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A method for second-order linear fuzzy two-point boundary value problems based on the Hukuhara differentiability

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Abstract

In this paper, we deal with second-order fuzzy linear two-point boundary value problems (BVP) under Hukuhara derivatives. Considering the first-order and second-order Hukuhara derivatives, four types of fuzzy linear two-point BVPs can be obtained where each may or may not have a solution. Therefore a fuzzy two-point (BVP) may have one, two, three, or four different kinds of solutions concerning this kind of derivative. To solve this fuzzy linear two-point (BVP), we convert each to two cases of crisp boundary value problems. We apply a standard method(numerical or analytical) to solve crisp two-point BVPs in their domain. Subsequently, the crisp solutions are combined to obtain a fuzzy solution to the fuzzy problems, and the solutions are checked to see if they satisfy the fuzzy issues. Conditions are presented under which fuzzy problems have the fuzzy solution and illustrated with some examples.

Keywords. Fuzzy number, Hukuhara differentiability, Fuzzy two-point boundary value problems, Fuzzy function.
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1. Introduction

Zadeh and others were the first to present and research fuzzy calculus and, arithmetic operations on fuzzy integers [12, 25, 30]. The search for fuzzy differential equations is an area of fuzzy mathematics. In many physics and engineering modeling , when all the parameters are precisely known, differential equations are employed. Fuzzy differential equations(FDEs) are used to describe uncertain issues, if at least one of the parameters, such as coefficients or boundary conditions, are unknown. Uncertain parameters are considered fuzzy numbers. As a result, FDEs, in some situations, may be considered a subset of uncertain differential equations. After being first introduced by Kaleva [18], fuzzy differential equations were afterwards developed by various researchers [5, 10, 11, 13, 26–28]. Regarding the Hukuhara derivative, Kaleva has developed fuzzy differential equations[18]. A highly fuzzy first-order initial value issue has been provided by Buckley and, others [6, 17, 24]. Bede and Gal [3] proposed generalized differentiability, which they later refined as detailed in [4, 20]. The interpretation of fuzzy differential equations becomes possible with this idea in several distinct ways We concentrated on fuzzy boundary value issues in this work.

One of the most important categories of differential equations is boundary value problems (BVP), which are found in many research branches including electrostatics, engineering, and other fields. For instance, the Schrodinger equation with fuzzy circumstances was solved the Heisenberg's uncertainty principle[23]. For instance, because of Heisenberg's uncertainty principle, the Schrodinger equation equipped with fuzzy conditions was solved in[15]. Fuzzy numbers and fuzzy functions are utilized to simulate these issues because one or more parameters and fuzzy state variables of (BVP) are often absent or ambiguous. These BVPs are known as fuzzy boundary value problems (FBVP). Some scholars have concentrated on finding (FBVPs). O'Regan et al. used fuzzy integral equations to work on the (FBVPs) based on the Hukuhara derivatives. However, their approach is not applicable to all FBVP classes. Khastan and Nieto used fuzzy generalized Hukuhara derivatives to get around this problem.[20, 31]. The stability and roundedness of solutions to non-linear differential equations of second order are stated and proved by Cemil Tun et al [29]. Remember that

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the Hukuhara minus serves as the foundation for the generalized Hukuhara derivatives. Additionally, some academics employ the crisp derivatives, also known as fuzzy inclusion differentials, to represent fuzzy inclusion[2, 9]. Bede et al [3, 4] and others [14, 22] proved that considering the Hukuhara derivative, some fuzzy two-point BVPs don't have solutions. Considering the new definitions and concepts of Hukuhara derivatives and certain conditions, overcome this problem. For this purpose, two kinds are taken for the first derivative, and for each kind, two kinds are considered for the second derivative. Therefore, to solve a FBVP, four models can be investigated Here, we offer a brand-new technique for discovering fuzzy BVP solutions in four different scenarios. Each of these, could or might not have a special solution. Then, regarding Hukuhara derivatives, an (FBVP) may have one, two, three, or four distinct types of solutions. We remember several notations and fundamental concepts from Section 2. In Section 3, considering the presented method, four examples of fuzzy linear two-point boundary value issues are discussed and described. In Section 4, we have introduced a methodology to solve fuzzy BVPs, and we provided some examples to demonstrate our findings.

2. Notations and basic concepts

Definition 2.1. [21] A fuzzy set u that requires the following conditions is called a fuzzy number:
(1.) The membership function of u is continuous, convex, and normal.
(2.) The support of u is bounded

Usually, fuzzy numbers are denoted by E^1. The fuzzy number, as the following, is entirely characterized by four real numbers $\alpha_1 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4$, is called a *trapezoidal fuzzy number*:

$$u(x) = \begin{cases} 0, & x < \alpha_1, \\ \frac{x-\alpha_1}{\alpha_2-\alpha_1}, & \alpha_1 \leq x < \alpha_2, \\ 1, & \alpha_2 \leq x \leq \alpha_3, \\ \frac{\alpha_4-x}{\alpha_4-\alpha_3}, & \alpha_3 \leq x < \alpha_4, \\ 0, & \alpha_4 < x. \end{cases} \tag{2.1}$$

It is mostly denoted in short as $u(\alpha_1, \alpha_2, \alpha_3, \alpha_4)_T$. The trapezoidal fuzzy numbers usually are denoted by $\mathbb{F}^T$. If $\alpha_2 = \alpha_3$, we attain the *triangular fuzzy number*.

Definition 2.2. [8] The fuzzy numbers are denoted by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$; $0 \leq r \leq 1$, that the following conditions hold,

- **1):** $\underline{u}(r)$ is a bounded left-continuous non- decreasing over $[0, 1]$.
- **2):** $\overline{u}(r)$ is a bounded left-continuous non- increasing over $[0, 1]$.
- **3):** $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$.

Assume $u = (\underline{u}(r), \overline{u}(r)), v = (\underline{v}(r), \overline{v}(r)) \in E^1$, $\alpha \in \mathbb{R}$, then the following hold:

$$\underline{(u+v)}(r) = (\underline{u}(r) + \underline{v}(r)),$$

$$\overline{(u+v)}(r) = (\overline{u}(r) + \overline{v}(r)),$$

$$\underline{\alpha u}(r) = \alpha \underline{u}(r), \ \overline{\alpha u}(r) = \alpha \overline{u}(r) \ \ if \ \ \alpha \geq 0,$$

$$\overline{\alpha u}(r) = \alpha \underline{u}(r), \ \underline{\alpha u}(r) = \alpha \overline{u}(r) \ \ if \ \ \alpha < 0.$$

Remark 2.3. If $u(r) = (\underline{u}(r), \overline{u}(r)) \in E^1$, we denote:

$$u_c(r) = \frac{\underline{u}(r) + \overline{u}(r)}{2}, \tag{2.2}$$

$$u_d(r) = \frac{\overline{u}(r) - \underline{u}(r)}{2}. \tag{2.3}$$

One can see that $\underline{u} = u_c - u_d$ and $\overline{u} = u_c + u_d$.

Lemma 2.4. *Suppose $u(r) = (\underline{u}(r), \overline{u}(r)) \in E^1$ and $v(r) = (\underline{v}(r), \overline{v}(r)) \in E^1$, also $k_1, k_2 \in \Re$. If $w = k_1 u + k_2 v$ then*

$$w_c = k_1 u_c + k_2 v_c, \tag{2.4}$$

$$w_d = |k_1| u_d + |k_2| v_d. \tag{2.5}$$

Proof. See[1]. □

Definition 2.5. [16] Let $u_1, u_2 \in E^1$. If there exits $u_3 \in E^1$ such that $u_1 = u_2 + u_3$ then u_3 is H-difference of u_1 and u_2, i.e $u_3 = u_1 \ominus u_2$.

Definition 2.6. [7] Suppose $F : [a, b] \to E^1$ is a fuzzy function, F is a differentiable function at $x_0 \in [a, b]$ if there exists a fuzzy number $F'(x_0) \in E^1$ such that

1): For each $h > 0$ enough near to zero, the $F(x_0 + h) \ominus F(x_0)$, $F(x_0) \ominus F(x_0 - h)$, and two following limits exist and are equal to $F'(x_0)$;

$$\lim_{h \to 0^+} \frac{F(x_0 + h) \ominus F(x_0)}{h}, \quad \lim_{h \to 0^+} \frac{F(x_0) \ominus F(x_0 - h)}{h},$$

or

2): For each $h < 0$ enough near to zero, the $F(x_0 + h) \ominus F(x_0)$, $F(x_0) \ominus F(x_0 - h)$, and two following limits exist and are equal to $F'(x_0)$;

$$\lim_{h \to 0^-} \frac{F(x_0 + h) \ominus F(x_0)}{h}, \quad \lim_{h \to 0^-} \frac{F(x_0) \ominus F(x_0 - h)}{h}.$$

Definition 2.7. [7] Consider $F : [a, b] \to E^1$, F is called (1)-differentiable on the interval $[a, b]$ wherever F is differentiable in Case (1) of the previous definition and its derivative is represented by $D_1^1 F(x)$, similarly for (2)-differentiable we have $D_2^1 F(x)$.

Theorem 2.8. *Suppose $F : [a, b] \to E^1$ is a fuzzy function, for each $r \in [0, 1]$ put $[F(x)]_r = [\underline{F}(x, r), \overline{F}(x, r)]$, then*

1: *$F'(x) = D_1^1 F(x) = (\underline{F}'(x, r), \overline{F}'(x, r))$, provided that F is (1)-differentiable,*

2: *$F'(x) = D_2^1 F(x) = (\overline{F}'(x, r), \underline{F}'(x, r))$, provided that F is (2)-differentiable,*

Proof. See[7]. □

The sufficient conditions for the existence of $D_1^1 F(x)$, $D_2^1 F(x)$ are given in the next theorem.

Theorem 2.9. *Assuming $F : [a, b] \to E^1$ is differentiable,*

1): *F is (1)-differentiable function $\Leftrightarrow (F_d)'(x, r) \geq 0$ for all $0 \leq r \leq 1$,*

2): *F is (2)-differentiable function $\Leftrightarrow (F_d)'(x, r) \leq 0$ for all $0 \leq r \leq 1$.*

Proof. Let F be (1)-differentiable due to Theorem 2.8, $(F')_d(x, r) \geq 0$, then $(F_d)'(x, r) = \frac{d}{dt}(\frac{\overline{F}(x,r) - \underline{F}(x,r)}{2}) = \frac{\overline{F}'(x,r) - \underline{F}'(x,r)}{2} = (F')_d(x, r) \geq 0$. Conversely, let $(F_d)'(x, r) \geq 0$ it means that $\frac{d}{dt}(\frac{\overline{F}(x,r) - \underline{F}(x,r)}{2}) \geq 0$, then $\frac{\overline{F}'(x,r) - \underline{F}'(x,r)}{2} \geq 0$. Therefore $F'(x, r) = (\underline{F}'(x, r), \overline{F}'(x, r))$ so F is (1)-differentiable.
Now let F is (2)-differentiable, due to Theorem 2.8, $(F')_d(x, r) \geq 0$ it means that $(F_d)'(x, r) = \frac{d}{dt}(\frac{\overline{F}(x,r) - \underline{F}(x,r)}{2}) = \frac{\overline{F}'(x,r) - \underline{F}'(x,r)}{2} = -(F')_d(x, r) \leq 0$. Conversely, let $(F_d)'(x, r) \leq 0$ it means that $\frac{d}{dt}(\frac{\overline{F}(x,r) - \underline{F}(x,r)}{2}) \leq 0$. $\frac{\overline{F}'(x,r) - \underline{F}'(x,r)}{2} \leq 0$ and $F'(x, r) = (\overline{F}'(x, r), \underline{F}'(x, r))$ hence F is (2)-differentiable. □

Remark 2.10. Regarding Theorems 2.8 and Eq.(2.9), the following statements are inferred about $F''(x_0)$:

1: Suppose F and F' are (1)-differentiable functions, $F''(x_0)$ is denoted as $D_{1,1}^2 F(x_0)$ and $[D_{1,1}^2 F(x_0)]_r = [\underline{F}''(x_0, r), \overline{F}''(x_0, r)]$.

2: Suppose F is (1)-differentiable function, F' is (2)-differentiable function then $F''(x_0)$ is denoted as $D^2_{1,2}F(x_0)$ and $[D^2_{1,2}F(x_0)]_r = [\overline{F}''(x_0,r), \underline{F}''(x_0,r)]$.

3: Suppose F is (2)-differentiable function, F' is (1)-differentiable function then $F''(x_0)$ is denoted as $D^2_{2,1}F(x_0)$ and $[D^2_{2,1}F(x_0)]_r = [\overline{F}''(x_0,r), \underline{F}''(x_0,r)]$.

4: If F and F' are (2)-differentiable functions, $F''(x_0)$ is denoted as $D^2_{2,2}F(x_0)$ and $[D^2_{2,2}F(x_0)]_r = [\underline{F}''(x,r), \overline{F}''(x,r)]$.

Theorem 2.9 is extended as follow:

Theorem 2.11. *Assuming $F : [a,b] \to E^1$ and $F' : [a,b] \to E^1$ are differentiable fuzzy functions, then:*

1): *$D^2_{1,1}F(x)$ exist $\Leftrightarrow (F_d)'(x,r)$ and $(F_d)''(x,r)$ are non-negative for each $r \in [0,1]$,*

2): *$D^2_{1,2}F(x)$ exist $\Leftrightarrow (F_d)'(x,r)$ is non-negative and $(F_d)''(x,r)$ is non-positive for each $r \in [0,1]$,*

3): *$D^2_{2,1}F(x)$ exist $\Leftrightarrow (F_d)'(x,r)$ and $(F_d)''(x,r)$ are non positive for each $r \in [0,1]$,*

4): *$D^2_{2,2}F(x)$ exist $\Leftrightarrow (F_d)'(x,r)$ is non-positive and $(F_d)''(x,r)$ is non-negative for each $r \in [0,1]$.*

Proof. For proof see Theorem 2.9 and Remark 2.10. □

3. Two-point fuzzy boundary value problems

Consider two-point fuzzy BVP

$$\begin{cases} y''(x) = p_1(x)y'(x) + p_2(x)y(x) + g(x), \\ y(0) = A, \\ y(l) = B, \end{cases} \tag{3.1}$$

where $x \in [0,l]$ and $A, B \in E^1$, $p_1(x), p_2(x)$ are real continuous functions and $g(x)$ is continuous fuzzy function. Many researchers, [13] demonstrate efficient methods to Eq.(3.1). In this paper, boundary values may be arbitrary fuzzy numbers and, the forcing function may be the arbitrary fuzzy function in parametric form. Due to the previous section, four cases are considered for solving Eq.(3.1) as follows:

Case 1.

y is (1-1)solution of fuzzy BVP (3.1) if $D^2_{1,1}y(x)$ exists and satisfies it. In this case we can rewrite Eq.(3.1) as follows:

$$(\underline{y}''(x,r), \overline{y}''(x,r)) = p_1(x)(\underline{y}'(x,r), \overline{y}'(x,r)) + p_2(x)(\underline{y}(x,r), \overline{y}(x,r)) + (\underline{g}(x,r), \overline{g}(x,r)).$$

It is clear that in the parametric form

$$(\underline{y}''(x,r), \overline{y}''(x,r)) = p_1(x)(\underline{y}'(x,r), \overline{y}'(x,r)) + p_2(x)(\underline{y}(x,r), \overline{y}(x,r)) + (\underline{g}(x,r), \overline{g}(x,r)).$$

Now we take $w = D^2_{1,1}y(x,r) = (\underline{y}''(x,r), \overline{y}''(x,r))$, $u = D^1_1 y(x,r) = (\underline{y}'(x,r), \overline{y}'(x,r))$ and $v = y(x,r) = (\underline{y}(x,r), \overline{y}(x,r))$, then we rewrite the above equation as:

$$w = p_1(x)u + p_2(x)v + g(x).$$

Along the Lemma 2.4 we conclude that:

$$w_c = p_1(x)u_c + p_2(x)v_c + g_c(x),$$
$$w_d = |p_1(x)|u_d + |p_2(x)|v_d + g_d(x),$$

where

$$w_c = \frac{(\underline{y}''(x) + \overline{y}''(x))}{2} = (\frac{\underline{y}(x) + \overline{y}(x)}{2})'' = y_c'',$$
$$u_c = \frac{(\underline{y}'(x) + \overline{y}'(x))}{2} = (\frac{\underline{y}(x) + \overline{y}(x)}{2})' = y_c',$$
$$w_d = \frac{(\overline{y}''(x) - \underline{y}''(x))}{2} = (\frac{\overline{y}(x) - \underline{y}(x)}{2})'' = y_d'',$$
$$u_d = \frac{(\overline{y}'(x) - \underline{y}'(x))}{2} = (\frac{\overline{y}(x) - \underline{y}(x)}{2}) = y_d',$$
$$v_c = y_c, \quad v_d = y_d.$$

Then for each r, we have two crisp boundary value problems as:

$$\begin{cases} y_c''(x,r) = p_1(x)y_c'(x,r) + p_2(x)y_c(x,r) + g_c(x), \\ y_c(0,r) = A_c(r), \\ y_c(l,r) = B_c(r), \end{cases} \tag{3.2}$$

and

$$\begin{cases} y_d''(x,r) = |p_1(x)|y_d'(x,r) + |p_2(x)|y_d(x,r) + g_d(x), \\ y_d(0,r) = A_d(r), \\ y_d(l,r) = B_d(r). \end{cases} \tag{3.3}$$

Remark 3.1. [19] Note that in the crisp case if

$$\begin{cases} y''(x) = a(x)y'(x) + b(x)y(x) + c(x), \\ y(0) = y_0, \\ y(l) = y_l, \end{cases} \tag{3.4}$$

with

1): $a(x), b(x), c(x) \in C[0,l]$,
2): $b(x) \geq 0$ on $[0,l]$,

then BVP(3.4) has a unique solution.

Proposition 3.2. *Assuming*

1): $p_1(x), p_2(x) \in C[0,l]$,
2): $g_c(x,r)$ *and* $g_d(x,r)$ *are continuous on* $D = [0,l] \times [0,1]$,
3): $p_2(x) > 0$ *for each* $x \in [0,l]$,

then both BVP (3.2) and BVP(3.3) have unique solutions that respectively denoted by $y_c(x,r)$,$y_d(x,r)$. *If* $y_d(x,r), y_d'(x,r) \geq 0$ *on* $D = [0,l] \times [0,1]$ *then fuzzy BVP(3.1) has a unique (1-1)solution.*

Proof. According to Remark(3.1), obviously, both BVP (3.2) and BVP (3.3) have unique solutions. Since $y_d(x,r) \geq 0, y_d'(x,r) \geq 0$ on D then by substituting into the Eq. (3.3), we have $y_d''(x,r) \geq 0$, therefor from Theorem 2.11, $y(x) = (y_c(x,r) - y_d(x,r), y_c(x,r) + y_d(x,r))$ is (1-1)solution to fuzzy BVP.(3.1) at $x \in [0,1]$. $\square$

Case 2.
y is (1-2)solution of fuzzy BVP(3.1) if $D^2_{1,2}y(x)$ exists and satisfies it. In this case we can rewrite Eq.(3.1) as follows:

$$[\overline{y}''(x,r), \underline{y}''(x,r)] = p_1(x)[\underline{y}'(x,r), \overline{y}'(x,r)] + p_2(x)[\underline{y}(x,r), \overline{y}(x,r)] + [\underline{g}(x,r), \overline{g}(x,r)].$$

It is clear that in the parametric form

$$(\overline{y}''(x), \underline{y}''(x)) = p_1(x)(\underline{y}'(x), \overline{y}'(x)) + p_2(x)(\underline{y}(x), \overline{y}(x)) + (\underline{g}(x,r), \overline{g}(x,r)).$$

Now we take $w = D^2_{1,2}y(x) = (\overline{y}''(x), \underline{y}''(x))$, $u = D^1_1 y(x) = (\underline{y}'(x), \overline{y}'(x))$, and $v = y(x) = (\underline{y}(x), \overline{y}(x))$ then we rewrite the above parametric form as

$$w = p_1(x)u + p_2(x)v + g(x).$$

Due to Lemma 2.4, we conclude that

$$w_c = p_1(x)u_c + p_2(x)v_c + g_c(x),$$
$$w_d = |p_1(x)|u_d + |p_2(x)|v_d + g_d(x),$$

where

$$w_c = \frac{(\underline{y}''(x) + \overline{y}''(x))}{2} = (\frac{\underline{y}(x) + \overline{y}(x)}{2})'' = y_c'',$$
$$u_c = \frac{(\underline{y}'(x) + \overline{y}'(x))}{2} = (\frac{\underline{y}(x) + \overline{y}(x)}{2})' = y_c',$$
$$w_d = \frac{(\underline{y}''(x) - \overline{y}''(x))}{2} = -(\frac{\overline{y}(x) - \underline{y}(x)}{2})'' = -y_d'',$$

$$u_d = \frac{(\overline{y}'(x) - \underline{y}'(x))}{2} = (\frac{\overline{y}(x) - \underline{y}(x)}{2})' = y_d',$$
$$v_c = y_c, \quad v_d = y_d.$$

For each r we have two crisp boundary value problems as

$$\begin{cases} y_c''(x,r) = p_1(x)y_c'(x,r) + p_2(x)y_c(x,r) + g_c(x,r), \\ y_c(0) = A_c(r), \\ y_c(l) = B_c(r), \end{cases} \tag{3.5}$$

and

$$\begin{cases} -y_d''(x,r) = |p_1(x)|y_d'(x,r) + |p_2(x)|y_d(x,r) + g_d(x,r), \\ y_d(0,r) = A_d(r), \\ y_d(l,r) = B_d(r). \end{cases} \tag{3.6}$$

Proposition 3.3. *Assuming that*

1): $p_1(x)$, $p_2(x)$ *are continuous on* $[0,l]$,
2): $g_c(x,r)$ *be continuous on* $D = [0,l] \times [0,1]$,
3): $p_2(x) > 0$ *on* $[0,l]$,
4): *would have existed a unique solution* $y_d(x,r)$ *to BVP (3.6) where* $y_d(x,r), y_d'(x,r) \geq 0$ *on* $D = [0,l] \times [0,1]$,

so (3.1) has the unique (1-2)solution.

Proof. According to Remark 3.1, it is obvious that BVP (3.5) has a unique solution. For each x in the domain, assuming there exists a unique solution $y_d(x,r)$ to BVP (3.6) where $y_d(x,r), y_d'(x,r) \geq 0$ on $D = [0,l] \times [0,1]$, then by substituting in to the Eq.(3.6), we have $y_d''(x) \leq 0$ then from Theorem 2.11 , $y(x) = (y_c(x,r) - y_d(x,r), y_c(x,r) + y_d(x,r)$ is (1-2)solution of fuzzy BVP (3.1) at $x \in [0,1]$. □

Case 3.
y is the (2-1)solution to fuzzy BVP(3.1) if $D^2_{2,1}y(x)$ exists and satisfies it. In this case, we can rewrite Eq. (3.1) as follows:

$$[\overline{y}''(x,r), \underline{y}''(x,r)] = p_1(x)[\overline{y}'(x,r), \underline{y}'(x,r)] + p_2(x)[\underline{y}(x,r), \overline{y}(x,r)] + [\underline{g}(x,r), \overline{g}(x,r)].$$

In a similar fashion to Case 1 and Case 2, for each r we have two crisp boundary value problems:

$$\begin{cases} y_c''(x,r) = p_1(x)y_c'(x,r) + p_2(x)y_c(x,r) + g_c(x,r), \\ y_c(0) = A_c(r), \\ y_c(l) = B_c(r), \end{cases} \tag{3.7}$$

and

$$\begin{cases} -y_d''(x,r) = -|p_1(x)|y_d'(x,r) + |p_2(x)|y_d(x,r) + g_d(x,r), \\ y_d(0,r) = A_d(r), \\ y_d(l,r) = B_d(r), \end{cases} \tag{3.8}$$

where
$y_c''(x,r) = (D^2_{2,1}y(x))_c$, $y_c'(x,r) = (D^1_2y(x))_c(r)$, $y_d''(x,r) = -(D^2_{2,1}y(x))_d$ and $y_d'(x,r) = -(D^1_2y(x))_d(r)$.

Proposition 3.4. *Assuming that*

1): $p_1(x)$, $p_2(x)$ *are continuous on* $[0,l]$,
2): $g_c(x,r)$ *be continuous on* $D = [0,l] \times [0,1]$,
3): $p_2(x) > 0$ *on* $[0,l]$,
4): *would have existed unique solution* $y_d(x,r)$ *to BVP (3.8) where* $y_d(x,r) \geq 0, y_d'(x,r) \leq 0$ *on* $D = [0,l] \times [0,1]$,

so fuzzy BVP (3.1) has the unique (2-1)solution.

Proof. According to Remark 3.1, it is obvious that BVP (3.7) has a unique solution. For each x in the domain, assuming there exists unique solution $y_d(x,r)$ to (3.6) where $y_d(x,r) \geq 0, y'_d(x,r) \leq 0$ on $D = [0,l] \times [0,1]$, then by substituting in to the Eq. (3.8), we have $y''_d(x,r) \leq 0$ then from Theorem 2.11, $y(x) = (y_c(x,r) - y_d(x,r), y_c(x,r) + y_d(x,r)$ is (2-1)solution of fuzzy BVP (3.1) at $x \in [0,1]$. □

Case 4.
y is a (2-2)solution to fuzzy BVP (3.1) if $D^2_{2,2}y(x)$ exists and satisfies it. In this case we can rewrite Eq.(3.1) as follows:

$$[\underline{y}''(x,r), \overline{y}''(x,r)] = p_1(x)[\overline{y}'(x,r), \underline{y}'(x,r)] + p_2(x)[\underline{y}(x,r), \overline{y}(x,r)] + [\underline{g}(x,r), \overline{g}(x,r)].$$

Then, for each r we have two crisp boundary value problems as

$$\begin{cases} y''_c(x,r) = p_1(x)y'_c(x,r) + p_2(x)y_c(x,r) + g_c(x,r), \\ y_c(0) = A_c(r), \\ y_c(l) = B_c(r), \end{cases} \tag{3.9}$$

and

$$\begin{cases} y''_d(x,r) = -|p_1(x)|y'_d(x,r) + |p_2(x)|y_d(x,r) + g_d(x,r), \\ y_d(0,r) = A_d(r), \\ y_d(l,r) = B_d(r), \end{cases} \tag{3.10}$$

where $y''_c(x,r) = (D^2_{2,2}y(x))_c,\ y'_c(x,r) = (D^1_2 y(x))_c(r)$ and $y''_d(x,r) = (D^2_{2,2}y(x))_d,\ y'_d(x,r) = -(D^1_2 y(x))_d(r)$.

Proposition 3.5. *Assume that*

1): $p_1(x)$, $p_2(x)$ *are continuous on* $[0,l]$,
2): $g_c(x,r)$ *and* $g_d(x,r)$ *are continuous on* $D = [0,l] \times [0,1]$,
3): $p_2(x) > 0$ *on* $[0,l]$,

then BVPs (3.9) and (3.10) have unique solutions that respectively denoted by $y_c(x,r)$,$y_d(x,r)$ *and let* $y_d(x,r) \geq 0, y'_d(x,r) \leq 0$ *on* D *then fuzzy BVP (3.1) has a unique (2-2)solution.*

Proof. According to Remark 3.1, it is obvious, that BVPs (3.9) and (3.10) have unique solutions. Since $y_d(x,r) \geq 0, y'_d(x,r) \leq 0$ on D then by substituting into the Eq. (3.10, we have $y''_d(x,r) \geq 0$ therefore from Theorem 2.11, $y(x) = (y_c(x,r) - y_d(x,r), y_c(x,r) + y_d(x,r))$ is (2-2)solution of fuzzy BVP (3.1) at $x \in [0,1]$. □

4. Numerical Results

Example 4.1. Suppose fuzzy two-point linear BVP is given as below:

$$\begin{cases} \frac{d^2}{dt^2}y(x,r) = \frac{d}{dt}y(x,r) + (r, 2-r), \quad 0 \leq x \leq 1, \\ y(0,r) = \tilde{0} = (-2+2r, 2-2r), \\ y(1,r) = (-4+r, 6-r). \end{cases} \tag{4.1}$$

where $0 \leq r \leq 1$.

1): Obtaining (1-1)solution to fuzzy BVP (4.1) is considered, we first rewrite Eq.(4.1) as

$$\begin{cases} D^2_{1,1}y(x,r) = D^1_1 y(x,r) + (r, 2-r), \quad 0 \leq x \leq 1, \\ y(0,r) = \tilde{0}, \\ y(1,r) = (-4+r, 6-r). \end{cases} \tag{4.2}$$

where $0 \leq r \leq 1$.
Due to Case.1 of Section 3, we apply a standard method to solve two crisp BVPs as follows:

$$\begin{cases} y''_c(x,r) = y'_c(x,r) + 1, \quad 0 \leq x \leq 1, \\ y_c(0,r) = 0, \\ y_c(1,r) = 1, \end{cases} \tag{4.3}$$

and

$$\begin{cases} y_d''(x,r) = y_d'(x,r) + (1-r), \quad 0 \le x \le 1, \\ y_d(0,r) = 2-2r, \\ y_d(1,r) = 5-r. \end{cases} \tag{4.4}$$

Solutions (4.3) and (4.4) respectively are

$$y_c(x,r) = 2\,\frac{\mathrm{e}^x}{\mathrm{e}-1} - x - 2\,(\mathrm{e}-1)^{-1},$$

and

$$y_d(x,r) = \frac{4e^x}{e-1} - \frac{2e(r-1)+3-e}{e-1} + x(r-1).$$

It is obvious that $y_d(x,r) \ge 0$ and $y_d'(x,r) \ge 0$. According to Proposition 3.1, equation (4.1) has a (1-1)solution on $x \in [0,1]$ that is $y(x,r) = (y_c(x,r) - y_d(x,r), y_c(x,r) + y_d(x,r))$. Figure 1(a) shows plots of $\underline{y}(x,0)$(lower bound) and $\overline{y}(x,0)$(upper bound)
Table.1 represent $y(0.2,r_i)$ for $r_i = 0.1i, i = 0,1,...,10$.

Table 1. The lower bound and upper bound of the fuzzy solution at $x = 0.2$.

i	r_i	$y_c(0.2,r_i)$	$y_d(0.2,r_i)$	$y_d'(0.2,r_i)$	$\underline{y}(0.2,r_i)$	$\overline{y}(0.2,r_i)$
0	0	0.0577	2.315	0.4227	−2.258	2.373
1	0.1	0.0577	2.135	0.3795	−2.078	2.193
2	0.2	0.0577	1.955	0.3373	−1.898	2.013
3	0.3	0.0577	1.775	0.2952	−1.718	1.833
4	0.4	0.0577	1.595	0.2530	−1.538	1;633
5	0.5	0.0577	1.415	0.2108	−1.358	1.473
6	0.6	0.0577	1.235	0.1678	−1.178	1.293
7	0.7	0.0577	1.055	0.1265	−0.998	1.113
8	0.8	0.0577	0.875	0.0843	−0.818	0.993
9	0.9	0.0577	0.695	0.0421	−0.638	0.753
10	1	0.0577	0.515	0.0324	−0.458	0.573

2): Obtaining (1-2)solution to (4.1) is considered, we first rewrite (4.1) as

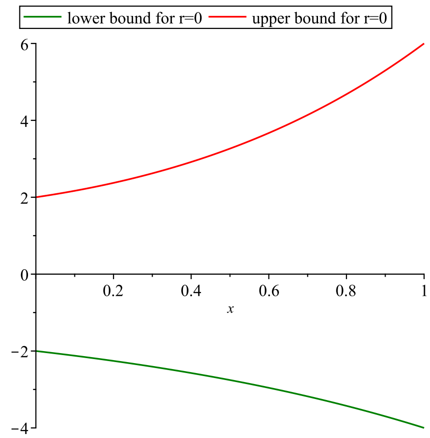
$$\begin{cases} D^2_{1,2}y(x,r) = D^1_1 y(x,r) + (r, 2-r), \quad 0 \le x \le 1, \\ y(0,r) = \tilde{0}, \\ y(1,r) = (-4+r, 6-r). \end{cases} \tag{4.5}$$

where $0 \le r \le 1$.
Due to Case.2 of Section 3, we apply a standard method to solve two crisp BVPs as follows:

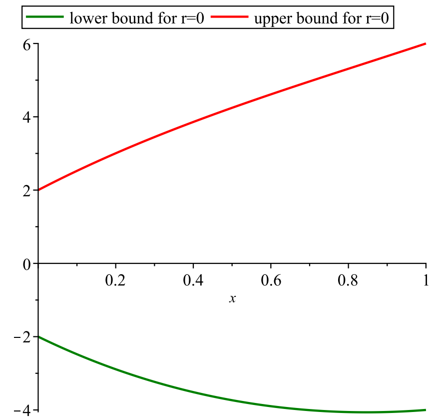
$$\begin{cases} y_c''(x,r) = y_c'(x,r) + 1, \quad 0 \le x \le 1, \\ y_c(0,r) = 0, \\ y_c(1,r) = 1, \end{cases} \tag{4.6}$$

and

$$\begin{cases} -y_d''(x,r) = y_d'(x,r) + (1-r), \quad 0 \le x \le 1, \\ y_d(0,r) = 2-2r, \\ y_d(1,r) = 5-r. \end{cases} \tag{4.7}$$



(a)



(b)

FIGURE 1. (a)(1-1)solution of equation(4.1), (b)(1-2)solution of equation(4.1).

Solutions (4.6) and (4.7) respectively are

$$y_c(x,r) = 2\frac{\mathrm{e}^x}{\mathrm{e}-1} - x - \frac{2}{\mathrm{e}-1},$$

and

$$y_d(x,r) = \frac{4e^{-x}}{e^{-1}-1} - \frac{2(e^{-1}r - e^{-1} - r + 3)}{e^{-1}-1} + x(r-1).$$

It is obvious that $y_d(x,r) \geq 0$ and $y_d'(x,r) \geq 0$. According to Proposition 3.2, equation (4.1) has a (1-2)solution on $x \in [0,1]$ that is $y(x,r) = (y_c(x,r) - y_d(x,r), y_c(x,r) + y_d(x,r))$ this solution for $r = 0$ is shown in Figure 1(b)

3): Suppose (2-1)solution to (4.1) is considered. We can then rewrite (4.1) as

$$\begin{cases} D^2_{2,1}y(x,r) = D^1_1 y(x,r) + (r, 2-r), \quad 0 \leq x \leq 1, \\ y(0,r) = \tilde{0}, \\ y(1,r) = (-4+r, 6-r), \end{cases} \tag{4.8}$$

where $0 \leq r \leq 1$.

we apply a standard method to solve two crisp BVPs as follows:

$$\begin{cases} y_c''(x,r) = y_c'(x,r) + 1, \quad 0 \leq x \leq 1, \\ y_c(0,r) = 0, \\ y_c(1,r) = 1, \end{cases} \tag{4.9}$$

and

$$\begin{cases} -y_d''(x,r) = -y_d'(x,r) + (1-r), \quad 0 \leq x \leq 1, \\ y_d(0,r) = 2-2r, \\ y_d(1,r) = 5-r. \end{cases} \tag{4.10}$$

Solutions (4.9) and (4.10) respectively are

$$y_c(x,r) = 2\,\frac{\mathrm{e}^x}{\mathrm{e}-1} - x - 2\,(\mathrm{e}-1)^{-1},$$

and

$$y_d(x,r) = \frac{2e^x(r+2) - 2(er - e + 2)r - 4}{e-1} + x(1-r).$$

Due to Theorem 2.11 since $y'_d(x,0) > 0$ then (4.1) does not have a (2-1)solution on $x \in [0,1]$.

4): Suppose a (2-2)solution to (4.1) is considered. We can then rewrite (4.1) as

$$\begin{cases} D^2_{2,2}y(x,r) = D^1_2 y(x,r) + (r, 2-r), \quad 0 \le x \le 1, \\ y(0,r) = \tilde{0}, \\ y(1,r) = (-4+r, 6-r), \end{cases} \tag{4.11}$$

where $0 \le r \le 1$.

Due to Case 4 of section 3, we apply a method to solve two crisp BVPs as follows:

$$\begin{cases} y''_c(x,r) = y'_c(x,r) + 1, \quad 0 \le x \le 1, \\ y_c(0,r) = 0, \\ y_c(1,r) = 1, \end{cases} \tag{4.12}$$

and

$$\begin{cases} y''_d(x,r) = -y'_d(x,r) + (1-r), \quad 0 \le x \le 1, \\ y_d(0,r) = 2-2r, \\ y_d(1,r) = 5-r. \end{cases} \tag{4.13}$$

Solutions (4.12) and (4.13) respectively are

$$y_c(x,r) = 2\,\frac{\mathrm{e}^x}{\mathrm{e}-1} - x - 2\,(\mathrm{e}-1)^{-1},$$

and

$$\frac{2e^{-x}(r+1)}{e^{-1}-1} - \frac{2(e^{-1}r - e^{-1}+2)}{e^{-1}-1} + x(1-r).$$

Hence $y_d(x,0) = x$ therefor $y'_d(x,0) > 0$. From Theorem 2.11, since $y'_d(x,r) > 0$ then (4.1) does not have (2-2)solution on $x \in [0,1]$.

Example 4.2. Consider the following two-point linear BVP

$$\begin{cases} y''(x) = cos(x)y'(x) + xy(x) + (r, 2-r), \quad 0 \le x \le 1, \\ y(0,r) = (r, 2-r), \\ y(1,r) = (2r, 4-r). \end{cases} \tag{4.14}$$

Suppose (1-1)solution to (4.14) is considered. We can rewrite fuzzy BVP (4.14) as

$$\begin{cases} D^2_{1,1}y(x,r) = cos(x)D^1_1 y(x,r) + xy(x) + (r, 2-r), \quad 1 \le x \le 1, \\ y(0,r) = (r, 2-r), \\ y(1,r) = (2r, 4-r), \end{cases} \tag{4.15}$$

where $0 \le r \le 1$.

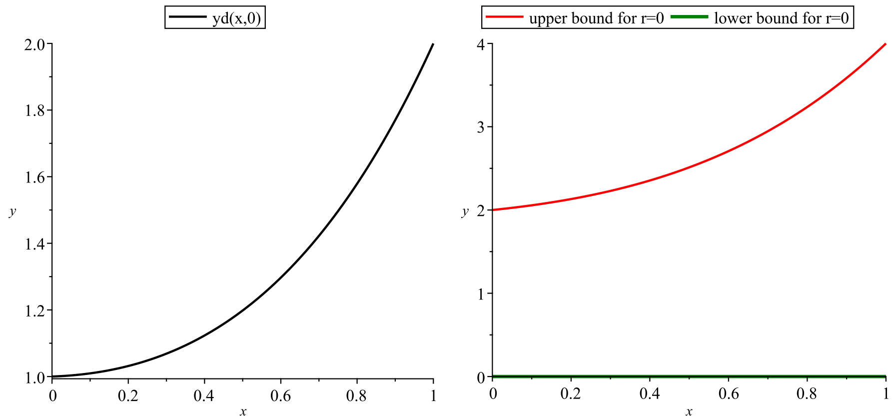
Due to Case.1 in Section 3, it is enough to solve two crisp BVPs as follows:

$$\begin{cases} y''_c(x,r) = cos(x)y'_c(x,r) + xy_c(x,r) + 1, \quad 0 \le x \le 1, \\ y_c(0,r) = 1, \\ y_c(1,r) = 2 + 0.5r, \end{cases} \tag{4.16}$$

and

$$\begin{cases} y''_d(x) = |cos(x)|y'_d(x) + |x|y_d(x) + 1 - r, \quad 0 \le x \le 1, \\ y_d(0,r) = 1-r, \\ y_d(1,r) = 2 - 1.5r. \end{cases} \tag{4.17}$$

Using standard methods (analytic or numerical), we conclude that $y_d(x,r) \ge 0, y'_d(x,r) \ge 0$ for all $1 \le x \le 2$ and $0 \le r \le 1$. Thus, the Eq. (4.14) has the unique (1-1)solution in its domain. Figure 2(a) shows $y_d(x,0.5) \ge 0, y'_d(x,0.5) \ge 0$ for $1 \le x \le 2$. Figure 2(b) indicates plots of $\underline{y}(x,0.5)$(lower bound) and $\overline{y}(x,0.5)$ (upper bound) of (1-1) solution of Eq. (4.15) in $[0,1]$.



(a) (b)

FIGURE 2. (a)Solution of (4.17) with $r = 0.5$, (b)(1-1)solution of (4.14) in$[0, 1]$ with r=0.

Now suppose (1-2)solution of fuzzy BVP (4.14) is considered. We can rewrite Eq. (4.14) as

$$\begin{cases} D^2_{1,2}y(x,r) = cos(x)D^1_1 y(x,r) + ty(x) + (r, 2-r), \quad 0 \le x \le 1, \\ y(0,r) = (r, 2-r), \\ y(1,r) = (2r, 4-r), \end{cases} \tag{4.18}$$

where $0 \le r \le 1$.

Due to Case 2 in section 3, it is enough to solve two crisp BVPs as follows:

$$\begin{cases} y''_c(x,r) = cos(x)y'_c(x,r) + ty_c(x,r) + 1, \quad 0 \le x \le 1, \\ y_c(0,r) = 1, \\ y_c(1,r) = 2 + 0.5r, \end{cases} \tag{4.19}$$

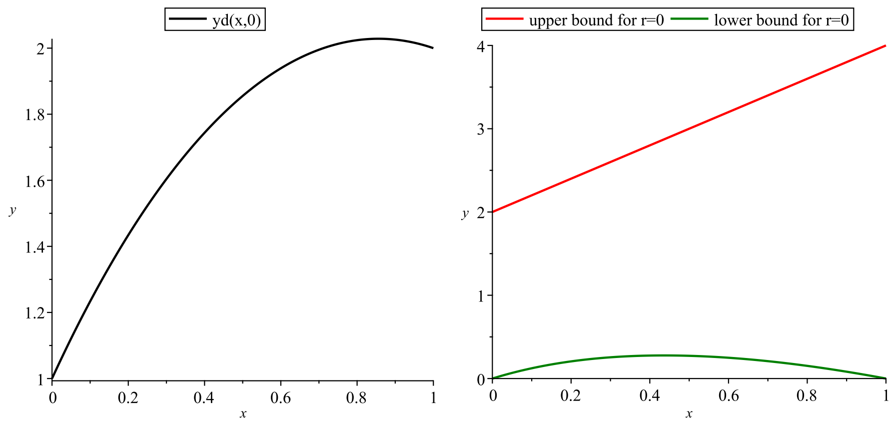
and

$$\begin{cases} -y''_d(x) = |cos(x)|y'_d(x) + |x|y_d(x) + 1 - r, \quad 0 \le x \le 1, \\ y_d(0,r) = 1 - r, \\ y_d(1,r) = 2 - 1.5r. \end{cases} \tag{4.20}$$

Using standard methods (analytic or numerical), we conclude that for each r, $y_d(x,r) \ge 0$ for each $x \in [0,1]$ but $y'_d(x,r) \ge 0$ for each $x \in [0, 0.856]$ while $y'_d(x,r) < 0$ for each $x \in [0.856, 1]$ see Figure 3(a) Hence the equation (4.14) has (1-2)solution on $[0, 0.856]$ but does not have (1-2)solution on $[0.856, 1]$. Figure 3(b) shows both graphs of $\underline{y}(x, 0)$(lower bound) and graph of $\overline{y}(x, 0)$(upper bound) to (4.18) where is (1-1)solution on $[0, 0.856]$.

5. CONCLUSION

Here, we have studied linear two-point fuzzy boundary value problems. Since there exists the first-kind and second-kind derivatives, hence we use the concept of Hukuhara differentiability and convert a linear fuzzy differential equation to four cases, where each case deals with crisp mathematics, where standard algorithms can be used. We hope to extend our method based on generalized differentiability fuzzy problems in the future.



(a) (b)

FIGURE 3. (a)Solution of (4.20) with $r = 0$, (b)Solution of (4.18) with $r = 0$.

References

[1] S. Abbasbandy, E. Babolian, and M. Alavi, *Numerical method for solving linear Fredholm fuzzy integral equations of the second kind*, Chaos Soliton Fract, *31* (2007), 138-146.

[2] E. Babolian, S. Abbasbandy, and M. Alavi, *Numerical Solution Of Fuzzy Differential Inclusion By Euler Method*, J. Sci. I. A. U (JSIAU), *18* (2009), 60-65.

[3] B. Bede and S. G. Gal, *Almost periodic fuzzy number-valued functions*, Fuzzy sets and Systems, *147* (2004), 385-403.

[4] B. Bede and S. G. Gal, *Generalizations of the differentibility of fuzzy number value functions with applications to fuzzy differential equations*, Fuzzy Sets and Systems, *151* (2005), 581-599.

[5] V. Bobylev, *A possibilistic argument for irreversibility*, Fuzzy Set Syst, *34* (1990), 7380.

[6] J. J. Buckley and T. Feuring, *Fuzzy differential equations*, Fuzzy Set Syst, *16* (2000), 4354.

[7] Y. Chalco-cano and H. Roman-Flores, *On new solutions of fuzzy differential equations*, Chaos, Soliton and Fractals, *38* (2008), 112-119.

[8] W. Cong-Xin and M. Ming, *Embedding Problem of fuzzy number space, Part I*, Fuzzy Sets and System, *44* (1991), 33-38.

[9] R. Dai and M. Chen, *On the structural stability for two-point boundary value problems of undamped fuzzy differential equations*, Fuzzy Sets and Systems Inpress, doi.org/10.1016/j.fss, (2022),03-014.

[10] P. Diamond, *Stability and periodicity in fuzzy differential equations*, IEEE Trans Fuzzy Syst, *8*(2000), 583590.

[11] P. Diamond, *Time-dependent differential inclusions, cocycle attractors and fuzzy differential equations*, IEEE Trans Fuzzy Syst , *7*(1999), 734740.

[12] D. Dubois and H. Prade, *Fuzzy number: an overview, in: The Analysis of Fuzzy Information*, Mathematics, j.c. Bezdek (Ed.), CRC Press, Boca Raton, FL, *1* (1987), 3-39.

[13] N. Gasilov, E. Amrahov, and A. G. Fatullayev, *Solution of linear differential equations with fuzzy boundary values*, Fuzzy Sets and Systems, *257* (2014), 169183.

[14] N. Gholami, T. Allahviranloo, S. Abbasbandy, and N. Karamikabir, *Fuzzy reproducing kernel space method for solving fuzzy boundary value problems*, Mathematical Sciences, *13* (2019), 97-103.

[15] A. K. Golmankhaneh and A. Jafarian, *About fuzzy Schrdinger equation in ICFDA*, 14th International Conference on Fractional Differentiation and Its Applications, IEEE, *2014* (2014), 1-5.

[16] M. Hukuhara, *Integration des applications measurables dont la valeur est un compact convexe*, Funkcialaj Ekvacioj, *10* (1967), 205-223.

[17] E. Hullermeier, *An approach to modeling and simulation of uncertain dynamical systems*, Int J Uncertain Fuzz Knowledge-based Syst, *5* (1997), 117137.

[18] O. Kaleva, *Fuzzy differential equations*, Fuzzy Set Syst, *24* (1987), 30117.

[19] H. B. Keller, *Numerical methods for two-point boundary value problems* Blaisdell. Waltham. MA, (1968) ,184pp. QA 372.K42

[20] A. Khastan and J. J. Nieto, *A boundary value problem for second order fuzzy differential equations*, Nonlinear Analysis, *72* (2010), 3583-3593.

[21] G. Klir, U. Clair ana B. Yuan, *Fuzzy set theory: foundations and applications*, Prentice-Hall Inc.; 1997.

[22] V. Lakshmikantham, K. N. Murty, and J. Turner, *Two-point boundary value problems associated with non-linear fuzzy differentiual equation*, Math. Inequal. Appl. *4* (2001), 727-533.

[23] X. A. Li, *General Purpose Variational Formulation for Boundary Value problems of orders Greater than Two*, Journal of Applied and Computational Mechanics, *7* (2021), 1788-1802.

[24] H. V. Long, J. J. Nieto, snd N. T. K. Son, *New approach for studying nonlocal problems related to differential systems and partial differential equations in generalized fuzzy metric spaces*, Fuzzy Sets and Systems, *331*(2018), 26-46

[25] M. Ma, M. Friedman and A. Kandel, *Numerical solution of fuzzy differential equations*, Fuzzy Set Syst, *105* (1999), 133138.

[26] S. Seikkala, *On the fuzzy initial value problem*, Fuzzy Set Syst, *24* (1987), 31930.

[27] V.I. Slyn'ko and C. Tun, *Instability of set differential equations*, J. Math. Anal. Appl. *467* (2018), 935-994.

[28] C. Tun and C. Tun , *A new boundedness theorem for a class of second order differential equations*, Arab. J. Sci. Eng. Sect. A Sci, *33*(2008),8392.

[29] C. Tun and O. Tun, *A note on the stability and boundedness of solutions to non-linear differential systems of second order*, Journal of the Association of Arab Universities for Basic and Applied Sciences, *24* (2017), 169-175.

[30] W. Voxman, *Canonical representation of discrete fuzzy numbers*, Fuzzy Set Syst, *118* (2001), 557-466.

[31] H. Wangon and R. Rodriguez Lpezb, *The existence of solutions to boundary value problems for interval-valued differential equations under gH-differentiability*, Information Sciences, *553* (2021), 225-246.