d^2\big)^{\frac{1}{2}}}\frac{|\psi'(w)\hat{x}(w)}{\left|\sin\left(\frac{\pi\psi(z)}{h}\right)\right|}|dw|\right|^2\le\frac{c_3c_1^2}{\left(\sinh\left(\frac{\pi d}{h}\right)\right)^2}, \tag{5.16}
$$

which is obtained by using (5.14), the bound $\sinh\left(\frac{\pi d}{h}\right) \leq \sin\left(\frac{\pi\psi(w)}{h}\right)$ on ∂D_E and the integrability of $|\hat{x^*}\psi'|$. For the second term, using (5.13), (5.6), and the assumptions on the coefficients of the differential equation, we obtain

$$\begin{aligned}\sum_{k=-M}^{N}\left|r_k^{(2)}\right|^2 = \sum_{k=-M}^{N}\Bigg|\sum_{j<-M,j>N}\Bigg[&\varepsilon\left(\frac{W(t_k)}{W(t_j)}\right)\delta_{jk}^{(3)}\\
&+h\left(\frac{W}{W(t_j)}\left(\frac{p_1}{\psi'}\right)+\frac{W'}{W(t_j)}\left(\frac{3\varepsilon}{\psi'}\right)\right)(t_k)\delta_{jk}^{(2)}\\
&+h^2\Bigg(\frac{W}{W(t_j)}\left(2\varepsilon\left(\frac{1}{\psi'}\right)\left(\frac{1}{\psi'}\right)''+\varepsilon\left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)'+\left(\frac{1}{\psi'}\right)\left(\frac{1}{\psi'}\right)'p_1+\left(\frac{1}{\psi'}\right)^2p_2\right)\\
&+\frac{W'}{W(t_j)}\left(3\varepsilon\left(\frac{1}{\psi'}\right)\left(\frac{1}{\psi'}\right)'+2\left(\frac{1}{\psi'}\right)^2p_1\right)+\frac{W''}{W(t_j)}\left(3\varepsilon\left(\frac{1}{\psi'}\right)^2\right)\Bigg)(t_k)\delta_{jk}^{(1)}\\
&+h^3\Bigg(\frac{W}{W(t_j)}\left(\varepsilon\left(\frac{1}{\psi'}\right)'''\left(\frac{1}{\psi'}\right)^2+\left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)''p_1+\left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)'p_2\right)\\
&+\frac{W'}{W(t_j)}\left(3\varepsilon\left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)''+2\left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)'p_1+\left(\frac{1}{\psi'}\right)^3p_2\right)\\
&+\frac{W''}{W(t_j)}\left(3\varepsilon\left(\frac{1}{\psi'}\right)^2\left(\frac{1}{\psi'}\right)'+\left(\frac{1}{\psi'}\right)^3p_1\right)+\frac{W'''}{W(t_j)}\varepsilon\left(\frac{1}{\psi'}\right)^3\Bigg)(t_k)\delta_{jk}^{(0)}\Bigg]\hat{x^*}(t_j)\Bigg|^2\\
&\leq c_1^2c_3'\sum_{k=-M}^{N}\left(\sum_{j<-M,j>N}\gamma_{kj}^2\sum_{j<-M,j>N}|\hat{x^*}(t_j)|^2\right)\leq\frac{c_1^2c_4}{h^2}e^{-2\beta_1Mh},\end{aligned}\tag{5.17}$$

where γ_{kj} is defined by

$$\gamma_{kj} = \max\left\{|\delta_{kj}^{(0)}|, |\delta_{kj}^{(1)}|, |\delta_{kj}^{(2)}|, |\delta_{kj}^{(3)}|\right\}.$$

Since the function $g(t,x)$ satisfies the Lipschitz condition, we have

$$\begin{aligned}\left|g(t_k,x(t_k))-g\left(t_k,\frac{\hat{x^*}(t_k)}{\psi'(t_k)}+v(t_k)\right)\right| &\leq l\left|x(t_k)-\frac{\hat{x^*}(t_k)}{\psi'(t_k)}-v(t_k)\right|\\
&= l\left|x(t_k)-\frac{\psi'(t_k)\big(x(t_k)-v(t_k)\big)}{\psi'(t_k)}-v(t_k)\right| = 0,\end{aligned}$$

$$\sum_{k=-M}^{N}\left|r_k^{(3)}\right|^2 = 0. \tag{5.18}$$

Now using (5.15), (5.16), (5.17), and (5.18), we obtain the result

$$\left|\left|\mathbf{A}\hat{\mathbf{x}^*}+\mathbf{K}(\hat{\mathbf{x}^*})-\mathbf{q}\right|\right| \leq c_1k_5M^{\frac{1}{2}}e^{-\sqrt{\pi d\beta_1 M}}.$$

□

Lemma 5.2. *Let the assumptions in lemma 5.1 be satisfied. If the matrix $\boldsymbol{A}$ is nonsingular, then*

$$\left|\left|\boldsymbol{A}^{-1}\right|\right| \leq \frac{9M^3}{\pi^3}\left(1+c_5M^{-1}\right), \tag{5.19}$$

holds for a constant c_5 and sufficiently large M.

Proof. The relation (5.19) can be proved using the results in [3] and [18]. □

Theorem 5.3. *Let x and $x_{M,N}$ be the exact and approximate solutions of equation (1.1) respectively. Also, let $\boldsymbol{c} = [0, c_{-M}, \cdots, c_N]$ be the solution of the system of equation (5.5). If the assumptions in lemma 5.1 and 5.2 are satisfied, then for $t \in \Gamma$,*

$$\left|x(t) - x_{M,N}(t)\right| \leq c_1 k_6 M^{\frac{7}{2}} e^{-\sqrt{\pi d \beta_1 M}}.$$

Proof. Let the function $\eta_{M,N}$ be defined by

$$\eta_{M,N}(t) = \frac{1}{\psi'(t)} \sum_{j=-M}^{N} \frac{W(t)}{W(t_j)} x^*(t_j)\psi'(t_j) S_j(t) = \frac{1}{\psi'(t)} \sum_{j=-M}^{N} \frac{W(t)}{W(t_j)} \hat{x^*}(t_j) S_j(t).$$

From (4.4)

$$\begin{aligned} x(t) - x_{M,N}(t) &= x(t) - x_{M,N}(t) + \eta_{M,N}(t) - \eta_{M,N}(t) \\ &= x(t) - U_{M,N}(t) - v(t) + \eta_{M,N}(t) - \eta_{M,N}(t), \end{aligned}$$

then using $x^*(t) = x(t) - v(t)$, we obtain

$$x(t) - x_{M,N}(t) = \left(x^*(t) - \eta_{M,N}(t)\right) + \left(\eta_{M,N}(t) - U_{M,N}(t)\right).$$

Now taking the absolute from both sides and using the triangular inequality we have

$$\left|x(t) - x_{M,N}(t)\right| \leq \left|x^*(t) - \eta_{M,N}(t)\right| + \left|\eta_{M,N}(t) - U_{M,N}(t)\right|. \tag{5.20}$$

By theorem 3.9 and proceeding as in [3]

$$\sup_{t\in\Gamma} \left|x^*(t) - \eta_{M,N}(t)\right| \leq c_1 c_6 M^{\frac{1}{2}} e^{-\sqrt{\pi d \beta_1 M}}. \tag{5.21}$$

We find a bound on the second term of the right-hand side of (5.20) as follows

$$\left|\eta_{M,N}(t) - U_{M,N}(t)\right| = \left| \frac{1}{\psi'(t)} \sum_{j=-M}^{N} \frac{W(t)}{W(t_j)} \left(\hat{x^*}(t_j) - c_j\right) S_j(t) \right|,$$

then using the cauchy schwartz inequality

$$\begin{aligned} \left|\eta_{M,N}(t) - U_{M,N}(t)\right| &\leq \left(\sum_{j=-M}^{N} \left|\hat{x^*}(t_j) - c_j\right|^2 \right)^{\frac{1}{2}} \left(\sum_{j=-M}^{N} \left| \frac{W(t)}{W(t_j)} \frac{S_j(t)}{\psi'(t)} \right|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{c_1}{4} \left(\sum_{j=-M}^{N} \left|\hat{x^*}(t_j) - c_j\right|^2 \right)^{\frac{1}{2}} \\ &= \frac{c_1}{4} \left|\left|\hat{\mathbf{x}^*} - \mathbf{c}\right|\right|, \end{aligned}$$

where the last relation obtained by the boundedness of $\frac{1}{\psi'(t)}$ and $\frac{W(t)}{W(t_j)} < c_1$. Finally, from (5.7) and (5.19), we have

$$\begin{aligned} \left|\left|\hat{\mathbf{x}^*} - \mathbf{c}\right|\right| &= \left|\left|\mathbf{A}^{-1}(\mathbf{A}\hat{\mathbf{x}^*} + \mathbf{K}(\hat{\mathbf{x}^*}) - \mathbf{q})\right|\right| \\ &\leq \left|\left|\mathbf{A}^{-1}\right|\right| \left|\left|\mathbf{A}\hat{\mathbf{x}^*} + \mathbf{K}(\hat{\mathbf{x}^*}) - \mathbf{q}\right|\right| \\ &\leq c_1 c_7 M^{\frac{5}{2}} e^{-\sqrt{\pi d \beta_1 M}}. \end{aligned} \tag{5.22}$$

Combining (5.21) with (5.22), we obtain the result

$$\left|x(t) - x_{M,N}(t)\right| \leq c_1 k_6 M^{\frac{7}{2}} e^{-\sqrt{\pi d \beta_1 M}}.$$

□

6. Numerical examples

In this section, four problems will be solved using the non-classical sinc-collocation method. In all examples we choose $\beta_1 = \beta_2 = 0.5$, $d = \frac{\pi}{2}$, $h = \sqrt{\frac{\pi d}{\beta_1 M}}$ and $N = \left[\left|\frac{\beta_1}{\beta_2} M + 1\right|\right]$.
The maximum absolute error of the presented method, is defined by

$$E = \max_{0 \le i \le 100} \left| x_{M,N}(t_i) - x(t_i) \right|,$$

where $x(t)$ and $x_{M,N}(t)$ are the exact and approximate solutions successively and t_i are equally spaced knots in the interval.
As it turned out, for any positive weight function, provided that $\frac{W(t)}{W(t_j)} \le c_1$, the convergence rate of the non-classical sinc-collocation method is exponential. But we tested different positive weights and found that the results were better for some weights than others. For this reason, we used the following weights to solve the examples:

$$W = W_1 = 1, \quad W = W_2 = 0.1 + \sin(\pi t), \quad W = W_3 = 1 + t - t^2.$$

All numerical results are obtained with Maple 12.

Example 6.1. Consider the following singularly perturbed boundary value problem [12, 23, 29]

$$-\varepsilon x'''(t) = -x(t) + 81\varepsilon^2 \cos(3t) + 3\varepsilon \sin(3t),$$
$$x(0) = 0, \qquad x'(0) = 9\varepsilon, \qquad x(1) = 3\varepsilon \sin(3),$$

with the exact solution $x(t) = 3\varepsilon \sin(3t)$.
In the Tables 1 and 2, the maximum absolute errors of the proposed method with different weights are given, and they are compared with the results of the ESM[12] method. Also, the Figures 1-3 show the maximum absolute error of the proposed method for different weights. These tables and figures show that the proposed method is more accurate compared to the classic sinc method by selecting some proper weight functions.

TABLE 1. The maximum absolute errors for example 6.1 with $\varepsilon = 1/16$.

	E			Other method	
M	W_1	W_2	W_3	N	ESM[12]
20	2.5(−6)	9.2(−8)	5.7(−7)	20	6.1(−5)
40	1.5(−8)	2.4(−10)	3.4(−9)	40	1.5(−6)
60	3.0(−10)	5.4(−12)	6.4(−11)	–	–
80	1.0(−11)	7.6(−14)	2.0(−12)	–	–

TABLE 2. The maximum absolute errors for example 6.1 with $\varepsilon = 1/64$.

	E			Other method	
M	W_1	W_2	W_3	N	ESM[12]
20	6.0(−7)	1.9(−8)	1.3(−7)	20	1.0(−6)
40	3.6(−9)	3.6(−11)	8.3(−10)	40	2.5(−6)
60	7.4(−11)	1.0(−12)	1.6(−11)	–	–
70	1.3(−11)	1.7(−13)	2.6(−12)	–	–

Example 6.2. Consider the following boundary value problem [19]

$$x'''(t) - t^2\big(x''(t) - x'(t)\big) + x(t)^2 = g(t),$$
$$x(0) = 0, \qquad x'(0) = -1, \qquad x'(1) = \sin(1),$$

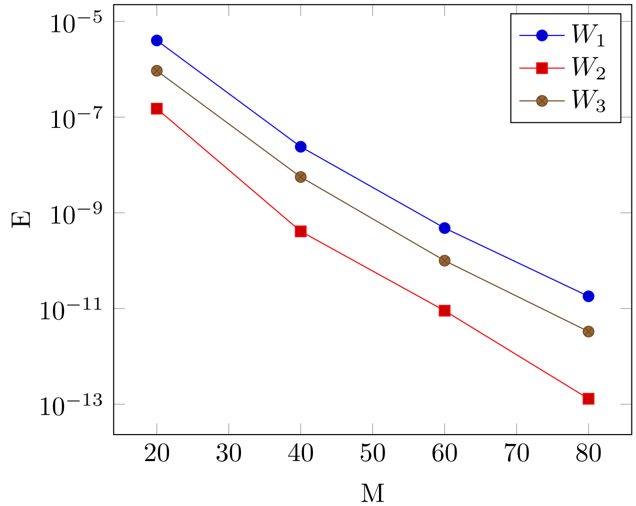

FIGURE 1. The maximum absolute error with $\varepsilon = 10^{-1}$ for example 6.1.

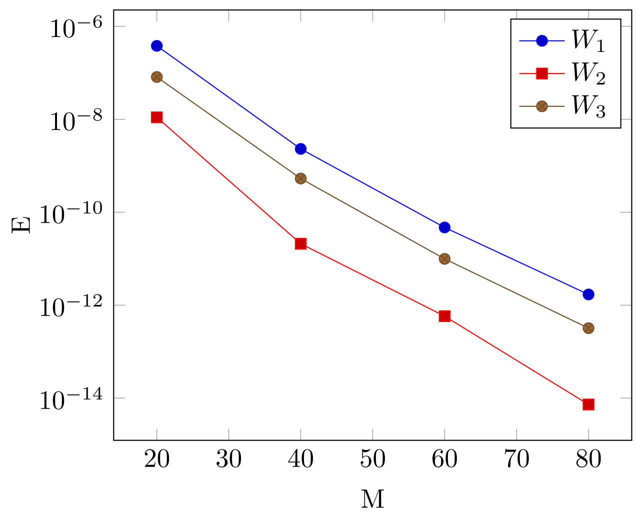

FIGURE 2. The maximum absolute error with $\varepsilon = 10^{-2}$ for example 6.1.

with the exact solution $x(t) = (t-1)\sin(t)$.
In the Table 3, the maximum absolute errors of the proposed method with different weights are given. This table shows that the proposed method is more accurate compared to the classic sinc method by selecting some proper weight functions. Also, this table shows the results of the method presented in [19] that our method is much more accurate compared to this method.

Example 6.3. Consider the following singular boundary value problem

$$x'''(t) - \frac{2}{t}x''(t) = x(t)^3 - 6e^t + 6te^t + 7t^2e^t + t^3e^t - t^9e^{3t},$$

$$x(0) = 0, \qquad x'(0) = 0, \qquad x'(1) = 4e,$$

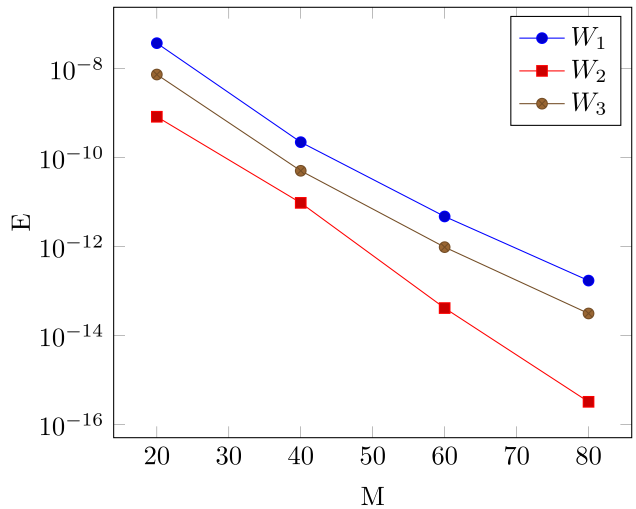

FIGURE 3. The maximum absolute error with $\varepsilon = 10^{-3}$ for example 6.1.

TABLE 3. The maximum absolute errors for example 6.2.

	E			Other methods	
M	W_1	W_2	W_3	N	method in[19]
5	5.0(−4)	6.1(−5)	1.8(−4)	32	7.9(−4)
10	3.2(−5)	6.5(−6)	1.2(−5)	64	6.5(−5)
20	1.6(−6)	4.0(−8)	2.3(−7)	128	1.6(−5)
30	1.3(−7)	5.4(−9)	7.2(−9)	256	4.1(−6)
40	1.5(−8)	1.5(−10)	4.4(−10)	512	1.1(−6)
50	2.0(−9)	2.7(−11)	5.4(−11)	−	−

with the exact solution $x(t) = t^3 e^t$.
In the Table 4, the maximum absolute errors of the proposed method with different weights are given. The table show that the proposed method is more accurate compared to the classic sinc method by selecting some proper weight functions.

TABLE 4. The maximum absolute errors for example 6.3.

	E		
M	W_1	W_2	W_3
10	9.6(−4)	1.0(−4)	5.7(−4)
30	4.3(−6)	1.2(−7)	9.1(−8)
50	5.7(−8)	5.9(−10)	4.2(−9)
70	1.4(−9)	1.7(−11)	5.2(−11)

Example 6.4. Consider the following singular singularly perturbed boundary value problems [7, 17]

$$\varepsilon x'''(t) + \frac{2}{t}x''(t) + x'(t) = -x(t) + \left(1 - \frac{2}{\varepsilon t}\right)\frac{\sin\left(\frac{t}{\sqrt{\varepsilon}}\right)}{\sin\left(\frac{1}{\sqrt{\varepsilon}}\right)},$$

$$x(0) = 0, \qquad x(1) = 1, \qquad x'(1) = \frac{\cos\left(\frac{1}{\sqrt{\varepsilon}}\right)}{\sqrt{\varepsilon}\sin\left(\frac{1}{\sqrt{\varepsilon}}\right)},$$

with the exact solution $x(t) = \frac{\sin\left(\frac{t}{\sqrt{\varepsilon}}\right)}{\sin\left(\frac{1}{\sqrt{\varepsilon}}\right)}$.

In the Table 5, the maximum absolute errors of the proposed method with different weight functions are given, and they are compared with the results of NQBSA[7] and QBSM[17] methods. The results of this table show that our method is much more accurate than other methods.

TABLE 5. The maximum absolute errors for example 6.4 with $\varepsilon = 10^{-2}$.

	E			Other methods		
M	W_1	W_2	W_3	N	$NQBSA$[7]	$QBSM$[17]
16	7.5(−2)	2.8(−2)	6.2(−2)	16	3.2(−3)	2.1(−2)
32	2.1(−3)	3.7(−4)	1.4(−3)	32	9.8(−4)	6.6(−3)
64	1.4(−5)	1.2(−6)	7.6(−6)	64	5.4(−5)	1.5(−3)
128	7.8(−9)	3.5(−10)	3.8(−9)	128	3.4(−6)	3.8(−4)

Remark: As the results of Examples 6.3 and 6.4 show, our method is efficient in controlling the singularity and works well in dealing with this type of problem, because the singularity of the equation occurs at the end point of the interval. And because most of the sinc grid points are gathered near the endpoints of the interval, it helps us to control the singularity well.

7. Conclusion

This article introduced a non-classical sinc-collocation method to solve the third-order boundary value problems. The properties of this method have been used to reduce the computations of these problems to some algebraic equations. It has been shown theoretically, that scheme is efficient and achieves exponential convergence. The convergence exponential rate of the proposed method shows that this method can achieve high accuracy with moderate computational effort. Finally, the applicability and accuracy of the method are checked on some examples. The results of these examples show that in our method, by selecting proper weights, the results are better than the classic sinc-collocation method. Also, our method is more accurate compared to other methods.

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