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Improved new qualitative results on stochastic delay differential equations of second order

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Abstract

This paper deals with a class of stochastic delay differential equations (SDDEs) of second order with multiple delays. Here, two main and novel results are proved on stochastic asymptotic stability and stochastic boundedness of solutions of the considered SDDEs. In the proofs of results, the Lyapunov-Krasovskii functional (LKF) method is used as the main tool. A comparison between our results and those are available in the literature shows that the main results of this paper have new contributions to the related ones in the current literature. Two numerical examples are given to show the applications of the given results.

Keywords. SDDEs, Second order, Multiple constant delays, Stability, Boundedness, Ito formula, LKF.
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1. Introduction

In the relevant literature, the qualitative theory of stochastic differential equations and SDDEs are active fields of research. For example, various qualitative properties of that kind of equations have been studied throughout numerous books or papers, see ([1–21, 41, 45, 46]).

As for some related works of this paper, qualitative properties of solutions called stability and boundedness of solutions of various nonlinear SDDEs of second order have been studied by some researchers (see, [1–6, 41]). In particular, Abou-El-Ela et al. [2] established two qualitative results on the motions of solutions of the following SDDE of second order:

$$x'' + g(x') + bx(t-\tau) + \sigma x\omega'(t) = p(t, x, x(t-h), x').$$

In [2], two results are derived, the first one is on the stochastic asymptotic stability of zero solution for this SDDE when $p \equiv 0$, while the second result studies the boundedness of solutions of this SDDE when $p \neq 0$. Here, an LKF is constructed and the proofs are provided by that LKF, and examples are also provided as numerical applications of the new results. Ademola et al. [5] investigated the stability and boundedness of the following nonlinear and nonautonomous SDE of second-order:

$$x'' + g(x, x')x' + f(x) + \sigma x\omega'(t) = p(t, x, x').$$

In [5], the second Lyapunov method is used for the nonlinear functions of this equation to obtain sufficient criteria that guarantee the stability and boundedness of solutions. Examples are also given to authenticate the correctness of the obtained results. Abou-El-Ela et al. [3] obtained a new result to the following nonlinear SDDE of second order with constant delay:

$$x'' + a(t)x' + b(t)f(x(t-\tau)) + g(t, x)\omega'(t) = 0.$$

In [3], the authors established sufficient conditions for the stochastic asymptotic stability of zero solution of this SDDE. The stochastic asymptotic stability result of that paper is proved by means of the LKF method and two

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examples are also provided as numerical applications of the given results. Later, Tunç and Tunç [41] dealt with the following SDDE of second order:

$$x'' + a(t)f(x,x')x' + b_0(t)g_0(x) + \sum_{i=1}^{n} b_i(t)g_i(x(t-\tau_i)) + g(t,x)\omega'(t) = 0.$$

The authors derived sufficient conditions to provide stochastic stability and stochastic asymptotic stability of zero solution of this equation. In [41], the technique of the proof is based on the LKF method and the derived results of [41] improve some former results on the same concepts in the relevant literature. Ademola et al. [4] considered the following nonlinear SDDE of second order with constant delay:

$$x'' + \psi(t)f(x,x')x' + g(x(t-\tau)) + \sigma x\omega'(t) = p(t,x,x',x(t-\tau)).$$

In [4], by using the LKF approach, similar results as in [41] were obtained on the stability and additionally on the boundedness of solutions. In addition to the sources above, till now, qualitative motions of trajectories of solutions of ordinary differential equations and DDEs of second order have been studied extensively, see, for example, [22–40, 42–44] and the references of these sources. In this article, we are concerned the following SDDE of second order with $(n+1)-$ time delays:

$$\begin{aligned} &x'' + g(x,x')x' + \sum_{i=1}^{n} b_i(t)h_i(x) + \sum_{i=1}^{n} f_i(x(t-\tau_i)) + f(t,x)\omega'(t) = r(t,x,x',x(t-\tau_0),x'(t-\tau_0)), \qquad (1.1)\\ &x(t) = \phi(t), t \in [-\tau, 0], \end{aligned}$$

where $x \in R$, $t \in [-\tau, \infty)$, $\tau_i > 0$, $(i = 0, 1, ..., n)$, are constant delays, $\tau = \max\{\tau_0, \tau_1, ..., \tau_n\}$, $g \in C(R^2, R)$, $b_i \in C(R^+, (0,\infty))$, $R^+ = [0,\infty)$, f_i, $h_i \in C(R,R)$, $f \in C(R^+ \times R, R)$, $r \in C(R^+ \times R \times R \times R \times R, R)$, $h_i(0) = f_i(0) = f(t,0) = 0$, and $\omega(t) \in R^m$ (a standard Wiener process, representing the noise). This continuity condition allows the existence of solutions of SDDE (1.1). In addiction, through this paper, it is supposed the existence and continuity of the derivatives

$$b_i'(t) = \frac{db_i(t)}{dt}, h_i'(x) = \frac{dh_i}{dx}, f_i'(x) = \frac{df_i}{dx}, g_{x'}(x,x') = \frac{\partial g(x,x')}{\partial x'}, i = 1, 2, ..., n.$$

Let throughout the paper x denote $x(t)$. We can convert SDDE (1.1) to the following system of SDDEs:

$$\begin{aligned} x' =& y,\\ y' =& -g(x,y)y - \sum_{i=1}^{n} b_i(t)h_i(x) - \sum_{i=1}^{n} f_i(x) + \sum_{i=1}^{n}\int_{t-\tau_i}^{t} f_i'(x(s))y(s)ds\\ &- f(t,x)\omega'(t) + r(t,x(t),y(t),x(t-\tau_0),y(t-\tau_0)). \qquad (1.2) \end{aligned}$$

The motivation for considering SDDE (1.1) comes from the papers of Abou-El-Ela et al. ([1–3]), Ademola et al. ([4, 5]), Adesina et al. [6], Tunç [41], which are focused on the stability and boundedness of solutions of SDDEs of second order, and those in the references of this paper ([7–40, 42–46]), which are related to the investigation of various properties of mathematical models such as ordinary, functional, stochastic, etc., of differential equations. If we compare SDDE (1.1) with the SDDEs mentioned above, then it can be easily seen that SDDE (1.1) is a different stochastic mathematical model from those SDDEs mentioned above and that can be seen in the relevant literature. In this paper, we define a new LKF and use some elementary inequalities to get some new criteria in relation to the stochastic stability, stochastic asymptotic stability, and uniform stochastic boundedness of solutions of SDDE (1.1).

The rest contents of this paper are organized as follows: Some basis informations, relevant definitions, lemmas and so on are given in section 2. Two new results on the stochastic asymptotic stability of zero solution and uniform stochastic boundedness of solutions of the system of SDDEs (1.2) are addressed in section 3. Lastly, section 4 concludes this paper with a conclusion.

2. Fundamental information

Consider a nonautonomous system of SDDEs:

$$dx(t) = f(t, x(t), x(t-\tau))dt + g(t, x(t), x(t-\tau))dB(t), \tag{2.1}$$

with initial data $\{x(\theta) : -\tau \leq \theta \leq 0\}$, $x_0 \in C([-\tau, 0], R^n)$. Here $f : R^+ \times R^{2n} \to R^n$ and $g : R^+ \times R^{2n} \to R^{n\times m}$ are measurable functions, $f(t, 0, 0) = g(t, 0, 0) = 0$ and $B(t) = (B_1(t), ..., B_m(t))^T$ is an $m-$dimensional Brownian motion, which is defined on the well-known probability space $(\Omega, F, F_{t\geq 0}, P)$. Suppose also that the functions f, g satisfy the local Lipschitz condition. Then, the system of SDDEs (2.1) has a unique maximal solution $x(t, x_0)$.

Definition 2.1. The zero solution of (2.1) is called to be stochastically stable or stable in probability, if for every pair $\varepsilon \in (0, 1)$ and $r > 0$, there exists a $\delta_0 = \delta_0(\varepsilon, r) > 0$ such that

$$Pr\{|x(t; t_0)| < r \text{ for all } t \geq 0\} \geq 1 - \varepsilon \text{ whenever } |x_0| < \delta_0.$$

Definition 2.2. The zero solution of (2.1) is said to be stochastically asymptotically stable (SAS) if it is stochastically stable and in addition if for every $\varepsilon \in (0, 1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that

$$Pr\{\lim_{t\to\infty} x(t; x_0) = 0\} \geq 1 - \varepsilon \text{ whenever } |x_0| < \delta.$$

Definition 2.3. A solution $x(t, x_0)$ of (2.1) is called to be uniformly stochastically bounded (USB) or bounded in probability, if it satisfies

$$E^{x_0}||x(t; t_0, x_0)|| \leq C(t_0, ||x_0||), \forall t \geq t_0,$$

where E^{x_0} denotes the expectation operator with respect to the probability law associated with x_0, and $C : R^+ \times R^n \to R^+$ is a constant function depending on t_0 and x_0. We say that the solutions of (2.1) are uniformly stochastically bounded if C is independent of t_0. Suppose that $C^{1,2}(R^+ \times R^n, R^+)$ denotes to the family of all nonnegative LKFs $V(t, x_t)$ defined on $R^+ \times R^n$, which are twice continuously differentiable in x and one in t. By Ito's formula, we have

$$dV(t, x_t) = LV_t(t, x_t) + V_x(t, x_t)g(t, x_t)dB(t),$$

where

$$LV(t, x_t) = \frac{\partial V(t, x_t)}{\partial t} + \frac{\partial V(t, x_t)}{\partial x} f(t, x_t) + \frac{1}{2} trace[g^T(t, x_t)V_{xx}(t, x_t)g(t, x_t)],$$

$dtdt = 0$, $dB_i dt = 0$, $dB_i dB_j = 0$, $dB_i dB_i = dt$, if $i \neq j$,$i, j = 1, 2, ..., n$,

$$V_t = \frac{\partial V}{\partial t}, V_x = \left(\frac{\partial V}{\partial x_1}, ..., \frac{\partial V}{\partial x_n}\right), V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{n\times n} = \begin{pmatrix} \frac{\partial^2 V}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 V}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_n \partial x_n} \end{pmatrix}_{n\times n}$$

with $x_t = x(t + \theta)$, $-r \leq \theta \leq 0$, $t \geq 0$. Let K represent the family of all continuous and nondecreasing functions μ from R^+ to R^+ such that $\mu(0) = 0$ and $\mu(r) > 0$ if $r > 0$.

Lemma 2.4. *(see [5]). Assume that there exist $V \in C^{1,2}(R^+ \times R^n, R^+)$ and $\phi \in K$ such that*

(A1) $V(t, 0) = 0$ for all $t \geq 0$;

(A2) $V(t, x_t) \geq \phi(||x(t)||)$, $\phi(r) \to \infty$ as $r \to \infty$;

(A3) $LV(t, x_t) \leq 0$ for all $(t, x) \in R^+ \times R^n$.

Then, the zero solution of the system of SDDEs (2.1) is stochastically stable. It should be noted that if $LV(t, x_t)$ is negative definite, then the zero solution of (2.1) is SAS.

3. SAS and boundedness

For the novel results as SAS and US boundedness of solutions of this research work, the following assumptions are given. Let μ, g_0, A_i, a_i, C_i, B_i, β_i, K, D_i, ρ, α_i, and r_0 be positive constants such that the following conditions hold:

$(D1)$ $\mu \geq g(x, y) \geq g_0 > 1, xg_y(x, y) \geq 0, \forall t \in R^+, y \neq 0, \forall x, y \in R,$

$(D2)$ $f_i(0)=0, A_i \geq \dfrac{f_i(x)}{x} \geq a_i, x \neq 0, \forall x \in R, \quad C_i \geq f_i'(x) > 0, \forall x \in R,$

$(D3)$ $B_i \geq b_i(t) \geq 1, b_i'(t) \leq 0, \forall t \in R^+,$

$(D4)$ $h_i(0)=0, D_i \geq \dfrac{h_i(x)}{x} \geq \alpha_i, x \neq 0, \forall x \in R, (i=1,2,...,n),$

$(D5)$ $f(t,0)=0, |f(t,x)| \leq \sqrt{2}K|x|, \forall t \in R^+, \forall x \in R,$

$(D6)$ $|r(t,x,y,x(t-\tau_0),y(t-\tau_0))| \leq \rho r_0, \forall t \in R^+, \forall x,y \in R.$

This first result of this paper is the following theorem. Let $r(.) = r(t,x,y,x(t-\tau_0),y(t-\tau_0)) = 0$.

Theorem 3.1. *If $(D1)-(D5)$ hold, then the zero solution of system of SDDEs (1.2) is SAS provided that*

$$2A+2B > K^2, g_0 > 1, \tau < \min\left[\frac{2A+2B-K^2}{2C}, \frac{g_0-1}{3C}\right],$$

where

$$A = \frac{1}{2}\sum_{i=1}^{n} a_i, B = \frac{1}{2}\sum_{i=1}^{n} \alpha_i, C = \frac{1}{2}\sum_{i=1}^{n} C_i.$$

Proof. We construct an LKF $\Upsilon = \Upsilon(t,x_t,y_t)$ by

$$\Upsilon = \sum_{i=1}^{n}\int_0^x f_i(\xi)d\xi + \frac{1}{2}y^2 + \sum_{i=1}^{n}\int_0^x b_i(t)h_i(\xi)d\xi + \int_0^x g(\xi,0)\xi d\xi + xy + \sum_{i=1}^{n}\gamma_i\int_{-\tau_i}^{0}\int_{t+s}^{t} y^2(\theta)d(\theta)ds, \tag{3.1}$$

where $\tau_1 > 0$, $\tau_2 > 0$,...,$\tau_n > 0$ are constant time-delays and the arbitrary positive constants γ_1,γ_2,...,γ_n will be chosen later.

By using $(D1)-(D3)$ and the LKF Υ, we observe the following relations, respectively:

$$\sum_{i=1}^{n}\int_0^x f_i(\xi)d\xi = \sum_{i=1}^{n}\int_0^x \frac{f_i(\xi)}{\xi}\xi d\xi \geq \sum_{i=1}^{n}\int_0^x a_i\xi d\xi = \frac{1}{2}\sum_{i=1}^{n} a_i x^2 = Ax^2,$$

where $A = \frac{1}{2}\sum_{i=1}^{n} a_i$;

$$\sum_{i=1}^{n}\int_0^x b_i(t)h_i(\xi)d\xi \geq \sum_{i=1}^{n}\int_0^x b_i(t)\frac{h_i(\xi)}{\xi}\xi d\xi \geq \sum_{i=1}^{n}\int_0^x \frac{h_i(\xi)}{\xi}\xi d\xi \geq \frac{1}{2}\sum_{i=1}^{n}\alpha_i x^2 = Bx^2,$$

where $B = \frac{1}{2}\sum_{i=1}^{n} \alpha_i$;

$$\int_0^x g(\xi,0)\xi d\xi \geq \frac{g_0}{2}x^2, \quad \sum_{i=1}^{n}\gamma_i\int_{-\tau_i}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds \geq 0.$$

According to the discussion above and (3.1), for an $\alpha > 0$, $\alpha \in R$, we can derive that

$$\Upsilon(t,x_t,y_t) \geq Ax^2 + Bx^2 + Cx^2 + \frac{1}{2}y^2 + xy \geq \alpha(x^2+y^2). \tag{3.2}$$

As for the next step, by virtue of $(D1)-(D3)$ and the LKF Υ, we observe the following inequalities:

$$\sum_{i=1}^{n}\int_0^x f_i(\xi)d\xi = \sum_{i=1}^{n}\int_0^x \frac{f_i(\xi)}{\xi}\xi d\xi \leq \sum_{i=1}^{n}\int_0^x A_i\xi d\xi = \frac{1}{2}\sum_{i=1}^{n} A_i x^2 = Dx^2,$$

$$\begin{aligned}\sum_{i=1}^{n}\int_{0}^{x}b_i(t)h_i(\xi)d\xi =&\sum_{i=1}^{n}\int_{0}^{x}b_i(t)\frac{h_i(\xi)}{\xi}\xi d\xi \le \frac{1}{2}\sum_{i=1}^{n}(B_iD_i)x^2 = Ex^2,\\ \int_{0}^{x}g(s,0)sds \le& Fx^2,\\ \gamma_i\int_{-\tau_i}^{0}\int_{t+s}^{t}y^2(\theta)d\theta ds =&\gamma_i\int_{t-\tau_i}^{t}(\theta-t+\tau_i)y^2(\theta)d\theta\\ \le&\gamma_i y^2\int_{t-\tau_i}^{t}(\theta-t+\tau_i)d\theta\\ =&\frac{1}{2}(\gamma_i\tau_i)y^2=\frac{1}{2}G_iy^2,\end{aligned}$$

where $D=\frac{1}{2}\sum_{i=1}^{n}A_i$, $E=\frac{1}{2}\sum_{i=1}^{n}(B_iD_i)$, $F=\frac{1}{2}\mu$ and $G_i=\frac{1}{2}(\gamma_i\tau_i)$. According to the date above, we have

$$\Upsilon(t,x_t,y_t)\le(D+E+F+2^{-1})x^2+\frac{1}{2}\sum_{i=1}^{n}G_iy^2. \tag{3.3}$$

By combining (3.2) and (3.3), it follows that the LKF Υ has upper and lower bounds as below:

$$\alpha(x^2+y^2)\le\Upsilon(t,x_t,y_t)\le(D+E+F+2^{-1})x^2+\sum_{i=1}^{n}G_iy^2.$$

From the derivative of the LKF Υ along the system of SDDEs (1.2), we obtain the following relationship:

$$\begin{aligned}L\Upsilon(t,x_t,y_t)=&-g(x,y)y^2+g(x,0)xy-g(x,y)xy+y^2+x\sum_{i=1}^{n}\int_{t-\tau_i}^{t}f_i'(x(s))y(s)ds\\ &+y\sum_{i=1}^{n}\int_{t-\tau_i}^{t}f_i'(x(s))y(s)ds-x\sum_{i=1}^{n}f_i(x)+\sum_{i=1}^{n}(\gamma_i\tau_i)y^2(t)-\sum_{i=1}^{n}\gamma_i\int_{t-\tau_i}^{t}y^2(s)ds\\ &+\sum_{i=1}^{n}b_i'(t)\int_{0}^{x}h_i(\xi)d\xi-x\sum_{i=1}^{n}b_i(t)h_i(x)+\frac{1}{2}f^2(t,x).\end{aligned}$$

By virtue of $(D1)-(D5)$, we derive the inequalities below for some terms of $L\Upsilon(t,x_t,y_t)$:

$$\begin{aligned}-g(x,y)y^2\le&g_0y^2,\\ -g(x,y)xy+g(x,0)xy=&-\left[\frac{g(x,y)-g(x,0)}{y}\right]xy^2=-xg_y(x,y)y^2\le 0,\\ x\sum_{i=1}^{n}\int_{t-\tau_i}^{t}f_i'(x(s))y(s)ds\le&\frac{1}{2}\sum_{i=1}^{n}(C_i\tau_i)x^2+\frac{1}{2}\sum_{i=1}^{n}C_i\int_{t-\tau_i}^{t}y^2(s)ds,\\ y\sum_{i=1}^{n}\int_{t-\tau_i}^{t}f_i'(x(s))y(s)ds\le&\frac{1}{2}\sum_{i=1}^{n}(C_i\tau_i)x^2+\frac{1}{2}\sum_{i=1}^{n}C_i\int_{t-\tau_i}^{t}y^2(s)ds,\\ h_i(0)=0,-\sum_{i=1}^{n}b_i(t)xh_i(x)=&-\sum_{i=1}^{n}b_i(t)\frac{h_i(x)}{x}x^2\le-\sum_{i=1}^{n}\alpha_ix^2,\quad x\ne0,\\ f_i(0)=0,-\sum_{i=1}^{n}xf_i(x)=&-\sum_{i=1}^{n}\frac{f_i(x)}{x}x^2\le-\sum_{i=1}^{n}a_ix^2,\quad x\ne0,\\ h_i(0)=0,\sum_{i=1}^{n}b_i'(t)\int_{0}^{x}h_i(\xi)d\xi=&\sum_{i=1}^{n}b_i'(t)\int_{0}^{x}\frac{h_i(\xi)}{\xi}\xi d\xi\le\sum_{i=1}^{n}b_i'(t)\alpha_ix^2\le0,x\ne0,f^2(t,x)\le2K^2x^2.\end{aligned}$$

According to the above inequalities and the derivative $L\Upsilon(t,x_t,y_t)$, we get

$$\begin{aligned}L\Upsilon(t,x_t,y_t) \le &- g_0y^2 + y^2 + \frac{1}{2}\sum_{i=1}^{n}(C_i\tau_i)x^2 + \frac{1}{2}\sum_{i=1}^{n}(C_i\tau_i)y^2 + \sum_{i=1}^{n}C_i\int_{t-\tau i}^{t}y^2(s)ds\\ &+ \sum_{i=1}^{n}(\gamma_i\tau_i)y^2 - \sum_{i=1}^{n}\gamma_i\int_{t-\tau i}^{t}y^2(s)ds - \sum_{i=1}^{n}a_ix^2 - \sum_{i=1}^{n}\alpha_ix^2 + K^2x^2\\ &= -\left[\sum_{i=1}^{n}a_i + \sum_{i=1}^{n}\alpha_i - \frac{1}{2}\sum_{i=1}^{n}(C_i\tau_i) - K^2\right]x^2 - \left[g_0 - 1 - \frac{1}{2}\sum_{i=1}^{n}(C_i\tau_i) - \sum_{i=1}^{n}(\gamma_i\tau_i)\right]y^2\\ &+ \sum_{i=1}^{n}(C_i-\gamma_i)\int_{t-\tau_i}^{t}y^2(s)ds.\end{aligned}$$

Let $\gamma_i = C_i > 0$. Then, we can observe that

$$\begin{aligned}L\Upsilon(t,x_t,y_t) \le &- \left[\sum_{i=1}^{n}a_i + \sum_{i=1}^{n}\alpha_i - \frac{1}{2}\sum_{i=1}^{n}(C_i\tau_i) - K^2\right]x^2 - \left[g_0 - 1 - \frac{3}{2}\sum_{i=1}^{n}(C_i\tau_i)\right]y^2\\ = &- \left[2A + 2B - \frac{1}{2}\sum_{i=1}^{n}(C_i\tau_i) - K^2\right]x^2\\ &- [g_0 - 1 - \frac{3}{2}\sum_{i=1}^{n}(C_i\tau_i)]y^2 \le -[2A+2B-K^2-2C\tau]x^2 - [g_0-1-3C\tau]y^2.\end{aligned}$$

Assume that $2A + 2B - K^2 - 2C\tau = \varepsilon_1$ and $g_0 - 1 - 3C\tau = \varepsilon_2$. Hence,

$$L\Upsilon(t,x_t,y_t) \le -\varepsilon_1x^2 - \varepsilon_2y^2 \le -\rho(x^2+y^2),$$

where $\rho = \min\{\varepsilon_1,\varepsilon_2\}$. According to the discussion above, the zero solution of (1.2) is SAS. The proof of Theorem 3.1 is completed. □

Example 3.2. For the case $r(.) \equiv 0$, we are concerned with the following nonlinear SDDE of second order:

$$x'' + (100 + exp(-(x')^2))x' + (1 + \frac{1}{1+t})(2x + x^2) + x(t-5^{-1}) + \frac{x(t-5^{-1})}{1+x^2(t-5^{-1})} + \sqrt{2}\frac{x}{1+x^4}\omega'(t) = 0. \tag{3.4}$$

As the next step, SDDE (3.4) is converted to the system below:

$$x' = y,$$

$$y' = -(100+exp(-y^2))y - (1+\frac{1}{1+t})(2x+x^2) - x - \frac{x}{1+x^2} + \int_{t-5^{-1}}^{t}\left[(1+\frac{1-x^2(s)}{(1+x^2(s))^2}\right]y(s)ds - \sqrt{2}\frac{x}{1+x^4}\omega'(t). \tag{3.5}$$

According to the systems (1.2) and (3.5), we have the data below, respectively:

$$g(x,y) = 100 + \exp(-y^2),$$

$$99 < 100 = g_0 \le g(x,y) = 100 + \exp(-y^2) \le 101,$$

$$f_1(x) = x + \frac{x}{1+x^2}, f_1(0) = 0,$$

$$\frac{f_1(x)}{x} = 1 + \frac{1}{1+x^2}, x \ne 0,$$

$$a_1 = 0.9 < 1 + \frac{1}{1+x^2} = \frac{f_1(x)}{x} \le 2 = A_2,$$

$$0 < f_1'(x) = 1 + \frac{1-x^2}{(1+x^2)^2} \leq 2 = C_1,$$

$$b(t) = 1 + \frac{1}{1+t},$$

$$1 \leq b(t) = \frac{1}{1+t} \leq 2,$$

$$b'(t) = -\frac{1}{(1+t)^2} \leq 0,$$

$$h_1(x) = 2x + x^2, h_1(0) = 0,$$

$$\frac{h_1(x)}{x} = 2 + x^2 \geq 2 = \alpha_1, x \neq 0,$$

$$f(t,x) = \sqrt{2}\frac{x}{1+x^4},$$

$$|f(t,x)| \leq \sqrt{2}|x|, K = 1,$$

$$2A + 2B = \sum_{i=1}^{1} a_i + \sum_{i=1}^{1} \alpha_i = a_1 + \alpha_1 + 2.9 > 1 = K^2,$$

$$\tau = \frac{1}{5} < \min\left[\frac{2A+2B-K^2}{2C}, \frac{g_0-1}{3C}\right] = \min\left[\frac{1.9}{4}, 66\right].$$

Thus, $(D1)-(D5)$ of Theorem 3.1 hold. Hence, the trivial solution of the system of SDDEs (3.5) is SAS.

Let $r(.) = r(t,x,y,x(t-\tau_0),y(t-\tau_0)) \neq 0$.

Theorem 3.3. *If $(D1)-(D6)$ hold, then the solutions of SDDE (1.2) are USB provided that*

$$2A + 2B > K^2, g_0 > 1, \tau < \min\left[\frac{2A+2B-K^2}{2C}, \frac{g_0-1}{3C}\right],$$

where

$$A = \frac{1}{2}\sum_{i=1}^{n} a_i, B = \frac{1}{2}\sum_{i=1}^{n} \alpha_i, C = \frac{1}{2}\sum_{i=1}^{n} C_i.$$

Proof. The basic tool in this proof is the LKF Υ in (3.1), which is used in the proof of Theorem 3.1. As the next step, depending upon $(D1)-(D6)$, from (3.1) and the system of SDDEs (1.2), we obtain the following inequalities:

$$\begin{aligned} L\Upsilon(t,x_t,y_t) &\leq -\rho(x^2+y^2) + |x||r(.)| + |y||r(.)| \\ &\leq -\rho(x^2+y^2) + r_0\rho|x| + r_0\rho|y| \\ &= -\frac{1}{2}\rho(x^2+y^2) - \frac{1}{2}\rho[|x|-r_0]^2 - \frac{1}{2}\rho[|y|-r_0]^2 + \rho r_0^2 \\ &\leq -\frac{1}{2}\rho(x^2+y^2) + \rho r_0^2, \forall \rho > 0, r_0 > 0. \end{aligned}$$

The rest of the proof is readily completed by following the way of Abou-El-Ela et al. ([2], Theorem 12). Therefore, we will ignore the details of the proof. □

Example 3.4. For the case $r(.) \neq 0$, we are interested in the following nonlinear SDDE of second order:

$$x'' + (100 + exp(-(x')^2))x' + (1 + \frac{1}{1+t})(2x + x^2) + x(t - 5^{-1})$$
$$+ \frac{x(t-5^{-1})}{1 + x^2(t-5^{-1})} + \sqrt{2}\frac{x}{1+x^4}\omega'(t)$$
$$= \frac{1}{16 + t + exp(x^2 + (x')^2 + x^2(t-6^{-1}) + (x'(t-6^{-1}))^2)}. \tag{3.6}$$

Next, SDDE (3.6) is converted to the following system of SDDEs:

$$x' = y,$$
$$y' = -(100 + \exp(-y^2))y - (1 + \frac{1}{1+t})(2x + x^2) - x - \frac{x}{1+x^2}$$
$$+ \int_{t-5^{-1}}^{t} \left[1 + \frac{1 - x^2(s)}{(1 + x^2(s))^2}\right] y(s)ds - \sqrt{2}\frac{x}{1+x^4}\omega'(t)$$
$$+ \frac{1}{16 + t + \exp(x^2 + y^2 + x^2(t-6^{-1}) + y^2(t-6^{-1}))}. \tag{3.7}$$

From (1.2) and (3.7), it follows that $g(x, y)$, $f_1(x)$, $b(t)$, $h_1(x)$, and $f(t, x)$ are from Example 3.2. So, we do not need to do any discussion for these functions. Namely, the verification of $(D1) - (D5)$ has been shown in Example 3.2. Next, we have

$$r(t, x, y, x(t-6^{-1}), y(t-6^{-1})) = \frac{1}{16 + t + \exp(x^2 + y^2 + x^2(t-6^{-1}) + y^2(t-6^{-1}))}.$$

Hence, we derive that

$$|r(t, x, y, x(t-6^{-1}), y(t-6^{-1}))| \leq \frac{1}{16} = \frac{1}{4} \times \frac{1}{4},$$

where $\rho = \frac{1}{4}$ and $r_0 = \frac{1}{4}$.

Finally, all $(D1) - (D6)$ of Theorem 3.3 are verified. Thus, the solutions of the system of SDDEs (3.7) are USB.

4. Conclusion

This research work considers a class of nonlinear SDDEs of second order, and it has two novel results and two related numerical examples as new contributions. The first result of our research work, Theorem 3.1, is related to the stochastic asymptotic stability of the zero solution. Next, the second result of our research work, Theorem 3.3, is related to the uniform stochastic boundedness of solutions. Both of the theorems, Theorem 3.1 and Theorem 3.3, include sufficient conditions. The consistent quality of the proposed model and the qualitative behaviors of its solutions are analyzed by means of a new LKF. Numerical applications of the given results are presented in two examples. The aim of this research work is to provide novel and useful contributions to the theory of SDDEs.

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